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A domination method for solving unbounded quadratic BSDEs

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Abstract

We introduce a domination argument which asserts that: if we can dominate the parameters of a quadratic backward stochastic differential equation (QBSDE) with continuous generator from above and from below by those of two BSDEs having ordered solutions, then also the original QBSDE admits at least one solution. This result is presented in a general framework: we do not impose any integrability condition on none of the terminal data of the three involved BSDEs, we do not require any constraint on the growth nor continuity of the two dominating generators.

As a consequence, we establish the existence of a maximal and a minimal solution to BSDEs whose coefficient H is continuous and satisfies $|H(t,y,z)| \leq \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2$, where α_t, β_t, θ_t are positive processes and the function f is positive, continuous and increasing (or even only positive and locally bounded) on $\mathbb{R}$. This is done with unbounded terminal value. In particular, we cover the classical case (f constant) obtained in ([11], [13], [26], [28]), the case f polynomial obtained in ([24]) and also the cases where the generator has super linear growth (not covered by the previous papers) such as $e^{|y|^k}|z|^p$ ($k \geq 0$, $0 \leq p < 2$), $e^{e^{|y|}}|z|^2$ and so on. In contrast to the works [11, 13, 24, 26, 28], we get the existence of a maximal and a minimal solution and we also cover the BSDEs with at most linear growth (take $f = 0$), and in particular some results obtained in [25, 14, 23]. We moreover establish the existence and uniqueness of solutions to BSDEs driven by $f(y)|z|^2$ when f is merely *locally integrable* on $\mathbb{R}$.

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1 Introduction

Let $(W_t)_{0\leq t\leq T}$ be a d–dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $(\mathcal{F}_t)_{0\leq t\leq T}$ the natural filtration of W augmented with $\mathbb{P}$-negligible sets. Let $H(t,\omega,y,z)$ be a real valued $\mathcal{F}_t$-progressively measurable process defined on $[0,\ T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. Let ξ be an $\mathcal{F}_T$-measurable $\mathbb{R}$-valued random variable. For $0 \leq t \leq T$, we consider the BSDE

$$Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \quad (eq(\xi, H))$$

ξ is called the terminal value and H is called the generator or the coefficient. A BSDE with data (ξ, H) will be labeled $eq(\xi, H)$ or BSDE (ξ, H) or BSDE(ξ, H).

Definition 1.1.

(i) We say that $eq(\xi, H)$ is quadratic if H has at most a quadratic growth in its z-variable.

(ii) A solution to $eq(\xi, H)$ is a process (Y, Z) which satisfies $eq(\xi, H)$ on $[0,\ T]$ and such that Y is continuous, $\int_0^T |Z_s|^2 ds < \infty$ $a.s$ and $\int_0^T |H(s, Y_s, Z_s)| ds < \infty$ $a.s.$

(iii) A positive solution is a solution (Y, Z) such that $Y_t \geq 0$. We Symmetrically define a negative solution. A bounded solution is a solution (Y, Z) such that Y is bounded.

Linear versions of $eq(\xi, H)$ appear as the equations for the adjoint process in stochastic control ([12]) as well as the model behind the Black & Scholes formula for the pricing and hedging of options in mathematical finance. Note also that $eq(\xi, H)$ is closely related to semilinear elliptic and parabolic partial differential equations (PDEs in short) of second order. The linear BSDEs can be solved more or less explicitly. When the coefficient H is uniformly Lipschitz, $eq(\xi, H)$ has a unique solution which can be constructed by using both Itô's representation theorem and a successive approximation procedure ([29]). Further developments on the BSDEs with various applications to stochastic control, partial dfferential equations and homogenization can be find for instance in [30, 31, 32]. We give hereafter a very short informal presentation on the relation between PDEs and BSDEs. Let L be a differential operator defined by

$$Lu := \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}(x)\partial^2_{ij}u + \sum_i b_i(x)\partial_i u.$$

For $t \in]0, T[$ and $x \in \mathbb{R}^d$, we consider the semilinear

PDE

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + Lu(t,x) + H(t,x,u(t,x),\nabla u(t,x)) = 0 \\ u(T,x) = g(x) \end{cases} \tag{1.1}$$

The probabilistic interpretation of the PDE (1.3) can be obained by the following system of SDE-BSDE:

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \\ Y_s^{t,x} = g(X_T) - \int_s^T H(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\ \qquad - \int_s^T Z_r dM_r^{X^{t,x}} \end{cases} \tag{1.2}$$

where $dM_r^{X^{t,x}} := \sigma(X_r^{t,x})dW_r$ is the martingale part of $X^{t,x}$.
Indeed, if we assume that the PDE (1.3) has a solution u which belongs to the class $\mathcal{C}^{1,2}(]0,\ T[\times\mathbb{R})$, then Itô's formula applied to $u(X_s^{t,x})$ shows that

$$(Y_s^{t,x}, Z_s^{t,x}) := \Big(u(s, X_s^{t,x}), \nabla_x u(s, X_s^{t,x})\Big)$$

is a solution to the BSDE part of equation (1.2). In particular, $u(t,x) = Y_t^{t,x}$.
Conversely, if $Y_s^{t,x}$ is the solution of the SDE-BDSE (1.2) satisfying certain conditions, then $Y_t^{t,x}$ is a solution to the PDE (1.3) in a viscosity sense.

The BSDEs, which we study here, can give for instance a probabilistic interpretation to second order semilinear parabolic and elliptic PDEs whose nonlinearity H has at most a quadratic growth in its gradient variable of the form $f(|u(t,x)|)|\nabla u(t,x)|^2$, with f measurable and locally bounded or even locally integrable in some situations. A simplest typical PDE which can be interpreted by the quadratic BSDEs we study here is the following nonlinear heat equation in $]0,T[\times\mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \dfrac{1}{2}\Delta u(t,x) + f(u(t,x)|\nabla u(t,x)|^2 = 0 \\ u(T,x) = g(x) \end{cases} \tag{1.3}$$

where f is a locally integrable function.

The Quadratic BSDEs were studied in many papers, among them one can cite the works [6, 7, 11, 13, 15, 19, 21, 24, 26, 28, 35]. In the present paper, we are concerned with the existence of solutions to BSDEs whose generator H satisfies $|H(t,y,z)| \le \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2$, where α_t, β_t, θ_t are positive processes and the function f is positive on $\mathbb{R}_+$ and locally bounded but not globally integrable on $\mathbb{R}$. We are motivated by the fact that the BSDEs driven by $H(t,y,z) = g(c_t,y) + f(|y|)|z|^2$ appears in stochastic differential utility, see [19]. This type of BSDEs are also related to quadratic PDEs appearing in financial markets, see [20].

Let us present another motivation : it has been recently shown in [6, 7] that the BSDEs driven by a generator H satisfying $|H(t,y,z)| \le \alpha+\beta|y|+\theta|z|+f(|y|)|z|^2$ have solutions when f is globally integrable and α, β, θ are constant. However, these two works can not cover the classical BSDEs driven by $|z|^2$, since they require the (global) integrability of f on the whole space $\mathbb{R}$. Thus, the following questions naturally arise:

1) Are there BSDEs whose generator H satisfying $|H(t,y,z)| \le \alpha_t+\beta_t|y|+\theta_t|z|+f(|y|)|z|^2$ that have solutions without assuming the global integrability of f?

2) If yes, what integrability condition we should require on the terminal value ξ?

The following example gives a positive answer to the first question. It moreover shows that neither global integrability of f nor integrability of ξ are necessary to the existence of solutions.

Example 1.1. Consider the BSDE

$$Y_t = \xi + \int_t^T \mathbb{1}_{\mathbb{R}_+}(Y_s)|Z_s|^2 ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T. \tag{1.4}$$

where for a set A, $\mathbb{1}_A$ denotes the indicator function of A.

Equation (1.4) is not covered by the works [11, 13, 19, 15, 21, 24, 26, 28, 35], since $\mathbb{1}_{\mathbb{R}_+}$ is neither constant nor continuous. It is also not covered by the works [6, 7, 19], since the function $\mathbb{1}_{\mathbb{R}_+}$ is not globally integrable. Nevertheless, equation (1.4) admits a solution *without any integrability* condition on ξ. Indeed, the function $u(y) := \frac{1}{2}(e^{2y}-1)\mathbb{1}_{\mathbb{R}_+}(y) + y\mathbb{1}_{\mathbb{R}_-^*}(y)$ belongs to the Sobolev space $W^2_{1,loc}(\mathbb{R})$ and solves the differential equation $\frac{1}{2}u''(y) - \mathbb{1}_{\mathbb{R}_+}(y)u'(y) = 0$ on $\mathbb{R}^*$. Therefore, using Itô-Krylov's formula for BSDEs (see [6, 7]) one can show that equation(1.4) has a solution if and only if the following equation has a solution.

$$\bar{Y}_t = u(\xi) - \int_t^T \bar{Z}_s dW_s, \quad 0 \le t \le T. \tag{1.5}$$

By Dudley's representation theorem [18], equation (1.5) has a solution without any integrability condition on $u(\xi)$. Since u is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$, we deduce that equation (1.4) has a solution *without any integrability of* ξ. If moreover, $u(\xi)$ is integrable then equation (1.4) has a unique solution (Y,Z) such that $u(Y)$ belongs to class (D). Other examples will be presented later.

The aim of this paper is to introduce a domination method which allows to solve $eq(\xi,H)$ when ξ is unbounded and $|H(t,y,z)| \le \alpha_t+\beta_t|y|+\theta_t|z|+f(|y|)|z|^2$, where α_t, β_t, θ_t are positive processes and the function f is positive on $\mathbb{R}_+$ and locally bounded but not globally integrable on $\mathbb{R}$. Our strategy is divided into five stages each of which has its own interest.

In step I, we establish existence, uniqueness and comparison of solutions to BSDEs driven by $f(y)|z|^2$ when f is locally integrable, this covers the classical case where f is constant. It will be shown that when $u_f(\mathbb{R}) = \mathbb{R}$, then the BSDE $(\xi, f(y)|z|^2)$ has a solution without any integrability condition on the terminal value, and when $u_f(\mathbb{R}) \neq \mathbb{R}$, then the condition $u_f(\xi)$ *integrable* is necessary to the existence of solutions for $eq(\xi, f(y)|z|^2)$. We also show that the uniqueness as well as the comparison hold for BSDE$(\xi, f(y)|z|^2)$ in the class of solutions such that $u_f(Y)$ belongs to class (D).

In order to explain the other steps, we precise some notation, definitions and assumptions.

Some notation. For given real numbers a and b, we set $a \wedge b := \min(a, b)$, $a \vee b := \max(a, b)$, $a^- := \max(0, -a)$ and $a^+ := \max(0, a)$.

For given positive processes α and β we denote

$$\xi_{\alpha,\beta} := \left(|\xi| + \int_0^T \alpha_s ds\right) e^{\int_0^T \beta_s ds}$$

We define $\xi^+_{\alpha,\beta}$ and $\xi^-_{\alpha,\beta}$ in likewise manner.

For $p > 0$, we denote by $\mathbb{L}^p_{loc}(\mathbb{R})$ ($\mathbb{L}^p_{loc}$ in short) the space of (classes) of functions u defined on $\mathbb{R}$ which are p-integrable on bounded set of $\mathbb{R}$. We also denote, $W^2_{p,loc}$:= the Sobolev space of (classes) of functions u defined on $\mathbb{R}$ such that both u and its generalized derivatives u' and u'' belong to $\mathbb{L}^p_{loc}(\mathbb{R})$.

$\mathcal{C}$:= the space of continuous and $\mathcal{F}_t$ –adapted processes.

$\mathcal{S}^p$:= the space of continuous, $\mathcal{F}_t$ –adapted processes φ such that $\mathbb{E}\left(\sup_{0\leq t\leq T} |\varphi_t|^p\right) < \infty$.

