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Blow-up in coupled solutions for a four dimensional semilinear elliptic system of Liouville type

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Abstract

We consider the existence of singular limit solution for a nonlinear elliptic system of Liouville type with Navier boundary conditions. We use the nonlinear domain decomposition method. Indeed, we build an exact solution of the problem on small balls centered on singularities and far from those balls so with a suitable choice of data on the edge balls and via a nonlinear Cauchy data matching, the obtained solution is a global one.

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1 Introduction and statement of the results

It is very rare that a real life phenomenon can be satisfactorily modeled by a single partial differential equation. Usually it takes a system of coupled partial differential equations to yield a reasonable model.

Although the real world seems in a muddle, many physical and biological phenomena can be described by using nonlinear differential equations. Mathematical properties of these equations are used to explain and predict such phenomena. For this reason, nonlinear systems have always received considerable attention in both mathematics, physics and biology.

A fundamental goal in the study of non-linear initial boundary value problems involving partial differential equations is to determine whether solutions to a given equation develop a singularity. Resolving the issue of blow-up is important, in part because it can have bearing on the physical relevance and validity of the underlying model.

Consider the simple model given by:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} &= D_{u_1}\Delta^2 u_1 + \rho_{u_1} f(u_1, u_2) \\ \dfrac{\partial u_2}{\partial t} &= D_{u_2}\Delta^2 u_2 + \rho_{u_2} g(u_1, u_2) \end{cases} \tag{1.1}$$

This is a "reaction-diffusion system of the activator-inhibitor type" that appears to account for many important types of pattern formation and morphogenesis observed in development. It has been widely used to model localization processes in nature, such as cell differentiation and morphogenesis [13] and biological pattern formation [17]. We explain the terms appearing in the system: D_{u_1} and D_{u_2} are the diffusion constants for the activator u_1 and the inhibitor u_2, respectively, while $\dfrac{\partial u_1}{\partial t}$ and $\dfrac{\partial u_2}{\partial t}$ describe the change of activator and inhibitor concentrations respectively, per time unit. The coefficients ρ_{u_1} and ρ_{u_2} are the corresponding cross-reaction coefficients, and the functions f and g express the reaction terms between u_1 and u_2.

Morphogenetic movements are characterized, at the tissue level, by a succession of cellular events, accurately located temporally and spatially. This phenomenon is modeled by partial differential equations systems that describe its global control and evolution, providing an explanation of how morphogenetic movements can be obtained from the integration of all the local dynamics of the cells. Our approach requires identification of the purported morphogens, measurement of their spatiotemporal concentrations and kinetics, and demonstration by knockouts or other genetic manipulations that they are essential components of the observed pattern formation.

The main purpose of this article is to present a rather efficient method to solve such a singular system of the reaction-diffusion system of the activator-inhibitor, also known as the time-independent Gierer-Meinhardt system. In particular, we take the functions f and g to be exponential functions and we consider the following elliptic system:

$$\begin{cases} \Delta^2 u_1 &= \rho^4 e^{u_1+\gamma_1 u_2} \quad \text{in } \Omega \\ \Delta^2 u_2 &= \rho^4 e^{u_2+\gamma_2 u_1} \quad \text{in } \Omega \\ \Delta u_i &= u_i = 0; \quad i=1,2 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^4$ be a regular bounded open domain in $\mathbb{R}^4$, γ_i and ρ are constants. We assume that $\gamma_1, \gamma_2 \in (0,1)$. We are then interested in the study of the existence of solutions with singular limits as the parameter ρ goes to 0.

1.1 A tour of the literature

Before stating our main result, summarized in Theorem 1.2 and Theorem 1.3 below, we survey some of main known important results in the field.

The system (1.2) is a natural generalization of the equation

$$\Delta^2 u = 6e^{4u} \quad \text{in } \mathbb{R}^4. \tag{1.3}$$

Equation (1.3) is invariant under translation, rotation, dilatation in the Euclidean space and the Kelvin transforms. In [14], Lin proved the following important classification result of finite-mass solutions of equation (1.3).

