Graduate J. Math. 3 (2018), 94 – 97

The weakly dependent strong law of large numbers revisited

Abdelmalek Abdesselam

Abstract

In this expository note we give a short, self-contained, and elementary proof of the strong law of large numbers under a power law decay hypothesis for joint second moments. The result is related to the classical one by Lyons. However, we also provide a rate of convergence. Our proof does not use maximal inequalities and is instead inspired by the method of multiscale large versus small field decompositions in constructive quantum field theory. As a hopefully entertaining application, we also include a short derivation of the so-called "Infinite Monkey Theorem".

MSC 2010. Primary: 60F15; Secondary: 81T08.

1 Introduction and main theorem

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of centered, real-valued, square integrable random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will denote the average of the first N variables by $A_N = \frac{X_1+\cdots+X_N}{N}$. The main result described in this article is as follows.

Theorem 1. *Suppose the sequence satisfies*

$$\exists\gamma>0, \exists K>0, \forall(m,n)\in\mathbb{N}^2,$$

$$\mathbb{E}X_mX_n \le \frac{K}{(1+|m-n|)^\gamma}\ .$$

Let β be a parameter in the interval $\left(\frac{1}{2},1\right)$ if $\gamma\ge 1$, or in the interval $\left(1-\frac{\gamma}{2},1\right)$ if $0<\gamma<1$. Then, with probability one, we have

$$|A_N| = O\left(\frac{\log N}{N^{1-\beta}}\right)\ .$$

Note that our hypothesis (for $m\neq n$) automatically holds in the case of negatively correlated variables. Also note that our hypothesis includes (when $m=n$) the requirement of uniformly bounded variances for the X_n, just as in [11, Corollary 11]. The result by Lyons allows more general bounds (there denoted by Φ_1) on second moments. However, in the power law case (i.e., $\Phi_1(x)=(1+x)^{-\gamma}$) the corresponding hypothesis, namely $\gamma>0$, is identical to ours. On the other hand, [11] proves the strong law of large numbers (SLLN), $A_N\to 0$ a. s., yet without rate of convergence such as the one provided by our theorem. The SLLN for dependent random variables (with or without almost sure rate of convergence) has been investigated in a number of relatively recent articles. In addition to [11], see for instance [2, 4–6, 8–10, 13, 14]. For example, [9, Theorem 2] implies the $\gamma>1$ part of our theorem but does not cover the case of long-range dependence $0<\gamma<1$.

In this note, we did not aim for maximal generality but rather for maximal simplicity. Indeed, in most of the literature we cited, the SLLN is proved by a two-step procedure where the intermediate stage consists in establishing a suitable maximal inequality. Our proof, inspired by the multiscale large versus small field decomposition method in constructive quantum field theory (see, e.g., [1] or [3]), is direct and bypasses the need for maximal inequalities. It is based on two simple ingredients. The first one is what one may loosely call multiscale (or dyadic) analysis, i.e., studying a random function (here $n\mapsto X_n$) in terms of its sums or averages on dyadic blocks. The latter are most easily visualized thanks to a dyadic tree. The second ingredient is combinatorial optimization in order to get good estimates. This involves the use of a very simple algorithm, namely, the greedy algorithm which can be summarized by the phrase "grab as much as you can, as soon as you can".

Note that it is also possible to prove the theorem using a maximal inequality. Probably the easiest way to do so would be to use [7, Lemma 1]. Instead, we set appropriate thresholds on sums corresponding to dyadic blocks which separate small from large field blocks and exploit a multiscale application of the Borel-Cantelli Lemma with suitable decay in both the scale direction and the spatial direction which together label the dyadic blocks. This is more in the spirit of large versus small field decompositions in field theory.

2 Proof of the theorem

For $\gamma > 0$, let us define the nondecreasing function $p_\gamma : [1, \infty) \to (0, \infty)$ as follows.

$$p_\gamma(x) = \begin{cases} \frac{1}{\gamma - 1} & \text{if} \quad \gamma > 1 \ , \\ (\log x) + 1 & \text{if} \quad \gamma = 1 \ , \\ \frac{x^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma < 1 \ . \end{cases}$$

Lemma 1. *For every finite set $J \subset \mathbb{N}$,*

$$\sum_{n \in J} \frac{1}{(1+n)^\gamma} \le p_\gamma(|J| + 1)$$

where $|J|$ denotes the cardinality of J.