$\mathcal{L}^2$:= the space of $\mathcal{F}_t$ –adapted processes φ satisfying $\int_0^T |\varphi_s|^2 ds < +\infty$ $\mathbb{P}$–a.s.

$\mathcal{M}^p$:= the space of $\mathcal{F}_t$–adapted processes φ satisfying $\mathbb{E}\left[\left(\int_0^T |\varphi_s|^2 ds\right)^{\frac{p}{2}}\right] < +\infty$.

BMO is the space of uniformly integrable martingales M satisfying $\sup_\tau ||\mathbb{E}\Big(|M_T - M_\tau|/\mathcal{F}_\tau\Big)||_\infty < \infty$, where the supremum is taken over all stopping times τ.

We say that the process $\varphi := (\varphi_s)_{0\leq s\ T}$ belongs to class (D) if $\sup_{0\leq\tau\leq T} |Y_\tau| < \infty$, where the supremum is taken over all stopping times τ such that $\tau \leq T$.

For a given locally integrable function f defined on $\mathbb{R}$, we put

$$u_f(y) : \int_0^y \exp\left(2\int_0^x f(r)dr\right) dx. \tag{1.6}$$

Since f locally integrable, then the transformation u_f defined in (1.6) is a one to one function, both u_f and its inverse belong to the Sobolev space $W^2_{1,loc}$. More details on the function u_f are given in Lemma 5.1 of Apendix.

Consider the following assumptions.

(A1) H is continuous in (y, z) for $a.e$ (t, ω) and satisfies,

$$|H(t, y, z)| \leq \alpha_t + \beta_t|y| + f(|y|)|z|^2$$

where α_t, β_t *are some* $(\mathcal{F}_t)$*-adapted processes which are positive and* f *is a real valued function which is continuous, increasing and positive on* $\mathbb{R}_+$

(A2) $u_f(\xi_{\alpha,\beta})$ *is integrable,*

where u_f is defined by (1.6).

In step II,we introduce the domination argument (Lemma 3.1 below) which is an abstract result that gives the existence of solutions to BSDEs without integrability condition on the terminal value. It asserts that: *if* (ξ_1, H_1) *and* (ξ_2, H_2) *are two BSDEs which respectively admit two solutions* (Y_1, Z_1) *and* (Y_2, Z_2) *such that* $(\xi_1, H_1, Y_1) \leq (\xi_2, H_2, Y_2)$, *then any Quadratic BSDE* (ξ, H) *with continuous generator and satisfying* $(\xi_1, H_1) \leq (\xi, H) \leq (\xi_2, H_2)$ *has at least one solution* (Y, Z) *such that* $Y_1 \leq Y \leq Y_2$. *Moreover, among all solutions which lie between* Y_1 *and* Y_2, *there are a maximal and a minimal solution.* The proof of this result do not need any a priori estimate nor approximation. It is based on the remarkable work [22] on the existence of reflected QBSDEs without any integrability condition on the terminal value. Actually, we derive the existence of our QBSDE from a suitable reflected QBSDE.

In steps III-IV, we use the domination argument to show that when assumptions (A1), (A2) are satisfied then $eq(\xi, H)$ admits a *maximal and a minimal solution* satisfying some conditions which will be precised later. We cover the results obtained in [11, 13, 24, 26] and also the cases where the generator has super linear growth such as $y|z|$, $e^{|y|}|z|^p$ $(0 \leq p < 2)$ $e^{|y|}|z|^2$ and more generally the case $|H(y, z)| \leq \alpha_t + \beta_t|y| + g(|y|)|z|^p + f(|y|)|z|^2$ where f and g are continuous, increasing and positive on $\mathbb{R}_+$, and $0 \leq p < 2$. In all these cases, the existence of solutions is established with an unbounded terminal value. Although the case where the function f is globally integrable is not covered by assumption (A1), one can again use the domination argument to show that if H satisfies (A1) with f globally integrable and $\xi_{\alpha,\beta}$ is integrable, then $eq(\xi, H)$ has a maximal and a minimal solution; this extends the result of [6, 7] to the case where α, β are processes and $\xi_{\alpha,\beta}$ is merely in $\mathbb{L}^1$.

In step V, we establish the existence of solutions to $eq(\xi, H)$ under the the following two assumptions:

(A3) H is continuous in (y, z) for $a.e$ (t, ω) and satisfies,

$$|H(t, y, z)| \leq \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2$$

where α_t, β_t are some $(\mathcal{F}_t)$-adapted processes which are positive and f is a real valued function which is continuous, increasing and positive on $\mathbb{R}_+$

(A4) $$\sup_{\pi\in\sum} \mathbb{E}\left(\Gamma^{\pi}_{0,T} u_f(\xi_{\alpha,\beta})\right)$$
$$:= \sup_{\pi\in\sum} \mathbb{E}\left(e^{\int_0^T \theta_r\pi_r dW_r - \frac{1}{2}\int_0^T \theta_r^2|\pi_r|^2 dr} u_f(\xi_{\alpha,\beta})\right) < +\infty$$

where

$$\sum := \Big\{\pi \in \mathcal{L}^2,\ |\pi| \in \{0,1\},\ a.e. \text{ and } \ ess\sup_{\omega} \int_0^T \theta_r^2|\pi_r|^2 dr < +\infty\Big\}$$

In this case, we again use the domination argument to reduce the solvability of $eq(\xi, H)$ to that of $eq\Big(u_f(\xi_{\alpha,\beta}),\ \theta_t|z|\Big)$ and then to deduce the existence of solutions to $eq(\xi, H)$. It should be noted that the works [6, 7, 11, 13, 19, 15, 24, 26, 28, 35] consider only the case $\theta_t = 0$. To the best of our knowlege, the case $\theta_t \neq 0$ is considered only in [7, 21] and in the present paper. We emphasize that assumptions (A1)-(A2) are covered by (A3)-(A4). However, for the sake of clarity and to make the paper easy to read, we separately treat these two situations.

1.1 Advantages of the method

Let us now present the domination method and its advantages. We consider the case when $\theta_t = 0$. In order to establish the existence of solutions, we first only assume (A1) then we prove that the solvability of $eq(\xi, H)$ is reduced to the positive solvability (i.e. $Y \geq 0$) of the two BSDEs $eq\Big(u_f(\xi^+_{\alpha,\beta}),\ 0\Big)$ and $eq\Big(u_f(\xi^-_{\alpha,\beta}),\ 0\Big)$ (see the proof of Proposition 3.1). We finally show that the latter two BSDEs have simultaneously positive solutions if and only if assumption (A2) is satisfied. This shows how assumption (A2) is not a priori imposed here but is generated along the proof. Our method makes it possible to control more precisely the integrability condition we should impose to the terminal value.

Here are some other advantages of our method: instead of $eq(\xi, H)$, we only work with the dominating equations $eq(\xi_1, H_1)$ and $eq(\xi_2, H_2)$ which are more simple than the initial one (ξ, H). In contrast to the papers [11, 13, 28, 26, 35], our method allows to get the existence of *a maximal and a minimal solution.* It moreover allows to deal with all one dimensional BSDEs up to quadratic ones and seems unify their treatment. Note also that, the three involved terminal data ξ, ξ_1 and ξ_2 are not necessary integrable, the two dominating coefficients H_1, H_2 are merely measurable and can have arbitrarily growth. Only $H(t, y, z)$ should be continuous on (y, z) and of at most quadratic growth in z. In return, the solutions lie in $\mathcal{C} \times \mathcal{L}^2$ and hence not necessary integrable.

1.2 Our approach versus the Literature

We summarize the results of some previous works in light of assumptions (A1)-(A2). The case where f is constant and α, β do not depend on ω has been considered in [26] where the existence of bounded solutions is established provided that the terminal value is bounded. In [13], the existence of solutions is obtained when α, β and f are constant ($f(y) = \frac{\gamma}{2}$) provided that $exp(\gamma|\xi|e^{\beta T})$ is integrable. The authors of [11] consider the case where f is constant($f(y) = \frac{\gamma}{2}$) and established the existence of solutions when $exp\Big[\gamma(|\xi| + \int_0^T \alpha_s ds)e^{\int_0^T \beta_s ds}\Big]$ is integrable. In [21], the authors consider a generalized QBSDE and the function f is replaced by a continuous process r_t, the existence of solution is then obtained provided, when $\alpha = \beta = 0$, that $\frac{1}{C_T}\Big(\exp(C_T|\xi|) - 1\Big)\mathbb{1}_{\{C_T>0\}} + |\xi|\mathbb{1}_{\{C_T=0\}}$, with $C_T = \sup_{s\leq T} r_s$. In [6, 7] the case where α, β are constant and f is globally integrable has been considered and the existence of solutions in $\mathcal{S}^2 \times \mathcal{M}^2$ was established provided that the terminal value is merely square integrable.

Let us briefly describe the principal methods used in some previous papers. When the function f is constant, two methods have been essentially developed in order to establish the existence of solutions. The first one is the monotone stability [11, 13, 26, 28]. The second approach is based on a fixed point argument and has been introduced in [35]. In this latter, the uniqueness is also obtained but it requires that the generator satisfies the so-called Lipschitz-quadratic condition. These two methods use some a priori estimates and approximations which are sometimes difficult to obtain. It should be noted that, the papers [11, 13, 26, 28, 35] consider the cases where the terminal value is bounded or at least with some exponential moments. Alternative methods were initiated in [21] then developed in [6, 7]. These methods are based on the nice work [22], where the existence of two barriers reflecting QBSDEs is established without any integrability condition on the terminal value. The ideas used in [6, 7, 21] are very close to the domination we develop here and they consist in deriving the existence of BSDEs from the existence of a suitable reflected BSDEs.

We now compare our method with those of [11, 13, 24, 26, 28, 35]. In the latter, the authors proceed as follows: they first impose some integrability (or boudedness) condition on the terminal value ξ. Next, they establish some a priori estimates for the solutions by using the integrability (or the boundedness) of ξ. This allows them to prove the existence of solutions by using a suitable approximation.

Our approach is completely different: in order to prove the existence of solutions, we only use Lemma 3.1 and some change of variables formulas. We do not need to establish any a priori estimates of the solutions. We do not need to construct any approximation. In contrast to the previous papers, the integrability of the terminal value is not a priori imposed but generated along the proof by solving an inverse problem. In contrast to the works [11, 13, 24, 26, 28, 35], we get the existence of a maximal and a minimal solution. Moreover, our results also cover and extend earlier results on BSDEs with at most linear growth, and in particular those obtained in [27, 25] and some of those obtained in [14, 23] (take $f = 0$). Note that Assumption A 3.3 of [33] is close to our Assumption ($A4$). To see this, take $f = 0$, $\alpha_s = |H(s, 0, 0)|$, $\beta_s = L_y$ (the Lipschitz constant in y of [33]) and $\theta_s = L_z$ (the Lipschitz constant in z of [33]).

The paper is organized as follows. In section 2, we establish the existence and uniqueness for the BSDE ($\xi, f(y)|z|^2$) when f is *locally integrable* on $\mathbb{R}$, we also give some examples of BSDEs which have solutions without any integrability of the terminal value ξ. In section 3, we begin by introducing the domination argument then we use it to establish the existence of solutions to $eq(\xi, H)$ under conditions (A1)-(A2) and also under assumption (A3)-(A4). Some integrability properties are also established for the solutions of $eq(\xi, H)$ under additional assumptions which will be specified below in section 3. In section 4, we treat the BSDEs with at most logarithmic growth $y \ln |y|$. Using the domination argument and some change of variables, we show that these equations can be solved by using the quadratic BSDEs and vice-versa. In section 5, some auxiliary results are given.

Since our approach consists in reducing the solvability of $eq(\xi, H)$ under assumptions (A1) [resp. (A3)] to the positive solvability ($Y \geq 0$) of $eq(u_f(\xi^+_{\alpha,\beta}), 0)$ [resp. $eq(u_f(\xi^+_{\alpha,\beta}), \theta_t|z|)$], the following two propositions which study the existence of positive solutions to these two simple BSDEs are then useful.