Theorem *[14] Let u be a solution of (1.3), satisfying the finite-mass condition*

$$\int_{\mathbb{R}^4} e^{4u} dx < \infty, \tag{1.4}$$

and $|u(x)| = o(|x|^2)$ at ∞. Then there exists some point $x^0 \in \mathbb{R}^4$ such that u is radially symmetric about x^0 and

$$u(x) = \ln\left(\frac{2\lambda}{1+\lambda^2|x-x^0|^2}\right).$$

This result is decisive for solving completely (1.3) under (1.4), because it reduces the problem to a simple ODE problem.

In [21], Wei and Ye constructed a non radial solution of Liouville equation (1.3) under (1.4) with the following asymptotic behavior:

$$u(x) = -\sum_{j=1}^{k} a_j (x_j - x_j^0)^2 - \alpha \ln|x| + c_0 + o(1), \ |x| > 1 \quad \text{and} \int_{\mathbb{R}^4} e^{4u(x)} dx = \frac{4\pi^2\alpha}{3},$$

for each fixed $x^0 \in \mathbb{R}^4, 1 \le k \le 4, \alpha \in (1 - \frac{k}{4}, 2)$ and $a_j > 0$ for $1 \le j \le k$.

In dimension 4, other authors were motivated by similar problems [2, 4, 9, 10]. Wei in [20] studied the behavior of solutions to the following nonlinear eigenvalue problem for the biharmonic operator Δ^2 in $\mathbb{R}^4$. More precisely, consider the following problem

$$\begin{cases} \Delta^2 u &= \lambda\, f(u) \quad \text{in} \quad \Omega \\ \Delta u &= u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

When $f(u) = e^u$, we can see that (1.5) is issued from the conformal geometry by prescribing the so-called Q-curvature on 4-dimensional Riemannian manifolds.
For more details and background material we refer to [1, 7, 16] and the references therein.

Before stating the result of [20], we will introduce some notations. Let $G(x,x')$ defined over $\Omega \times \Omega$, be the Green function associated to the bi-laplacian operator with a Navier boundary conditions, which is the solution of

$$\begin{cases} \Delta_x^2 G(x,x') &= 64\,\pi^2\, \delta_{x=x'} \quad \text{in } \Omega \\ \Delta_x G(x,x') &= G(x,x') = 0 \quad \text{on } \partial\Omega \end{cases}$$

and denote by $H(x,x') = G(x,x') + 8\ln|x-x'|$ its smooth part. Consider now the functional E defined on the set $\{(x^1,...,x^m) \in \Omega^m;\ x^i \neq x^j \text{ for all } 1 \leq i \neq j \leq m\}$ by

$$E(x^1,...,x^m) = \sum_{j=1}^{m} H(x^j,x^j) + \sum_{j\neq l} G(x^j,x^l)$$

and denote by u^* the function defined on $\Omega \setminus \{x^1, \ldots, x^m\}$ by

$$u^*(x) = \sum_{j=1}^{m} G(x,x^j). \tag{1.6}$$

In [20], the author proved the following result:

Theorem *[20] Let Ω be a smooth bounded domain in $\mathbb{R}^4$ and f a smooth nonnegative increasing function such that*

$$e^{-u} f(u) \quad \textit{tends to}\ 1, \ \textit{as}\ u \longrightarrow +\infty. \tag{1.7}$$

For u_λ solution of (1.5), denote by $\Sigma_\lambda = \lambda \int_\Omega f(u_\lambda)dx$. Then there are only three possibilities:

i) the $\{\Sigma_\lambda\}$ accumulate to 0. Then $\|u_\lambda\|_{L^\infty(\Omega)} \longrightarrow 0$ as $\lambda \longrightarrow 0$.

ii) the $\{\Sigma_\lambda\}$ accumulate to $+\infty$. Then $u_\lambda \longrightarrow +\infty$ as $\lambda \longrightarrow 0$.

iii) the $\{\Sigma_\lambda\}$ accumulate to $64\ \pi^2\ m$, for some positive integer m. Then the limiting function $u^* = \lim_{\lambda\to 0} u_\lambda$ has m blow-up points, $\{x^1,...,x^m\}$, where $u_\lambda(x^i) \longrightarrow +\infty$ as $\lambda \to 0$.

Moreover, $(x^1,...,x^m)$ is a critical point of E.