Proof: The inequality is trivial if $J = \varnothing$. Otherwise the left-hand side is maximized, for fixed $|J|$, when $J = \{1, \ldots, |J|\}$. A simple sum/integral comparison gives the upper bound $\int_0^{|J|} \frac{dx}{(1+x)^\gamma}$ and the lemma follows from the evaluation of the integral in all three cases for γ. □

Lemma 2. *For every nonempty set $J \subset \mathbb{N}$,*

$$\sum_{(m,n) \in J^2} \frac{1}{(1 + |m-n|)^\gamma} \le |J| \times (1 + 2p_\gamma(|J|)) \ .$$

Proof: By separating the cases corresponding to the relative positions of m and n, one sees that the left-hand side is equal to

$$|J| + 2 \sum_{m \in J} \sum_{k \in J_m} \frac{1}{(1+k)^\gamma}$$

where $J_m = \{n - m \mid n \in J \text{ and } n > m\}$. We then apply Lemma 1 to the sum over J_m, together with the nondecreasing property of p_γ and the obvious inequality $|J_m| + 1 \le |J|$, in order to conclude. □

For any nonempty finite set $J \in \mathbb{N}$, we have, by hypothesis,

$$\begin{aligned} \mathbb{E}\left(\sum_{n \in J} X_n\right)^2 &= \sum_{(m,n) \in J^2} \mathbb{E} X_m X_n \\ &\le K \sum_{(m,n) \in J^2} \frac{1}{(1+|m-n|)^\gamma} \ . \end{aligned}$$

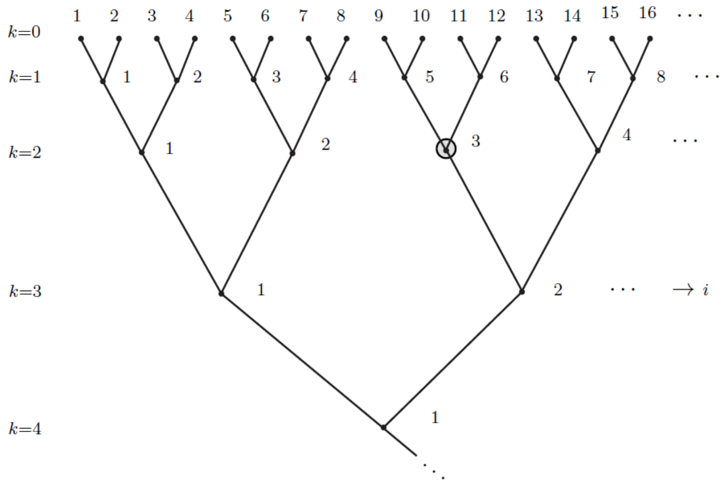
For every $c > 0$, we have, using Chebychev's Inequality and Lemma 2,

$$\mathbb{P}\left(\left|\sum_{n \in J} X_n\right| \ge c\right) \le K c^{-2} |J| \times (1 + 2p_\gamma(|J|)) \ . \quad (2.1)$$

For every $(k, i) \in \mathbb{N}_0 \times \mathbb{N}$, we define the dyadic block

$$B_{k,i} = \left\{ n \in \mathbb{N} \mid (i-1)2^k + 1 \le n \le i 2^k \right\} \ .$$

It is convenient to visualize them using an infinite tree as in the figure:



For example, the node circled in grey with coordinates $(k, i) = (2, 3)$ corresponds to the block

$$B_{2,3} = \{9, 10, 11, 12\} \subset \mathbb{N}.$$

The numbers indicated on the tree refer to the horizontal i coordinate. The k coordinate indicates the depth. Finally, the set of leafs of the tree is a visualization of the set $\mathbb{N}$ which labels the random variables X_n. Depending on the realized sample $\omega \in \Omega$, we will call $B_{k,i}$ a bad block (or large field block) if

$$\left| \sum_{n \in B_{k,i}} X_n \right| \ge 2^{\beta k} i^\beta \ .$$

Otherwise, we say $B_{k,i}$ is a good block (or small field block). By (2.1) with $c = 2^{\beta k} i^\beta$, we have $\forall (k,i) \in \mathbb{N}_0 \times \mathbb{N}$,

$$\mathbb{P}\left(B_{k,i} \text{ is bad}\right) \le \ K \ i^{-2\beta} \times 2^{(1-2\beta)k} (1 + 2p_\gamma(2^k))$$

and therefore

$$\sum_{(k,i) \in \mathbb{N}_0 \times \mathbb{N}} \mathbb{P}\left(B_{k,i} \text{ is bad}\right) < \infty \ .$$

Indeed, the sum over i converges by the hypothesis $\beta > \frac{1}{2}$. The sum over k is also convergent as can easily be checked in all three cases for γ. For instance, in the long-range dependence case when $0 < \gamma < 1$, bounding $\sum_{k \ge 0} 2^{(1-2\beta)k}(1 + 2p_\gamma(2^k))$ amounts to bounding $\sum_{k \ge 0} 2^{(1-2\beta)k} \ 2^{(1-\gamma)k} < \infty$, because of the assumption $\beta > 1 - \frac{\gamma}{2}$. By the first Borel-Cantelli Lemma, it is thus immediate that the (random) set $F \subset \mathbb{N}_0 \times \mathbb{N}$ of bad block labels is almost surely finite.