1.3 Two basic BSDEs

The following proposition is useful in studying the solvability of ($eq(\xi, H)$ when assumption (A1) is satisfied. It characterizes the existence of positive solutions to a BSDE driven by a null generator.

Proposition 1.1. ***The BSDE*** $(\zeta, 0)$.
I) According to Dudley's theorem [18], the following BSDE has a solution for any $\mathcal{F}_T$–measurable random variable ζ.

$$y_t := \zeta - \int_t^T z_s dW_s \tag{1.7}$$

Furthermore:

(i) If ζ is integrable, then equation (1.7) *has a unique solution (y, z) such that y belongs to class (D) given by $Y_t = \mathbb{E}(\zeta/\mathcal{F}_t)$. Furthermore, z belongs to $\mathcal{M}^p$ for each $0 < p < 1$.*

(ii) If ζ is positive and (1.7) *has a positive solution, then ζ is necessary integrable.*

(iii) If $\zeta \neq 0$, then for any process z, $(0, z_t)$ could not be a solution to the BSDE $(\zeta, 0)$.

Proof. Assertion (i) can be proved by using a usual localization and Fatou's lemma. We prove Assertion (ii). Dudley's representation theorem allows us to show that $eq(\zeta, 0)$ has at least a solution (y, z) in $\mathcal{C} \times \mathcal{L}^2$. Since ζ is integrable, then $y_t := \mathbb{E}(\zeta/\mathcal{F}_t)$ is a solution which belongs to class (D). It follows that the stochastic integral $\int_0^{\cdot} z_s dW_s$ is a uniformly integrable martingale. Using Proposition 4.7, Chap. IV of [34] (see also [16]) and the Burkholder-Davis-Gundy inequality we show that z belongs to $\mathbb{M}^p$, for each $0 < p < 1$. We shall prove that the process (y, z) we just constructed is actually the unique solution such that y belongs to class (D). Let (y^1, z^1) and (y^2, z^2) be two solutions such that y^1 and y^2 belong to class (D). It follows that $y^1 - y^2$ belongs to class (D) and hence the stochastic integral $(\int_0^t (z^1_s - z^2_s) dW_s)_{0 \leq t \leq T}$ is a martingale in class (D). It follows that $y^1 = y^2$. Using the Burkholder-Davis-Gundy inequality, we show that $\mathbb{E}\left[\left(\int_0^T |z^1_s - z^2_s|^2\right)^{p/2}\right] = 0$. Assertion ($ii$) is proved. We shall prove (iii). Let $\zeta \neq 0$. Assume that there exists a process z such that $(0, z)$ is a solution to $eq(\zeta, 0)$. Then for any $t \leq T$, $0 = \zeta - \int_t^T z_s dW_s$ which implies that $\zeta = 0$ by putting $t = T$. □

The following proposition, which is taken from [21] (Proposition 6.1 of [21]), gives a necessary and sufficient condition which ensures the existence of positive solutions to the BSDEs driven by the generator $\theta_t|z|$. The result is useful when assumption (A3) is satisfied.

Proposition 1.2. *([21], Proposition 6.1).* ***The BSDE*** $(\zeta,\ \theta_t|z|^2)$. *Let ζ be a positive $\mathcal{F}_T$–measurable random variable. The BSDE*

$$y_t = \zeta + \int_t^T \theta_s |z_s| ds - \int_t^T z_s dW_s, \qquad t \leq T \tag{1.8}$$

has a positive solution if and only if

$$\sup_{\pi \in \sum} \mathbb{E}\left(\Gamma^{\pi}_{0,T}\zeta\right) < +\infty,$$

with notation as in (A4). In this case, there exist $\bar{z} \in \mathcal{L}^2$ and $\bar{y}_t := ess \sup_{\pi \in \sum} \mathbb{E}(\Gamma^{\pi}_{t,T}\zeta|\mathcal{F}_t)$ such that $(\bar{y}, \bar{z})$ is the minimal solution of Equation (1.8). Furthermore, $\bar{y}_t \geq \mathbb{E}(\zeta|\mathcal{F}_t) \geq 0$, for each $t \in [0, T]$.

2 The BSDE($\xi, f(y)|z|^2$)

The BSDE($\xi, f(y)|z|^2$) will be used in order to solve the general equation (ξ, H) with $|H(t, y, z)| \leq \alpha_t |y| +$

$\beta_t|y| + \theta_t|z| + f(|y|)|z|^2$. However, since $eq(\xi, f(y)|z|^2)$ is interesting itself and do not need the domination argument, we give in this subsection a complete study of this equation in the case where the function f is locally integrable but not necessary continuous. A characterization of the existence of solution is given for this equations. This is related to the function u_f, and for instance, when $u_f(\mathbb{R}) = \mathbb{R}$ then the BSDE$(\xi, f(y)|z|^2)$ has a solution for each terminal value ξ. No integrability is required to the terminal value ξ. We start this section by some examples which are covered by the present work and not covered by those of [6, 7, 11, 13, 21, 24, 26, 28, 35].

2.1 Some examples of QBSDEs with measurable terminal value

Example 2.1. Let f_1 be a bounded function which is globally integrable on $\mathbb{R}$. We assume that f is bounded by 1 for simplicity. Clearly, the generator $H_1(y,z) := f_1(y)|z|^2$ satisfies $|H(y,z)| \leq |z|^2$. Hence, $H_1(y,z)$ is of quadratic growth. It was shown in [6, 7] that the QBSDE $(\xi, f_1(y)|z|^2)$ has a solution without any integrability condition on the terminal value ξ. Moreover, when ξ is square integrable then $eq(\xi, f_1(y)|z|^2)$ has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$.

Indeed, for a given locally integrable function f, the transformation u_f defined in (1.6) is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$. Both u_f and its inverse belong to the Sobolev space $W^2_{1,loc}$. Using Itô-Krylov's formula for BSDEs (see [6, 7]), it follows that $eq(\xi, f_1(y)|z|^2)$ has a solution if and only if $eq(u_{f_1}(\xi), 0)$ has a solution. Since f_1 is globally integrable, then u_{f_1} and its inverse are uniformly Lipschitz. It follows that $u_{f_1}(\xi)$ is square integrable if and only if ξ is square integrable. Therefore, $eq(\xi, f_1(y)|z|^2)$ has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ whenever ξ is merely square integrable. This also shows that the convexity of the generator is not necessary to the uniqueness.

Example 2.2. The functions

$$f_2(y) := e^y \text{ and } f_3(y) := \mathbb{1}_{\{y>0\}} + \mathbb{1}_{\{y\leq 0\}}e^y \tag{2.1}$$

are not globally integrable on $\mathbb{R}$. However, the same argument shows that $eq(\xi, f_i(y)|z|^2)$ has a solution without any integrability of ξ, for $i \in \{2,3\}$. Note that f_2 is neither globally integrable nor bounded.

2.2 Existence of BSDE$(\xi, f(y)|z|^2)$ with f locally integrable

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally integrable function. The goal of this part is to explain how some $eq(\xi, f(y)|z|^2)$ has solutions without integrability of ξ and others need the integrability of ξ.

Consider the QBSDE

$$Y_t = \xi + \int_t^T f(Y_s)|Z_s|^2 ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{2.2}$$

The function u_f defined by (1.6) belongs to the Sobolev space $W^2_{1,loc}(\mathbb{R})$. Therefore, applying Itô-Krylov's formula to u_f (see [6, 7]), one can show that (Y,Z) is a solution to equation (2.2) if and only if $(\bar{Y}, \bar{Z}) := (u_f(Y), u_f'(Y)Z)$ is a solution to the BSDE

$$\bar{Y}_t = u_f(\xi) - \int_t^T \bar{Z}_s dW_s, \quad 0 \leq t \leq T. \tag{2.3}$$

According to Dudley's representation theorem, $eq(u_f(\xi), 0)$ has a solution for any $\mathcal{F}_T$-measurable random variable $u_f(\xi)$. No integrability is required to $u_f(\xi)$. But, our problem is to solve $eq(\xi, f(y)|z|^2)$. This is the subject of the following proposition.

Proposition 2.1. *Let f be a locally integrable function and ξ a $\mathcal{F}_T$-measurable random variable.*

(i) If $u_f(\mathbb{R}) = \mathbb{R}$, then $eq(\xi, f(y)|z|^2)$ has a solution. No integrability is needed for ξ.

(ii) Let $u_f(\mathbb{R}) \neq \mathbb{R}$. If f is positive, $u_f(y)$ is then increasing and $\lim_{y\to\infty} u_f(y) = +\infty$. Assume that $\lim_{y\to-\infty} u_f(y) = c > -\infty$. Then, necessary $\bar{u}_f(\xi) := u_f(\xi) - c$ is integrable. In this case, $eq(\xi, f(y)|z|^2)$ has a solution given by $Y_s = \bar{u}_f^{-1}(\mathbb{E}[\bar{u}_f(\xi)/\mathcal{F}_s])$. The case f negative goes similarly.

Proof. Assertion (i) is simple. We prove assertion (ii). The function $\bar{u}_f(y) := u_f(y) - c$ belongs to the Sobolev space $W^2_{1,loc}(\mathbb{R})$ and it is one to one from $\mathbb{R}$ into $\mathbb{R}_+$. Hence, Itô-Krylov's formula applied to $\bar{u}_f$ shows that (Y,Z) is a solution to equation (2.2) if and only if $(\bar{Y}, \bar{Z}) := (\bar{u}_f(Y), \bar{u}_f'(Y)Z)$ is a solution to the BSDE

$$\bar{Y}_t = \bar{u}_f(\xi) - \int_t^T \bar{Z}_s dW_s, \quad 0 \leq t \leq T. \tag{2.4}$$

Since $\bar{Y} := \bar{u}_f(Y)$ is positive, Therefore, $eq(\xi, f(y)|z|^2)$ has a solution if and only if $\bar{u}_f(\xi)$ is integrable. We then deduce that for any $s \leq T$, $Y_s = \bar{u}_f^{-1}(\mathbb{E}[\bar{u}_f(\xi)/\mathcal{F}_s])$.

2.3 Uniqueness and comparison for $eq(\xi, f(y)|z|^2)$ with f locally integrable

Proposition 2.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be locally integrable. Let u_f be the function defined in Lemma 5.1-I, v the function defined in Lemma 5.1-II and w the fonction defined in Lemma 5.1-III. Assume that $u_f(\xi)$ is integrable. Then, the BSDE$(\xi, f(y)|z|^2)$ has a unique solution (Y,Z) such that $u_f(Y)$ belongs to class (D). Moreover,*

(i) If $v(Y)$ belongs to class (D), then $\mathbb{E}\int_0^T |Z_s|^2 ds < \infty$,

(ii) If $w(Y)$ belongs to class (D), then $\mathbb{E}\int_0^T |f(Y_s)||Z_s|^2 ds < \infty$.

Proof. The BSDE$(\xi, f(y)|z|^2)$ has a solution if and only if the BSDE $(u_f(\xi), 0)$ has a solution. But $eq(u_f(\xi), 0)$ has a solution by Dudley's representation theorem. This gives the existence of solutions. We prove the uniqueness. Let Y^1 and Y^2 be two solutions of $eq(\xi, f(y)|z|^2)$ such that $u_f(Y^1)$ and $u_f(Y^2)$ belong to class (D). Arguing as in the proof of Proposition 1.1, we show that $u_f(Y^1) = u_f(Y^2)$ which implies that $Y^1 = Y^2$ since u_f is one to one. Arguing again as in the proof of Proposition 1.1, we show that $\int_0^T |Z_s^1 - Z_s^2|^2 ds = 0$ $a.s$, since u_f' is strictly positive.

We prove (i). Let v be the map defined in Lemma 5.1-II). For $N > 0$, let

$$\tau_N := \inf\{t > 0 : |Y_t| + \int_0^t |v'(Y_s)|^2 |Z_s|^2 ds \geq N\} \wedge T.$$

Set $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. Since the map $x \mapsto v(|x|)$ belongs to $W^2_{1,loc}(\mathbb{R})$, then thanks to Itô-Krylov's formula for BSDEs (see [6, 7]), we have for any $t \in [0, T]$,

$$\begin{aligned} &v(|Y_0|) \\ &= v(|Y_{t\wedge\tau_N}|) \\ &\quad + \int_0^{t\wedge\tau_N} \Big[\text{sgn}(Y_s) v'(|Y_s|) f(Y_s)|Z_s|^2 - \frac{1}{2} v''(|Y_s|)|Z_s|^2\Big] ds \\ &\quad - \int_0^{t\wedge\tau_N} \text{sgn}(Y_s) v'(|Y_s|) Z_s dW_s\,. \end{aligned}$$

Lemma 5.1-II) allows us to deduce that for any $N > 0$,

$$\frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_N} |Z_s|^2 ds \leq \mathbb{E}\Big[v(|Y_{t\wedge\tau_N}|)\Big] \leq \sup_{\tau\leq T} \mathbb{E}\Big[v(|Y_\tau|)\Big]$$

where the supremum in the first right hand term, is taken over all stopping times $\tau \leq T$.

Since the process $[v(|Y_t|)]$ belongs to class (D), the proof is completed by using Fatou's lemma.

We prove (ii). Without loss, we assume that f is positive. Let w be the map defined in Lemma 5.1-III). For $N > 0$, let

$$\tau_N := \inf\{t > 0 : |Y_t| + \int_0^t |w'(Y_s)|^2 |Z_s|^2 ds \geq N\} \wedge T.$$

Set $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. Since the map $x \mapsto w(|x|)$ belongs to $W^2_{1,loc}(\mathbb{R})$, then thanks to Itô-Krylov's formula for BSDEs (see [6, 7]), we have for any $t \in [0, T]$,

$$\begin{aligned} &w(|Y_0|) \\ &= w(|Y_{t\wedge\tau_N}|) \\ &\quad + \int_0^{t\wedge\tau_N} \Big[\text{sgn}(Y_s) w'(|Y_s|) f(Y_s)|Z_s|^2 - \frac{1}{2} w''(|Y_s|)|Z_s|^2\Big] ds \\ &\quad - \int_0^{t\wedge\tau_N} \text{sgn}(Y_s) w'(|Y_s|) Z_s dW_s. \end{aligned}$$

Assumption **(A1)** and Lemma 5.1-III) allow us to show that for any $N > 0$,

$$\begin{aligned} \frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_N} f(Y_s)|Z_s|^2 ds &\leq \mathbb{E}[w(|Y_{t\wedge\tau_N}|)] \\ &\leq \sup_{\tau\leq T} \mathbb{E}\Big[w(|Y_\tau|)\Big] \end{aligned}$$

where the supremum in the first right hand term, is taken over all stopping times $\tau \leq T$.
Since $w(Y)$ belongs to class (D), the proof is completed by sending N to ∞ and using Fatou's lemma. Proposition 2.2 is proved. □

Remark 2.1. (i) Propositions 2.1 and 2.2 give the existence of solution in $\mathcal{C} \times \mathcal{L}^2$ with and without integrability of $u_f(\xi)$. However, if one wants to get more integrability of the solutions, then it is sufficient to impose more integrability on $u_f(\xi)$. For instance, when ξ is bounded then the solution (Y, Z) is such that Y is bounded and $(\int_0^t Z_s dW_s)_{0\leq t\leq T}$ is a BMO martingale.

(ii) Note that, in contrast to [6, 7], the integrability of $u_f(\xi)$ is not equivalent to that of ξ. This fact generate many supplementary difficulties in our situation.

Proposition 2.3. (Comparison)
Let ξ_1, ξ_2 be $\mathcal{F}_T$–measurable. Let f_1, f_2 be elements of $\mathbb{L}^1_{loc}(\mathbb{R})$. Assume that $u_{f_1}(\xi_1)$ and $u_{f_2}(\xi_2)$ are integrable. Let (Y^{f_1}, Z^{f_1}), (Y^{f_2}, Z^{f_2}) be respectively the unique solution in class (D) of $eq(\xi_1, f_1(y)|z|^2)$ and $eq(\xi_2, f_2(y)|z|^2)$. Assume that $\xi_1 \leq \xi_2$ a.s. and $f_1 \leq f_2$ a.e. Then $Y_t^{f_1} \leq Y_t^{f_2}$ for all t $\mathbb{P}$–a.s.

Proof. According to Proposition 2.2, the solutions Y^{f_1} and Y^{f_2} belong to class (D). Arguing as in the proof of Proposition 2.2, one can show that the processes $\left(\int_0^t u'_{f_1}(Y_s^{f_1}) Z_s^{f_1} dW_s\right)_{0\leq t\leq T}$ and $\left(\int_0^t u'_{f_2}(Y_s^{f_2}) Z_s^{f_2} dW_s\right)_{0\leq t\leq T}$ are uniformly integrable martingales. Using the Burkholder-Davis-Gundy inequality and the fact that $f_1 \leq f_2$, we show that the process $\left(\int_0^t u'_{f_1}(Y_s^{f_2}) Z_s^{f_2} dW_s\right)_{0\leq t\leq T}$ is a uniformly integrable martingale. The rest of the proof can be performed as that of Proposition 3.2. in [7]. □

3 BSDE (ξ, H)

To deal with more general BSDEs, we need the domination argument which will be present in the following subsection.

3.1 The domination argument

The domination argument implicitly appears in [6, 7] in a particular situation where the two dominating solutions belong to $\mathcal{S}^2 \times \mathcal{M}^2$, the function f is globally integrable on $\mathbb{R}$, the three terminal values are square integrable, the two dominating coefficients H_1, H_2 are of quadratic growth in z and of linear growth in y. Here, this argument is presented in a general framework, that is: the two dominating solutions lie in $\mathcal{C} \times \mathcal{L}^2$, the three involved terminal data ξ, ξ_1 and ξ_2 are not necessary integrable, the two dominating coefficients H_1, H_2 are merely measurable and can have arbitrarily growth. Only $H(t, y, z)$ should be continuous on (y, z) and of at most quadratic growth in z.

Definition 3.1. (Domination conditions) We say that the data (ξ, H) satisfy a domination condition if there exist two $(\mathcal{F}_t)$ progressively measurable processes H_1 and H_2, two $(\mathcal{F}_T)$ measurable random variables ξ_1 and ξ_2 such that:

(D1) $\xi_1 \le \xi \le \xi_2$

(D2) $eq(\xi_1, H_1)$ and $eq(\xi_2, H_2)$ have two solutions (Y^1, Z^1) and (Y^2, Z^2) such that:

- **(a)** $Y^1 \le Y^2$,
- **(b)** for every (t, ω), $y \in [Y_t^1(\omega),\ Y_t^2(\omega)]$ and $z \in \mathbb{R}^d$,
  - (i) $H_1(t, y, z) \le H(t, y, z) \le H_2(t, y, z)$
  - (ii) $|H(t, \omega, y, z)| \le \eta_t(\omega) + C_t(\omega)|z|^2$

where C and η are $\mathcal{F}_t$-adapted processes such that C is continuous and η satisfies for each ω, $\int_0^T |\eta_s(\omega)|ds < \infty$.

Lemma 3.1. *(Existence by domination) Let H be continuous in (y, z) for a.e. (t, ω). Assume moreover that (ξ, H) satisfy the domination conditions (D1)–(D2). Then,*

(i) The BSDE (ξ, H) has at least one solution (Y, Z) such that $Y^1 \le Y \le Y^2$.

(ii) Among all solutions which lie between Y^1 and Y^2, there exist a maximal and a minimal solution.

This lemma directly gives the existence of solutions. We do not need any a priori estimates nor approximation. The idea of the proof consists in deriving the existence of solutions for the BSDE without reflection from solutions of a suitable QBSDE with two reflecting barriers. To this end, we use the remarkable result of Essaky & Hassani ([22], Theorem 3.2) which establishes the existence of solutions for reflected QBSDEs without assuming any integrability condition on the terminal value. For the self-contained, this result is stated in Theorem 5.1 in Appendix.

Proof. (of Lemma 3.1) Using Theorem 3.2 in [22] (see Theorem 5.1 in Apendix) with $L = Y^{H_1}$ and $U = Y^{H_2}$, there exists a process (Y, Z, K^+, K^-) such that (Y, Z) belongs to $\mathcal{C} \times \mathcal{L}^2$ and (Y, Z, K^+, K^-) satisfies the following reflected BSDE, for $t \in [0,\ T]$,

$$\begin{cases} (i) & Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \\ & \qquad + \int_t^T dK_s^+ - \int_t^T dK_s^- \\ (ii) & \forall\, t \le T, \quad Y_t^{H_1} \le Y_t \le Y_t^{H_2}, \\ (iii) & \int_0^T (Y_t - Y_t^{H_1})dK_t^+ = \int_0^T (Y_t^{H_2} - Y_t)dK_t^- \\ & \qquad = 0, \text{ a.s.}, \\ (iv) & K_0^+ = K_0^- = 0, \quad K^+, K^- \text{ are continuous} \\ & \qquad \text{nondecreasing}, \\ (v) & dK^+ \perp dK^-. \end{cases} \tag{3.1}$$

Moreover, equation (3.1) has a minimal solution and a maximal solution.

It remains to show that $dK^+ = dK^- = 0$. Since $Y_t^{H_2}$ is a solution to the BSDE $eq(\xi_2, H_2)$, then Tanaka's formula applied to $(Y_t^{H_2} - Y_t)^+$ shows that

$$\begin{aligned} &(Y_t^{H_2} - Y_t)^+ \\ &= (Y_0^{H_2} - Y_0)^+ \\ &\quad + \int_0^t \mathbb{1}_{\{Y_s^{H_2} > Y_s\}}[H(s, Y_s, Z_s) - H_2(s, Y_s^{H_2}, Z_s^{H_2})]ds \\ &\quad + \int_0^t \mathbb{1}_{\{Y_s^{H_2} > Y_s\}}(dK_s^+ - dK_s^-) \\ &\quad + \int_0^t \mathbb{1}_{\{Y_s^{H_2} > Y_s\}}(Z_s^{H_2} - Z_s)dW_s \\ &\quad + L_t^0(Y^{H_2} - Y) \end{aligned}$$

where $L_t^0(Y^{H_2} - Y)$ is the local time at time t and level 0 of the semimartingale $(Y^{H_2} - Y)$.

Since $(Y_t^{H_2} - Y_t)^+ = (Y_t^{H_2} - Y_t)$, then by identifying the terms of $(Y_t^{H_2} - Y_t)^+$ with those of $(Y_t^{H_2} - Y_t)$, one can show that:

$$(Z_s - Z_s^{H_2})\mathbb{1}_{\{Y_s^{H_2} = Y_s\}} = 0 \quad \text{for } a.e.\ (s, \omega)$$

and

$$\begin{aligned} &\int_0^t \mathbb{1}_{\{Y_s^{H_2} = Y_s\}}(dK_s^+ - dK_s^-) \\ &= L_t^0(Y^{H_2} - Y) \\ &\quad + \int_0^t \mathbb{1}_{\{Y_s^{H_2} = Y_s\}}[H_2(s, Y_s^{H_2}, Z_s^{H_2}) - H(s, Y_s, Z_s)]ds \end{aligned}$$

Since $\int_0^t \mathbb{1}_{\{Y_s^{H_2} = Y_s\}}dK_s^+ = 0$, it holds that

$$\begin{aligned} 0 &\le L_t^0(Y^{H_2} - Y) \\ &\qquad + \int_0^t \mathbb{1}_{\{Y_s^{H_2} = Y_s\}}[H_2(s, Y_s^{H_2}, Z_s^{H_2}) - H(s, Y_s, Z_s)]ds \\ &= -\int_0^t \mathbb{1}_{\{Y_s^{H_2} = Y_s\}}dK_s^- \\ &\le 0 \end{aligned}$$

Hence, $\int_0^t \mathbb{1}_{\{Y_s^{H_2}=Y_s\}} dK_s^- = 0$, which implies that $dK^- = 0$. Similar arguments allow to show that $dK^+ = 0$. Therefore (Y, Z) is a solution to the (non reflected) BSDE $eq(\xi, H)$. □

As byproducts of the domination lemma, we establish in this section the existence of solutions to $eq(\xi, H)$ first when (A1)-(A2) are satisfied and next when (A3)-(A4) hold. Some integrability properties of the solutions are also established under additional assumptions which will be specified below.