Assuming finiteness of F, let

$$N_F = \max\left(\cup_{(k,i)\in F} B_{k,i}\right) \in \{-\infty\} \cup \mathbb{N}.$$

The theorem is then a consequence of the following observation.

Lemma 3. *If $N \geq 4N_F - 1$, then*

$$\left|\sum_{n=1}^{N} X_n\right| \leq \left(\frac{\log N}{\log 2} + 1\right) \times N^{\beta} .$$

Proof: Note that N can be uniquely written as

$$N = 2^{k_1} + \cdots + 2^{k_l}$$

with $k_1 > \cdots > k_l \geq 0$. The k's correspond to the positions of the ones in the binary representation of N. Define

$$\begin{aligned} i_1 &= 1 \\ i_2 &= 2^{k_1-k_2} + 1 \\ i_3 &= 2^{k_1-k_3} + 2^{k_2-k_3} + 1 \\ &\vdots \\ i_l &= 2^{k_1-k_l} + 2^{k_2-k_l} + \cdots + 2^{k_{l-1}-k_l} + 1 . \end{aligned}$$

Then $B_{k_1,i_1}, \ldots, B_{k_l,i_l}$ form a set partition of $\{1, \ldots, N\}$. It is the partition provided by the greedy algorithm, namely, $B_{k_1,i_1} = \{1, 2, \ldots, 2^{k_1}\}$ is the biggest dyadic block inside $\{1, 2, \ldots, N\}$ and starting from 1, while $B_{k_2,i_2} = \{2^{k_1} + 1, \ldots, 2^{k_1} + 2^{k_2}\}$ is next biggest one can form, etc.

Provided all the blocks $B_{k_1,i_1}, \ldots, B_{k_l,i_l}$ are good, one can write the estimates

$$\begin{aligned} \left|\sum_{n=1}^{N} X_n\right| &= \left|\sum_{s=1}^{l} \sum_{n\in B_{k_s,i_s}} X_n\right| \\ &\leq \sum_{s=1}^{l} \left|\sum_{n\in B_{k_s,i_s}} X_n\right| \\ &\leq \sum_{s=1}^{l} 2^{\beta k_s} i_s^{\beta} = \sum_{s=1}^{l} \left(2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}\right)^{\beta} \\ &\leq l N^{\beta} \end{aligned}$$

by construction. Since $k_1 > \cdots > k_l \geq 0$, we have $l \leq k_1 + 1$. But $2^{k_1} \leq N$, so we obtain $l \leq \frac{\log N}{\log 2} + 1$ and the desired inequality follows.

All that remains is to show that the hypothesis $N \geq 4N_F - 1$ is enough to guarantee that all the blocks $B_{k_1,i_1}, \ldots, B_{k_l,i_l}$ are good. This is essentially a geometric argument based on the dyadic tree. If $F = \varnothing$, then $N_F = \infty$ and the hypothesis $N \geq 4N_F - 1$ is moot. However, in that case there is nothing more to prove since all blocks are good. We now assume $F \neq \varnothing$ (and of course finite). The condition $N \geq 4N_F - 1$ and the greedy algorithm chosen for the construction of $B_{k_1,i_1}, \ldots, B_{k_l,i_l}$ ensure that all the bad blocks are strict subsets of B_{k_1,i_1}. Indeed, let 2^r be the smallest power of two such that $2^r \geq N_F$. Thus all bad blocks should be subsets of $B_{r,1}$. By construction $2^{k_1} \leq N < 2^{k_1+1}$ and therefore $4N_F \leq N+1 \leq 2^{k_1+1}$. From $N_F \leq 2^{k_1-1}$, we deduce $2^r \leq 2^{k_1-1}$, i.e., $r + 1 \leq k_1$. Since all bad blocks are strict subsets of B_{k_1,i_1}, none of the blocks $B_{k_1,i_1}, \ldots, B_{k_l,i_l}$ can be bad and we are done.