3.2 BSDE(ξ, H) with $|H(t, y, z)| \leq \alpha_t + \beta_t|y| + f(|y|)|z|^2$

The goal is to solve $eq(\xi, H)$. If one try to follow the proofs given in [11, 13], we should establish some a priori estimates of solutions then use a suitable approximation. This way is not efficient in our situation and in particular the exponential transformation can not be applied because f is not constant. Furthermore, in contrast to the domination argument, the method of [11, 13] can not be applied when the integrability of the terminal value is not a priori fixed. For the same reason, the argument developed in [24, 15, 35] are also not effective in our situation. Note moreover that the methods used in [11, 13, 24, 15, 35] can not allow to prove the existence of a maximal and a minimal solution. The method used in [6, 7] does not work in our situation since f is not globally integrable. The questions which then arise are : what condition we should impose to the terminal value ξ in order to get the existence of solution when f is not globally integrable? How do we proceed in this case ? This is the subject of the next subsection. To prove the existence of solutions to $eq(\xi, H)$, our strategy consists in using Lemma 3.1 which allows to work without any a priori integrability condition on the terminal value ξ. Therefore, we start by assuming that only (A1) is satisfied. Let

$$g(t, y, z) := \alpha_t + \beta_t|y| + f(|y|)|z|^2 \tag{3.2}$$

According to Lemma 3.1, to establish the existence of solutions to $eq(\xi, H)$, it is enough to show that $eq(\xi^+, g)$ has a solution (Y^g, Z^g) and $eq(-\xi^-, -g)$ has a solution (Y^{-g}, Z^{-g}) such that

$$Y^{-g} \leq Y^g. \tag{3.3}$$

In [6, 7], the fact that ξ is assumed square integrable and f is globally integrable make simple the solvability of $eq(\xi^+, g)$ and $eq(-\xi^-, -g)$ in $\mathcal{S}^2 \times \mathcal{M}^2$ from which we easly deduce inequality (3.3). The difficulty which arises in our situation is:

how to prove inequality (3.3) *when we do not have any information on the integrability of* ξ *nor on the integrability of the solutions?*

We emphasize that the comparison theorem does not work in this situation. Note that, in order to establish the existence of solution to $eq(\xi, H)$, we need to prove inequality (3.3). Since the comparison theorem does not work, we proceed as follows: we assume that (A1) holds, *we force* Y^{-g} *to be negative and* Y^g *to be positive then we deduce the integrability condition [namely (A2)] which we should impose to the terminal value* ξ *in order to get the existence of solution.* This shows that assumption (A2) is not a priori imposed (as done in the previous papers) but generated by solving an *inverse problem.* Lemma 3.1 plays a key role in our proof. In particular, it allows us to reduce the solvability of $eq(\xi^+, g)$ to that of $eq(\xi^+_{\alpha,\beta}, 0)$ from which we derive assumption (A2).

3.2.1 The BSDE $(\xi, \alpha_t + \beta_t|y| + f(|y|)|z|^2)$

The goal of this subsection is to show that the BSDE (ξ^+, g) has a positive solution and the BSDE $(\xi^-, -g)$ has a negative solution. The method used in [6, 7] to show this consists in reducing the solvability of these two QBSDEs to the solvability of two BSDEs having linear growth. These computations do not work in our situation, since f is not globally integrable. The proof of following proposition shows how to get the existence of a positive solution to $eq(\xi^+, g)$ by using the domination argument when we do not have a priori information on the integrability of the terminal value ξ. It also allows to see how to determine the integrability we should impose to ξ by solving an inverse problem.

Proposition 3.1. *Let* ξ *be an* $\mathcal{F}_T-$*measurable random variable. Let* f *be as in assumption (A1) and* g *be the function defined by* (3.2).

(i) The BSDE (ξ^+, g) *has a positive solution if the BSDE* $\left(u_f(\xi^+_{\alpha,\beta}),\ 0\right)$ *has a positive solution* (Y, Z) *such that* $Y \geq u_f\left[e^{\int_0^T \beta_s ds}\left(\int_0^T \alpha_s ds\right)\right]$.

(ii) The BSDE $(-\xi^-, -g)$ *has a negative solution if the BSDE* $\left(u_f(\xi^-_{\alpha,\beta}),\ 0\right)$ *has a positive solution* (Y, Z) *such that* $Y \geq u_f\left[e^{\int_0^T \beta_s ds}\left(\int_0^T \alpha_s ds\right)\right]$.

(iii) If moreover assumption (A2) is satisfied, then $eq(\xi^+, g)$ *has a positive solution and* $eq(\xi^-, -g)$ *has a negative solution.*

Proof. (i) For the simplicity of notations, we assume that α and β are constant. Note that (Y, Z) is a positive solution to $eq(\xi^+, g)$ if and only if (Y, Z) is a positive solution to the BSDE

$$Y_t = \xi^+ + \int_t^T (\alpha + \beta Y_s + f(Y_s)|Z_s|^2)ds - \int_t^T Z_s dW_s. \tag{3.4}$$

Therefore, it is enough to prove that:
if $eq(u_f[e^{\beta T}(\xi^+ + \alpha T)],\ 0)$ has a solution (Y^0, Z^0) such that $Y^0 \geq u_f(\alpha T e^{\beta T})$ then equation (3.4) has a positive solution.

Return back to BSDE (3.4). By putting $(Y^1_t, Z^1_t) := (Y_t + \alpha t, Z_t)$, we see that equation (3.4) has a positive solution if and only if the BSDE

$$\begin{aligned} Y^1_t &= \xi^+ + \alpha T + \int_t^T \Big[\beta(Y^1_s - \alpha s) \\ &\quad + f(Y^1_s - \alpha s)|Z^1_s|^2\Big] ds - \int_t^T Z^1_s dW_s \end{aligned} \quad (3.5)$$

has a solution (Y^1, Z^1) such that for each t, $Y^1_t \geq \alpha t$.

Consider now the BSDE

$$Y_t = \alpha T - \int_t^T Z_s dW_s \quad (3.6)$$

Clearly

- $(Y, Z) = (\alpha T, 0)$ is a solution to the BSDE (3.6),
- $0 \leq \xi^+ + \alpha T \leq \xi^+ + \alpha T$,
- for any $y \geq \alpha T$, $0 \leq \beta(y - \alpha s) + f(y - \alpha s)|z|^2 \leq \beta y + f(y)|z|^2$, since f is increasing.

Therefore, using Lemma 3.1 [with $\xi_1 = 0$, $\xi = \xi^+ + \alpha T = \xi_2$, $H_1 = 0$ and $H_2 = \beta y + f(y)|z|^2$], we show that equation (3.5) has a solution (Y^1, Z^1) satisfying $Y^1 \geq \alpha T$ if the BSDE

$$Y^2_t = \xi^+ + \alpha T + \int_t^T \Big[\beta(Y^2_s) + f(Y^2_s)|Z^2_s|^2\Big] ds - \int_t^T Z^2_s dW_s \quad (3.7)$$

has a solution (Y^2, Z^2) satisfying $Y^2 \geq \alpha T$.

But, Itô's formula shows that (Y^2, Z^2) is a solution to equation (3.7) satisfying $Y^2 \geq \alpha T$ if and only if the process $(Y^3_t, Z^3_t) := (Y^2_t e^{\beta t}, Z^2_t e^{\beta t})$ is a solution to the BSDE

$$\begin{aligned} Y^3_t &= (\xi^+ + \alpha T)e^{\beta T} + \int_t^T f(Y^3_s e^{-\beta s})|Z^3_s|^2 e^{-\beta s} ds \\ &\quad - \int_t^T Z^3_s dW_s \end{aligned} \quad (3.8)$$

satisfying $Y^3 \geq \alpha T e^{\beta T}$.

Since f is increasing and continuous then, as previously, we again use Lemma 3.1 [with $\xi_1 = 0$, $\xi = (\xi^+ + \alpha T)e^{\beta T} = \xi_2$, $H_1 = 0$ and $H_2 = f(y)|z|^2$], to show that equation (3.8) has a solution (Y^3, Z^3) such that $Y^3 \geq \alpha T e^{\beta T}$ if $eq([\xi^+ + \alpha T]e^{\beta T}, f(y)|z|^2)$ has a solution (Y^4, Z^4) satisfying $Y^4 \geq \alpha T e^{\beta T}$. Applying Itô's formula to $u_f(Y^4_t)$, we show that $eq([\xi^+ + \alpha T]e^{\beta T}, f(y)|z|^2)$ has a solution (Y^4, Z^4) such that $Y^4 \geq \alpha T e^{\beta T}$ if and only if $eq(u_f[\xi^+ + \alpha T]e^{\beta T}), 0)$ has a solution (Y^5, Z^5) satisfying $Y^5 \geq u_f(\alpha T e^{\beta T})$. According to Proposition 1.1, the latter is equivalent to the fact that $u_f[e^{\beta T}(\xi^+ + \alpha T)]$ is integrable. Assertion (i) is proved.

Assertion (ii) can be proved similarly, since (Y, Z) is a negative solution to $eq(-\xi^-, -g)$ if and only if $(Y', Z') := (-Y, -Z)$ is a positive solution to the BSDE

$$Y'_t = \xi^- \int_t^T \Big[\alpha + \beta Y'_s + f(Y'_s) Z^2_s\Big] ds - \int_t^T Z'_s dW_s \quad (3.9)$$

Proposition 3.1 is proved. □

3.2.2 The BSDE (ξ, H) with $|H(t, y, z)| \leq \alpha + \beta|y| + f(|y|)|z|^2$

The following theorem is deduced from Proposition 3.1 and Lemma 3.1. It covers the previous results established in [11, 13, 15, 24, 26] and many others situations which are not covered by the previous works on QBSDE. For instance, we cover the cases: $H(y, z) = y|z|$ and also $H(y, z) = \alpha_t + \beta_t|y| + h(|y|)|z|^p + f(|y|)|z|^2$ with $0 < p < 2$ and f, h continuous, increasing and positive on $\mathbb{R}_+$.

Theorem 3.1. (i) *Assume that H and ξ satisfy (A1)-(A2). Then, $eq(\xi, H)$ has at least one solution (Y, Z) which satisfies for any t,*

$$-u_f^{-1}\left(\mathbb{E}\left[u_f(\xi^-_{\alpha,\beta})/\mathcal{F}_t\right]\right) \leq Y_t \leq u_f^{-1}\left(\mathbb{E}\left[u_f(\xi^+_{\alpha,\beta})/\mathcal{F}_t\right]\right). \quad (3.10)$$

In particular, we have

$$-\mathbb{E}\left[u_f\left(\xi^-_{\alpha,\beta}\right)\right] \leq \mathbb{E}[u_f(Y_t)] \leq \mathbb{E}\left[u_f\left(\xi^+_{\alpha,\beta}\right)\right].$$

(ii) *Among all solutions satisfying* (3.10)*, there are a maximal and a minimal solution. Note also that, among all solutions satisfying $Y^{-g} \leq Y \leq Y^g$, there also exists a maximal and a minimal solution.*

The following remark will be used later. It can be proved as Theorem 3.1.

Remark 3.1. Under the assumptions of Theorem 3.1, it also holds that

$$\begin{aligned} &-u_f^{-1}\left(\mathbb{E}\left[u_f\left(\xi^-_{\alpha,\beta}\right)/\mathcal{F}_t\right]\right) \\ &\leq \left(Y_t + \int_0^t \alpha_s ds\right) e^{\int_0^t \beta_s ds} \\ &\leq u_f^{-1}\left(\mathbb{E}\left[u_f\left(\xi^+_{\alpha,\beta}\right)/\mathcal{F}_t\right]\right). \end{aligned}$$

And in particular

$$\begin{aligned} &-\mathbb{E}\left[u_f\left(\xi^-_{\alpha,\beta}\right)\right] \\ &\leq \mathbb{E}\left(u_f\left[\left(Y_t + \int_0^t \alpha_s ds\right) e^{\int_0^t \beta_s ds}\right]\right) \\ &\leq \mathbb{E}\left[u_f\left(\xi^+_{\alpha,\beta}\right)\right]. \end{aligned}$$

Corollary 3.1. *Theorem 3.1 remains valid when the function f is continuous but not necessary increasing. In this case, assumption (A2) should be slightly modified as follows:*

(A2bis) $\mathbb{E}\left[u_\phi\left(\xi_{\alpha,\beta}\right)\right] < \infty$ *where* $\phi(y) := \sup_{0 \leq x \leq y} f(x)$.

Proof. Since the function ϕ is increasing and continuous, the result follows. □

Remark 3.2. Note that when f is merely locally bounded but not necessary continuous, then Theorem 3.1 remains valid with the following condition in place of assumption (A2).

(A2ter) $\mathbb{E}\left[u_{\varphi}\Big(\xi_{\alpha,\beta}\Big)\right] < \infty$, where φ is the smallest continuous, increasing function such that $\varphi \geq f$.

Proposition 3.2. *(Integrability property of solutions)* (i) *Let the assumptions of Theorem 3.1 be satisfied. Assume moreover that there exists $p > 1$ such that*

$$\mathbb{E}\Big[\Big(u_f[\xi_{\alpha,\beta}]\Big)^p\Big] < \infty. \tag{3.11}$$

then the BSDE(ξ, H) has a solution (Y, Z) which satisfies

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left[u_f\left([|Y_t| + \int_0^t \alpha_s ds]e^{\int_0^t \beta_s ds}\right)\right]^p\right) \leq \mathbb{E}\Big([u_f(\xi_{\alpha,\beta})]^p\Big).$$

(ii) *If moreover, $v(|Y|)$ belongs to class (D) and $\sup_{0\leq s\leq T}\mathbb{E}\left[|Y_s|v'(|Y_s|)\right] < \infty$ then, the BSDE(ξ, H) has a solution (Y, Z) which satisfies*

$$\mathbb{E}\int_0^T |Z_s|^2 ds < \infty, \tag{3.12}$$

here v is the function defined in Lemma 5.1-II).

Remark 3.3. (i) Let v be the function defined in Lemma 5.1-II). Since for y large enough, $v'(|y|) \leq \Big[u'(|y|)\Big]^p$ $|y|v'(|y|) \leq \Big[u'(|y|)\Big]^p$ and we know by Theorem 3.1 that

$$|Y_t| \leq u_f^{-1}\left(\mathbb{E}\left[u_f\Big(\xi_{\alpha,\beta}\Big)/\mathcal{F}_t\right]\right),$$

then the conditions $v(|Y|)$ belongs to class (D) and $\sup_{0\leq s\leq T}\mathbb{E}\left[|Y_s|v'(|Y_s|)\right] < \infty$ are satisfied when

$$\sup_{0\leq t\leq T}\mathbb{E}\left\{(\alpha_t + \beta_t)\left(u_f'\left[u_f^{-1}\left(\mathbb{E}\left[u_f\left(\xi_{\alpha,\beta}\right)/\mathcal{F}_t\right]\right)\right]\right)^p\right\} < \infty, \tag{3.13}$$

and

$$\mathbb{E}\left(\left[\sup_{t\leq T}\left(u_f'\left[u_f^{-1}\left(\mathbb{E}\left[u_f\Big(\xi_{\alpha,\beta}\Big)/\mathcal{F}_t\right]\right)\right]\right)\right]^p\right) < \infty. \tag{3.14}$$

(ii) Conditions (3.13) and (3.14), which seem be complicated, cover those used in previous works. For instance, when α_t, β_t and f are constant with $f(y) = \frac{\gamma}{2}$ (see [13]), one can take $u_f(y) = exp(\gamma y)$. And in this case, conditions (3.13) and (3.14) become $\mathbb{E}[e^{p\gamma(\xi_{\alpha,\beta})}] < \infty$ for some $p > 1$, which is the condition imposed in [13].

Proof. (of Theorem 3.1). The existence of (a maximal and a minimal) solutions follows from Proposition 3.1 and Lemma 3.1. Indeed, the two solutions we constructed in Proposition 3.1 satisfy $Y^{-g} \leq Y^g$. We then use Lemma 3.1 with $\xi_1 = -\xi^-$, $\xi_2 = \xi^+$, $H_1 = -g$ and $H_2 = g$ to get the existence of solutions. We shall prove estimate (3.10). For simplicity, we assume that α and β are constant. Since $u_f[e^{\beta T}(\xi^+ + \alpha T)]$ is integrable, then the solution we constructed in the proof of assertion (i) of Proposition 3.1 satisfies for any t,

$$u_f([Y^g + \alpha t]e^{\beta t}) \leq \mathbb{E}(u_f[e^{\beta T}(\xi^+ + \alpha T)]/\mathcal{F}_t).$$

This shows the first inequality of (3.10). A similar argument allows us to prove the second inequality. Theorem 3.1 is proved. □

Proof of Proposition 3.2. Assertion (i). Note that the solution (Y^5, Z^5) we constructed in the proof of Proposition 3.1 satisfies the BSDE

$$Y_t^5 = u_f[e^{\beta T}(\xi^+ + \alpha T)] - \int_t^T Z_s^5 dW_s \tag{3.15}$$

Since there exists $p > 1$ such that inequality (3.11) holds, then equation (3.15) has a unique solution (Y^5, Z^5) such that (details can be found in [3, 4]):

$$\mathbb{E}(\sup_{0\leq t\leq T}[Y_t^5]^p) \leq \mathbb{E}\left(\left[u_f\left(e^{\beta T}[|\xi| + \alpha T]\right)\right]^p\right).$$

But, since $u_f[e^{\beta t}(Y_t^g + \alpha_t)] \leq Y^5$, we then have,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big[u_f\Big(\Big[Y_t^g + \int_0^t \alpha_s ds\Big]e^{\int_0^t \beta_s ds}\Big)\Big]^p\Big)$$
$$\leq \mathbb{E}\left(\left[u_f\left(e^{\beta T}[|\xi| + \alpha T]\right)\right]^p\right).$$

Similarly, we get

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}\Big[u_f\Big(\Big[(-Y_t^g) + \int_0^t \alpha_s ds\Big]e^{\int_0^t \beta_s ds}\Big)\Big]^p\Big)$$
$$\leq \mathbb{E}\left(\left[u_f\left(e^{\beta T}[|\xi| + \alpha T]\right)\right]^p\right).$$

Assertion (i) is proved.

We shall prove assertion (ii). Let v be the function defined in Lemma 5.1-II). For $N > 0$, let $\tau_N := \inf\{t > 0 : |Y_t| + \int_0^t |v'(Y_s)|^2|Z_s|^2 ds \geq N\} \wedge T$. Set $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. Since the map $x \mapsto v(|x|)$ belongs to $W^2_{1,loc}(\mathbb{R})$ [actually the map $x \mapsto v(|x|)$ belongs to $\mathcal{C}^2(\mathbb{R})$, since f is assumed to be continuous], then thanks to Itô-Krylov's formula for BSDEs (see [7]), we have for any $t \in [0, T]$

$$\begin{aligned} &v(|Y_0|)\\ &= v(|Y_{t\wedge\tau_N}|)\\ &\quad + \int_0^{t\wedge\tau_N}\Big[\text{sgn}(Y_s)v'(|Y_s|)H(s, Y_s, Z_s) - \frac{1}{2}v''(|Y_s|)|Z_s|^2\Big]ds\\ &\quad - \int_0^{t\wedge\tau_N}\text{sgn}(Y_s)v'(|Y_s|)Z_s dW_s\,. \end{aligned}$$

Assumption **(A1)** and Lemma 5.1-II) allow us to show that for any $N > 0$,

$$\frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_N}|Z_s|^2 ds \le \mathbb{E}\Big[v(|Y_{t\wedge\tau_N}|)\Big] + \mathbb{E}\int_0^T [\alpha_t v'(|Y_s|) + \beta_t |Y_s| v'(|Y_s|)] ds$$

Since the processes $v(|Y|)$ and $|Y|v'(|Y|)$ belong to class (D), the proof is completed by using Fatou's lemma. Proposition 3.2 is proved. □

Corollary 3.2. *(BMO property) (i) Let (A1) be satisfied. Assume moreover that ξ, $\int_0^T \alpha_s ds$ and $\int_0^T \beta_s ds$ are bounded. Then, every solution (Y, Z) satisfying inequalities* (3.10) *is such that Y is bounded and the process $(\int_0^t Z_s dW_s)_{0\le t\le T}$ is a BMO martingale.*

Proof. (i) Let (Y, Z) be a solution to $eq(\xi, H)$ such that Y satisfies inequalities (3.10). Since ξ, $\int_0^T \alpha_s ds$ and $\int_0^T \beta_s ds$ are bounded, then clearly Y is bounded. Arguing as in the proof of Proposition 3.2 (ii), one can show that Z belongs to $\mathcal{M}^2$. We shall prove that the process $(\int_0^t Z_s dW_s)_{0\le t\le T}$ is a BMO martingale. Let v be the function defined in Lemma 5.1-II). Since the map $x \longmapsto v(|x|)$ belongs to $W^2_{1,loc}(\mathbb{R})$, then Itô-Krylov's formula for BSDEs ([7]) shows that for any $\mathcal{F}_t$–stopping time $\tau \le T$,

$$\begin{aligned} v(|Y_T|) &= v(|Y_\tau|) \\ &+ \int_\tau^T \Big[\frac{1}{2} v''(|Y_s|)|Z_s|^2 - \operatorname{sgn}(Y_s) v'(|Y_s|) H(s, Y_s, Z_s)\Big] ds \\ &+ \int_\tau^T \operatorname{sgn}(Y_s) v'(|Y_s|) Z_s dW_s\,. \end{aligned}$$

Since Y is bounded and Z belongs to $\mathcal{M}^2$, it follows that the stochastic integral in the right hand side term of the previous equality is a square integrable $\mathcal{F}_t$–martingale. Passing to conditional expectation, one can show that there exist positive constants K_1 and K_2 such that

$$\mathbb{E}\left(\int_\tau^T |Z_s|^2 ds/\mathcal{F}_\tau\right) \le K_1 + \mathbb{E}\left(\int_\tau^T [(\alpha_s + \beta_s|Y_s|) v'(|Y_s|)] ds/\mathcal{F}_\tau\right)$$

We complete the proof by noticing that the processes Y, $\int_0^T \alpha_s ds$ and $\int_0^T \beta_s ds$ are bounded. Assertions (ii) and (iii) can be proved similarly. □

Remark 3.4. Let $|H(t, y, z)| \le \alpha_t + \beta_t|y| + f(y)|z|^2$, with f continuous, positive and increasing but not globally integrable. Then, the condition which ensures the existence of solution is:

$$\mathbb{E}\big(u_f(\xi^+_{\alpha,\beta})\big) < \infty \text{ and } \mathbb{E}\left(\exp\left[(\xi^-)e^{\int_0^T \beta_s ds} 2f(0)\right]\right) < \infty \quad (3.16)$$

Proof. We assume that α and β are constants for simplicity. As previously, thanks to Lemma 3.1, $eq(\xi, H)$ has a solution if $eq(\xi^+, \alpha + \beta|y| + f(y)|z|^2)$ has a positive solution and $eq(-\xi^-, -[\alpha + \beta|y| + f(y)|z|^2])$ has a negative solution. The first inequality in (3.16) can be proved as in Proposition 3.1. To prove the second inequality of (3.16), we see that (Y, Z) is a negative solution to $eq(-\xi^-, -[\alpha + \beta|y| + f(y)|z|^2])$ if $(Y', Z') := (-Y, -Z)$ is a positive solution to the BSDE

$$Y'_t = \xi^- + \int_t^T [\alpha + \beta Y'_s + f(-Y'_s)|Z'_s|^2] ds - \int_t^T Z'_s dW_s.$$

Since f is increasing, we use Lemma 3.1 to show that the previous BSDE has a positive solution if the BSDE $Y^1_t = \xi^- + \int_t^T [\alpha + \beta Y^1_s + 2f(0)|Z^1_s|^2] ds - \int_t^T Z^1_s dW_s$ has a positive solution. The sequel of the proof goes as in Theorem 3.1. □

The following corollary extends the result of [6, 7] to the case where $\xi_{\alpha,\beta}$ is only integrable and the coefficients α, β, γ are positive processes.