□

3 Application

With graduate students in mind, we conclude this article by providing an entertaining application of our theorem, namely, a quick proof of the so-called Infinite Monkey Theorem (IMT). We refer the reader to the excellent Wikipedia entry on the IMT for the history of this metaphor and connections to literature, e.g., the work of Jorge Luis Borges, as well as to popular culture.

Not counting spaces, *Hamlet* by William Shakespeare [12] contains 140655 characters (Word document character count from the heading "ACT 1" to "They exit, marching, after the which, a peal of ordnance are shot off." in the cited digital source). Using the old ASCII-128 encoding of characters with seven bits, this corresponds to a binary string of $7 \times 140655 = 984585$ bits. Now imagine a monkey sitting in front of a typewriter with only two keys: one for the digit '0' and one for the digit '1'. The monkey is randomly typing a text for an infinite amount of time. A strong form of the IMT would then say that

> *With practical certainty (or probability 1), the monkey will not only manage to type the exact text of Hamlet but will do so infinitely many times with an asymptotic frequency of success exactly given by* $2^{-984585}$.

Let $L \geq 1$ be an integer and $W = (w_1, \ldots, w_L) \in \{0, 1\}^L$. For example L could be 984585 and W could be *Hamlet* in binary encoding. We introduce the sequence of Bernoulli random variables $(M_n)_{n\in\mathbb{N}}$ where M_n is the n-th digit typed by the infinite monkey. We define a new sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ by letting

$$Y_n = \begin{cases} 1 & \text{if } (M_n, M_{n+1}, \ldots, M_{n+L-1}) = W , \\ 0 & \text{otherwise} . \end{cases}$$

Finally, we recenter and let $X_n = Y_n - \mathbb{E}Y_n$. Since Y_p and Y_q are independent for $q \geq p + L$, the X_n trivially satisfy the hypotheses of our theorem for any $\gamma > 0$. The IMT, as formulated above, follows immediately.

References

[1] A. Abdesselam and V. Rivasseau, An explicit large versus small field multiscale cluster expansion. Rev. Math. Phys. **9** (1997), no. 2, 123–199.

[2] T. K. Chandra and S. Ghosal, Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables. Acta Math. Hungar. **71** (1996), no. 4, 327–336.

[3] J, Dimock, The renormalization group according to Balaban. II. Large fields. J. Math. Phys. **54** (2013), no. 9, 092301, 85 pp.

[4] P. Doukhan, O. Klesov and G. Lang, Rates of convergence in some SLLN under weak dependence conditions. Acta Sci. Math. (Szeged) **76** (2010), no. 3–4, 683–695.

[5] I. Fazekas and O. Klesov, A general approach to the strong laws of large numbers. Theory Probab. Appl. **45** (2002), no. 3, 436–449.

[6] T.-C. Hu, A. Rosalsky and A. Volodin, On convergence properties of sums of dependent random variables under second moment and covariance restrictions. Statist. Probab. Lett. **78** (2008), no. 14, 1999–2005.

[7] N. Kôno, Classical limit theorems for dependent random sequences having moment conditions. In: *Probability theory and mathematical statistics (Tbilisi, 1982)*, pp. 315–319, Lecture Notes in Math. 1021, Springer, Berlin, 1983.

[8] V. M. Korchevsky, On the strong law of large numbers for sequences of random variables without the independence condition. Vestnik St. Petersburg Univ. Math. **44** (2011), no. 4, 268–271.

[9] V. M. Korchevsky, On the strong law of large numbers for sequences of dependent random variables with finite second moments. J. Math. Sci. (N.Y.) **206** (2015), no. 2, 197–206.

[10] A. Kuczmaszewska, The strong law of large numbers for dependent random variables. Statist. Probab. Lett. **73** (2005), no. 3, 305–314.

[11] R. Lyons, Strong laws of large numbers for weakly correlated random variables. Michigan Math. J. **35** (1988), no. 3, 353–359.

[12] W. Shakespeare, *Hamlet*. Circa 1600. Digital text by the Folger Shakespeare Library, available at https://www.folgerdigitaltexts.org/download/doc/Ham.doc

[13] S. H. Sung, Maximal inequalities for dependent random variables and applications. J. Inequal. Appl. (2008), Art. ID 598319, 10 pp.

[14] S. Yang, C. Su and K. Yu, Keming, A general method to the strong law of large numbers and its applications. Statist. Probab. Lett. **78** (2008), no. 6, 794–803.

Abdelmalek Abdesselam
Department of Mathematics,
P. O. Box 400137, University of Virginia,
Charlottesville, VA 22904-4137, USA.
E-mail address: malek@virginia.edu