Corollary 3.3. *Let (A1) be satisfied with f globally integrable and locally bounded. Assume that $\xi_{\alpha,\beta}$ is integrable. Then, $eq(\xi, H)$ has a solution such that for every t*

$$-\mathbb{E}\left(\xi^-_{\alpha,\beta}/\mathcal{F}_t\right) \;\le\; Y_t \;\le \mathbb{E}\left(\xi^+_{\alpha,\beta}/\mathcal{F}_t\right).$$

(ii) Among all solutions satisfying (3.10)*, there are a maximal and a minimal solution. Note also that, among all solutions satisfying $Y^{-g} \le Y \le Y^g$, there also exists a maximal and a minimal solution where the function g is defined by* (3.2)*.*

Proof. Let g be the function defined by (3.2). Since f is globally integrable u_f and its inverse are uniformly Lipschitz. Hence, arguing as in the proof of Proposition 3.1 one show that $eq(\xi^+, g)$ has a positive solution Y^g such that: $0 \le Y^g_t \le \mathbb{E}\left(\xi^+_{\alpha,\beta}/\mathcal{F}_t\right)$. Symmetrically, we show that $eq(-\xi^-, -g)$ has a negative solution Y^{-g} which satisfied : $-\mathbb{E}\left(\xi^-_{\alpha,\beta}/\mathcal{F}_t\right) \le Y^{-g}_t \le 0$. Since f is locally bounded, then according to the proof of Theorem 4.1 of [7], there exists a continuous function $\bar{f}$ such that $f \le \bar{f}$. Therefore, we can apply Lemma 3.1 with $\eta_t = \alpha_t + \beta_t(|Y^{-g}_t| + |Y^g_t|)$ and $C_t = \sup_{s\le t}\sup_{a\in[0,1]} |\bar{f}(aY^{-g}_s + (1-a)Y^g_s)|$. Corollary 3.3 is proved. □

3.2.3 BSDE(ξ, H) with $|H(t, y, z)| \le \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2$

As in the previous subsection, we use Lemma 3.1 to reduce the solvability of $eq(\xi, H)$ to the positive solvability ($Y \ge 0$) of the simple BSDEs $(u_f(\xi^+_{\alpha,\beta}), \theta_t|z|)$ and $(u_f(\xi^-_{\alpha,\beta}), \theta_t|z|)$ then apply Proposition 1.2 to conlude. We put

$$h(t, y, z) := \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2 \qquad (3.17)$$

Proposition 3.3. *Assume that* (A3), (A4) *are satisfied. Then,*

(i) $eq(\xi, H)$ *has at least one solution such that*

$$-u_f^{-1}\left(ess\sup_{\pi\in\sum}\mathbb{E}\left(\Gamma^{\pi}_{t,T}u_f(\xi^-_{\alpha,\beta})/\mathcal{F}_t\right)\right) \le Y_t \le u_f^{-1}\left(ess\sup_{\pi\in\sum}\mathbb{E}\left(\Gamma^{\pi}_{t,T}u_f(\xi^+_{\alpha,\beta})/\mathcal{F}_t\right)\right) \quad (3.18)$$

(ii) Among all solutions satisfying inequalities (3.18) *there are a maximal and a minimal solution.*

(iii) Assume moreover that $\exp(\int_0^T\theta_s^2ds)$ *is integrable, and* $\xi_{\alpha,\beta}$ *is bounded. Then all solutions satisfying inequalities* (3.18) *are bounded.*

Remark 3.5. One may wonder what is the usefulness of assumption (A3) since it can be reduced to assumption (A1) by the operation $\alpha_t+\beta_t|y|+\theta_t|z|+f(|y|)|z|^2)\le\alpha_t+\frac{1}{2}\theta^2+\beta_t|y|+[\frac{1}{2}+f(|y|)]|z|^2)$. It should be noted that in this case the integrability requested to the terminal value will be higher.

Proof. (of Proposition 3.3.) Since H satisfies (A3), then according to Lemma 3.1, it is enough to show that $eq(\xi^+,\alpha_t+\beta_t|y|+\theta_t|z|+f(|y|)|z|^2)$ has a positive solution. Arguing as in the proof of Theorem 3.1, one can show that $eq(\xi^+,\alpha_t+\beta_t|y|+\theta_t|z|+f(|y|)|z|^2)$ has a positive solution if $eq(u_f(\xi^+_{\alpha,\beta}),\,\theta_t|z|)$ has a solution which is greater than $u_f(\alpha Te^{\beta T})$. Now, $eq(u_f(\xi^+_{\alpha,\beta}),\,\theta_t|z|)$ has a solution thanks to assumption (A4) and Proposition 1.2. Inequality (3.18) can be derived by using the same arguments which has been used to prove inequality (3.10). Proposition 3.3 is proved. □

Remark 3.6. (i) Taking $f=0$ in Proposition 3.18, we get the existence of one dimensional BSDEs with a stochastic linear growth. This covers some results obtained in [27, 25, 14, 23, 33].

(ii) We emphasize that Proposition 1.8 combined with Lemma 3.1 allows to directly prove the existence of solutions to BSDEs whose generators satisfy

$$|H(t,y,z)|\le\alpha_t+\beta_t|y|+\theta_t|z| \quad (3.19)$$

and ξ satisfies assumption (A3) with $f=0$.

This constitute a new result on the existence of solutions to BSDEs with at most linear growth which for instance covers the recent result [25], with a simpler proof.

Remark 3.7. Since the transformation u_f does not impact Z, we then have:
If $|H(t,y,z)|\le\alpha_t+\beta_t|y|+\theta_t|z|+f(|y|)|z|^2$ with α, β, f satisfy (A3) and $\mathbb{E}\int_0^T e^{q\gamma_s}ds<\infty$ for some $q>0$, then arguing as in Proposition 3.3 and using [3, 4], one can show that $eq(\xi,H)$ has a solution (Y,Z) in $\mathcal{S}^p\times\mathcal{M}^p$ provided that $\exp(\frac{1}{2}\int_0^T\gamma_se^{\beta_s}ds)u_f(\xi_{\alpha,\beta})$ is p–integrable for some $p>1$.

The following corollary is not difficult to establish.

Corollary 3.4. *(BMO property)*

(i) Let (A3) *be satisfied. Assume moreover that* ξ, $\int_0^T\alpha_sds$, $\int_0^T\beta_sds$ *and* $\int_0^T\gamma_s^2ds$ *are bounded. Then, every solution* (Y,Z) *satisfying inequalities* (3.18) *is such that* Y *is bounded and the process* $(\int_0^tZ_sdW_s)_{0\le t\le T}$ *is a BMO martingale.*

(ii) When $|H(t,y,z)|\le\alpha_t+\theta_t|z|+f(|y|)|z|^2$ *and* ξ, $\int_0^T\alpha_sds$, $\int_0^T\gamma_s^2ds$ *are bounded, then we have the same conclusion as* (i) *with* f *locally integrable and increasing but not continuous.*

(iii) When $|H(t,y,z)|\le\theta_t|z|+f(|y|)|z|^2$ *and* ξ, $\int_0^T\alpha_sds$, $\int_0^T\gamma_s^2ds$ *are bounded, then we have the same conclusion as* (i) *with* f *merely locally integrable but neither increasing nor continuous.*

The following Corollary can be proved by combining the proof of Corollary 3.3 with that of Proposition 3.3.

Corollary 3.5. *Let (A3) be satisfied with* f *globally integrable and locally bounded. Assume that* $\xi_{\alpha,\beta}$ *satisfies (A4). Then,*

(i) $eq(\xi,H)$ *has a solution such that for every* t

$$-ess\sup_{\pi\in\sum}\mathbb{E}\left(\Gamma^{\pi}_{t,T}(\xi^-_{\alpha,\beta})/\mathcal{F}_t\right)\le Y_t\le ess\sup_{\pi\in\sum}\mathbb{E}\left(\Gamma^{\pi}_{t,T}(\xi^+_{\alpha,\beta})/\mathcal{F}_t\right)$$

(ii) Among all solutions satisfying the above inequalities, there are a maximal and a minimal solution. Note also that, among all solutions satisfying $Y^{-h}\le Y\le Y^h$, *there exist maximal and a minimal solutions where the function* g *is defined by* (3.17).

4 Quadratic BSDEs and BSDEs with logarithmic nonlinearity

The aim of this subsection is to study the BSDE (ξ,H) with H continuous and with $lLogl$-growth. That is : there exist positive constants a, b and c such that for every t,y,z

$$|H(t,y,z)\le a+b|y|+c|y||\ln|y|| \quad (4.1)$$

Using Lemma 3.1, we show that that of $eq(\xi,H)$ is equivalent to the solvability of $eq(\xi,\alpha+\beta|y|+\frac{\gamma}{2}|z|^2)$, for suitable α, β and γ. According to Theorem 3.1, the BSDE $(\xi,\alpha+\beta|y|+\frac{\gamma}{2}|z|^2)$ has a solution when $exp(\gamma e^{\alpha T}|\xi|)$ is integrable, and we have the following propositions.

Proposition 4.1. *The BSDE* $(\xi,\alpha+\beta|y|+\frac{\gamma}{2}|z|^2)$ *has a solution if and only if the BSDE* $(e^{\gamma\xi},\gamma\alpha y+\gamma\beta y|\ln y|)$ *has a solution.*

Proof. Let

$$g(t,y,z) := \alpha + \beta|y| + \frac{\gamma}{2}|z|^2. \tag{4.2}$$

Applying Itô's formula to $u(y) := e^{\gamma y}$, we show that (Y_t, Z_t) is a solution to $eq(\xi, g)$ if and only if $(\bar{Y}_t, \bar{Z}_t) := (e^{\gamma Y_t}, \gamma e^{\gamma Y_t} Z_t)$ is a solution to $eq(e^{\gamma\xi}, \gamma\alpha y + \gamma\beta y|\ln y|)$. Indeed: Let (Y, Z) be a solution of $eq(\xi, g)$. By Itô's formula we have,

$$\begin{aligned}
e^{\gamma Y_t} &= e^{\gamma Y_T} + \int_t^T \gamma e^{\gamma Y_s} g(s, Y_s, Z_s)ds \\
&\quad - \int_t^T \gamma e^{\gamma Y_s} Z_s dW_s - \frac{\gamma^2}{2}\int_t^T e^{\gamma Y_s}|Z_s|^2 ds \\
&= e^{\gamma\xi} + \int_t^T \gamma e^{\gamma Y_s}(\alpha + \beta|Y_s| + \frac{\gamma}{2}|Z_s|^2)ds \\
&\quad - \int_t^T \gamma e^{\gamma Y_s} Z_s dW_s - \frac{\gamma^2}{2}\int_t^T e^{\gamma Y_s}|Z_s|^2 ds \\
&= e^{\gamma\xi} + \int_t^T \gamma e^{\gamma Y_s}(\alpha + \beta|Y_s|)ds \\
&\quad - \int_t^T \gamma e^{\gamma Y_s} Z_s dW_s
\end{aligned}$$

It is clear that $\bar{Y} > 0$ and $(\bar{Y}, \bar{Z})$ satisfies the BSDE

$$\bar{Y}_t = e^{\gamma\xi} + \int_t^T \gamma\left(\alpha\bar{Y}_s + \beta\bar{Y}_s|\ln\bar{Y}_s|\right)ds - \int_t^T \bar{Z}_s dW_s.$$

Proposition 4.1 is proved. □

Proposition 4.2. *Let ξ be an $\mathcal{F}_T-$measurable random variable. Let G be defined by*

$$G(y) := a + b|y| + c|y||\ln|y||$$

(i) If $eq(e^{(a+b+2c)e^{cT}}(\xi^+ + 1)^{e^{cT}}, 0)$ has a positive solution, then $eq(\xi^+, G)$ has a positive solution.

(ii) (Y, Z) is a negative solution of $eq(-\xi^-, -G)$ if and only if $(-Y, -Z)$ is a positive solution to $eq(\xi^-, G)$. Therefore, if $eq(e^{(a+b+2c)e^{cT}}(\xi^- + 1)^{e^{cT}}, 0)$ has a positive solution, then $eq(-\xi^-, -G)$ has a negative solution.

(iii) If $|\xi|^{e^{cT}}$ is integrable, then $eq(\xi^+, G)$ has a positive solution and $eq(-\xi^-, -G)$ has a negative solution, and therefore $eq(\xi, H)$ has at least one solution (Y, Z) which belongs to $\mathcal{S}^{e^{cT}} \times \mathcal{M}^2$. Moreover, according to [8], the uniqueness holds in $\mathcal{S}^{e^{2cT}+1} \times \mathcal{M}^2$ provided that $|\xi|^{e^{2cT}+1}$ is integrable.

Proof. Let $G(y) := a + b|y| + c|y||\ln|y||$. Let Y^G be a positive solution to $eq(\xi^+, G)$. This is equivalent to say that Y^G is a positive solution to the BSDE $(a + by + cy|\ln y|)$. Applying Itô's formula to the function $u(Y_t^G) := \ln(Y_t^G + 1)$, we obtain

$$\begin{aligned}
u(Y_t^G) &= \ln(\xi^+ + 1) \\
&\quad + \int_t^T \Bigg(\left[a + bY_s^G + c\,Y_s^G|\ln(Y_s^G)|\right]\frac{1}{1+Y_s^G} \\
&\quad + \frac{1}{2}\frac{|Z_s^G|^2}{(1+Y_s^G)^2}\Bigg)ds \\
&\quad - \int_t^T \frac{1}{1+Y_s^G} Z_s^G dW_s.
\end{aligned}$$

The process $(\bar{Y}, \bar{Z}) := (\ln(1+Y^G), \frac{Z^G}{1+Y^G})$ satisfies the BSDE

$$\bar{Y}_t = \ln(\xi^+ + 1) + \int_t^T \bar{H}(s, \bar{Y}_s, \bar{Z}_s)ds - \int_t^T \bar{Z}_s\, dW_s\,,$$

where

$$\begin{aligned}
&\bar{H}(t,y,z) \\
&:= \left[a + b(e^y - 1) + c\,(e^y - 1)\,|\ln(e^y - 1)|\right]\frac{1}{e^y} + \frac{1}{2}|z|^2.
\end{aligned}$$

Since the function $x|\ln(x)| < 1$ for each x in $[0, 1]$ and strictly increasing in $[1, +\infty)$, we then have $\left(x|\ln(x)|\right)\frac{1}{1+x} \leq 1 + |\ln(x+1)|$. Hence

$$\left(a + bx + cx|\ln(x)|\right)\frac{1}{1+x} \leq a + b + c + c|\ln(x+1)|$$

It follows that

$$0 \leq \bar{H}(t,y,z) \leq a + b + c + c\,y + \frac{1}{2}|z|^2$$

According to Lemma 3.1, it is enough to show that $eq(\ln(\xi^+ + 1), a + b + c + c\,y + \frac{1}{2}|z|^2)$ has a positive solution. But from Proposition 3.1, $eq(\ln(\xi^+ + 1), a + b + c + c\,y + \frac{1}{2}|z|^2)$ has a positive solution when

$$eq(e^{(a+b+2c)e^{cT}} e^{e^{cT}\ln(\xi^+ + 1)}, 0)$$

has a positive solution, which is equivalent to say that $eq(e^{(a+b+2c)e^{cT}}(\xi^+ + 1)^{e^{cT}}, 0)$ has a positive solution. This implies that $(\xi^+ + 1)^{e^{cT}}$ is integrable. Assertions (ii) can be proved as assertion (i). Lemma 3.1 allows to establish existence of solutions of assertion (iii). □

Remark 4.1. (Uniqueness). According to assertion (iii) of Proposition 4.2, $eq(\xi, a + b|y| + c|y||\ln|y||)$ has a unique solution (Y, Z) which belongs to $\mathcal{S}^{e^{2cT}+1} \times \mathcal{M}^2$ provided that $|\xi|^{e^{2cT}+1}$ is integrable. Therefore, $eq(\xi, \alpha + \beta|y| + \frac{\gamma}{2}|z|^2)$ has a unique solution provided that $\exp\left(\gamma\xi(e^{2\beta T} + 1)\right)$ is integrable. We moreover have, $\sup_{0\leq s\leq T}\exp\left(\gamma|Y_s|(e^{2\beta_s} + 1)\right)$ is integrable. This gives a simple proof to the uniqueness of $eq(\xi, \alpha + \beta|y| + \frac{\gamma}{2}|z|^2)$ without using the convexity (in z) of the generator.

5 Appendix.

We recall the result of Essaky & Hassani ([22]) on the two barriers reflecting QBSDEs. It establishes the existence of solutions for reflected QBSDEs without assuming any integrability condition on the terminal datum. This result is used in the proof of Lemma 3.1.

Theorem 5.1. ([22], Theorem 3.2) *Let L and U be continuous processes and ξ be a $\mathcal{F}_T$–measurable random variable. Assume that*

- *for every $t \in [0,\ T]$, $L_t \leq U_t$*
- *$L_T \leq \xi \leq U_T$.*
- *there exists a continuous semimartingale which passes between the barriers L and U.*
- *The generator h is continuous in (y, z) and satisfies for every (s, ω), every $y \in [L_s(\omega), U_s(\omega)]$ and every $z \in \mathbb{R}^d$.*

$$|h(s, \omega, y, z)| \leq \eta_s(\omega) + C_s(\omega)|z|^2$$

where η and C are two $\mathcal{F}_t$–adapted processes such that $\mathbb{E}\int_0^T \eta_s ds < \infty$ and C is continuous.

Then, the following RBSDE has a maximal and a minimal solution.

$$\begin{cases} (i) & Y_t = \xi + \int_t^T h(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \\ & \quad + \int_t^T dK_s^+ - \int_t^T dK_s^- \text{ for all } t \leq T \\ (ii) & \forall\, t \leq T,\ L_t \leq Y_t \leq U_t, \\ (iii) & \int_0^T (Y_s - L_s)dK_s^+ = \int_0^T (U_s - Y_s)dK_s^- = 0, a.s., \\ (iv) & K_0^+ = K_0^- = 0, \quad K^+, K^- \text{ are continuous nondecreasing.} \\ (v) & dK^+ \perp dK^- \end{cases}$$

The following lemma allows to remove the quadratic term from $eq(\xi, \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2$

Lemma 5.1. I) *Let $f \in \mathbb{L}^1_{loc}(\mathbb{R})$ but not necessarily continuous. Then the function*

$$u_f(x) := \int_0^x \exp\left(2\int_0^z f(r)dr\right)dz$$

satisfies the differential equation $\frac{1}{2}u_f''(x) - f(x)u_f'(x) = 0$ a.e on $\mathbb{R}$, and has the following properties:

(j) u_f is a one to one function. Both u and its inverse u_f^{-1} are locally Lipschitz, that is for every $R > 0$ there exist two positive constants m_R and M_R such that, for any $|x|, |y| \leq R$,

$$m_R\,|x-y| \leq |u_f(x) - u_f(y)| \leq M_R\,|x-y|$$

(jj) Both u_f and its inverse function u_f^{-1} belong to $W^2_{1,\,loc}(\mathbb{R})$. If moreover f is continuous, then both u_f and u_f^{-1} belong to $\mathcal{C}^2(\mathbb{R})$.

II) *Set*

$$K(y) := \int_0^y \exp\left(-2\int_0^z f(r)dr\right)dz.$$

Then, the function

$$*v(x) := \int_0^x K(y)\exp\left(2\int_0^y f(r)dr\right)dy$$

satisfies the differential equation $\frac{1}{2}v''(x) - f(x)v'(x) = \frac{1}{2}$ a.e. on $\mathbb{R}$ and has the following properties:

(jjj) v and v' are positive on $\mathbb{R}_+$ and v belongs to $W^2_{1,\,loc}(\mathbb{R})$.

(jv) The map $x \longmapsto v(|x|)$ belongs to $W^2_{1,\,loc}(\mathbb{R})$, and belongs to $\mathcal{C}^2(\mathbb{R})$ if f is continuous.

III) *Set $G(z) := \int_0^z f(x)e^{-2\int_0^x f(r)dr}dx$. The function*

$$w(y) := \int_0^y G(z)e^{2\int_0^x f(r)dr}dz \tag{5.1}$$

has the following properties :

(vj) the map $x \longmapsto w(|x|)$ belongs to $W^2_{1,\,loc}$

(vjj) w satisfies the following differential equation

$$\frac{1}{2}w''(x) - f(x)w'(x) = \frac{1}{2}f(x) \quad a.e. \text{ on } \mathbb{R}.$$

Proof. I) Clearly, u_f and its inverse u_f^{-1} are continuous, one to one, strictly increasing functions and we have $\frac{1}{2}u_f''(x) - f(x)u_f'(x) = 0$ *a.e.* on $\mathbb{R}$. Since $u_f'(x) := \exp(2\int_0^x f(t)dt)$, then, for every $|x| \leq R$

$$\exp\left(-2\|f\|_{\mathbb{L}^1([-R,\ R])}\right) \leq |u_f'(x)| \leq \exp\left(2\|f\|_{\mathbb{L}^1([-R,\ R])}\right). \tag{5.2}$$

This shows that u_f and u_f^{-1} are locally Lipschitz.

We prove (jj). Using inequality (5.2), one can show that both u_f and u_f^{-1} belong to $\mathcal{C}^1$. Since the second generalized derivative u_f'' satisfies $u_f''(x) = 2f(x)u_f'(x)$ for *a.e.* x, we get that u_f'' belongs to $\mathbb{L}^1_{loc}(\mathbb{R})$. Therefore u_f belongs to $W^2_{1,\,loc}(\mathbb{R})$. Using again assertion (j), we prove that u_f^{-1} belongs to $W^2_{1,\,loc}(\mathbb{R})$.

II) Obviously v and v' are positive on $\mathbb{R}_+$ and v satisfies the differential equation $\frac{1}{2}v''(x) - f(x)v'(x) = \frac{1}{2}$ *a.e.* on $\mathbb{R}$. Since f is locally integrable on $\mathbb{R}$, one can easily check that v belongs to $W^2_{1,\,loc}(\mathbb{R})$. This proves assertions (jjj), from which we deduce assertion (jv). The proof of III) is similar. □

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