In [4], the authors considered the following problem

$$\begin{cases} \Delta^2 u &= \rho^4\, e^u \quad \text{in} \quad \Omega \\ u &= \Delta u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.8}$$

They constructed a non-minimal solution with singular limit as the parameter ρ goes to 0. Their results can be stated as follows:

Theorem *[4] Let Ω be a smooth open subset of $\mathbb{R}^4$ and $x^1,...,x^m \in \Omega$ be given points. Assume that $(x^1,\ldots,x^m)$ is a nondegenerate critical point of E, then there exist $\rho_0 > 0$ and $(u_\rho)_{\rho\in(0,\rho_0)}$ a one parameter family of solutions of (1.8), such that*

$$\lim_{\rho\to 0} u_\rho = u^* \ \ \textit{in}\ \ \mathcal{C}^{4,\alpha}_{loc}(\Omega - \{x^1,\ldots,x^m\}).$$

In dimension 2, if we consider the corresponding Dirichlet problem on a bounded domain Ω in $\mathbb{R}^2$,

$$-\Delta u = \rho^2 e^u \ \text{ in }\ \Omega \subset \mathbb{R}^2, \quad u = 0 \ \text{ on }\ \partial\Omega. \tag{1.9}$$

It is well known that as the parameter ρ tends to 0, non-minimal solutions exist and they have singular limits. In [5], Baraket and Pacard proved

Theorem *[5] Let Ω be a smooth open subset of $\mathbb{R}^2$ and $z^1,...,z^m \in \Omega$. Assume that $(z^1,\ldots,z^m)$ is a nondegenerate critical point of the function*

$$F : (z^1,...,z^m) \in \mathbb{C}^m \longrightarrow \sum_j h(z^j,z^j) + \sum_{j\neq l} g(z^j,z^l)$$

then there exist $\rho_0 > 0$ and $(u_\rho)_{\rho\in(0,\rho_0)}$ a one parameter family of solutions of (1.9), such that

$$\lim_{\rho\to 0} u_\rho = u^* := \sum_{j=1}^{m} g(.,z^j) \ \ \textit{in}\ \ \mathcal{C}^{2,\alpha}_{loc}(\Omega - \{z^1,\ldots,z^m\}).$$

Here g is the Green's function defined as the solution of

$$\begin{cases} -\Delta_z g(z,z') = 8\pi\delta_{z=z'} & \text{in } \Omega \\ g(z,z') = 0 & \text{on } \partial\Omega, \end{cases}$$

and h is its smooth part defined by

$$h(z,z') = g(z,z') + 4\ln|z-z'|.$$

Some generalizations can be found in [3, 6, 12].

Recently, the authors in [19] considered the following problem

$$\begin{cases} -\Delta u_1 &=& \rho^2 e^{u_1+\gamma_1 u_2} & \text{in} & \Omega\subset\mathbb{R}^2 \\ -\Delta u_2 &=& \rho^2 e^{u_2+\gamma_2 u_1} & \text{in} & \Omega\subset\mathbb{R}^2 \\ u_1 &=& u_2 = 0 & \text{on} & \partial\Omega. \end{cases}$$

They proved the existence of singular limit solutions with blow-up on different points as $\rho\to 0$ using the nonlinear domain decomposition method.

This paper extends this result in dimension four.

Theorem 1.1. *Let Ω be a regular domain of $\mathbb{R}^4$ and $x^1, \dots, x^m \in \Omega$ be given disjoint points. We suppose that $(x^1, .., x^p, .., x^m)$ is a nondegenerate critical point of the function*

$$\mathcal{F}(x^1, \dots, x^m) = \frac{1}{2\gamma_1}\sum_{i=1}^{p} H(x^i,x^i) + \frac{1}{2\gamma_2}\sum_{j=p+1}^{n} H(x^j,x^j) + \sum_{i=1,\, j=p+1}^{i=p,\, j=m} G(x^i,x^j).$$

Then there exist γ_0 in $(0,1)$ such that for $\gamma_1,\ \gamma_2 \in (0,\gamma_0)$, there exist $\rho_0>0$ and $(u_i^\rho)_{0<\rho\le\rho_0}$ a one parameter family of solutions of (1.2), such that

$$\begin{cases} \lim\limits_{\rho\longrightarrow 0} u_1^\rho &=& \sum\limits_{i=1}^{p} G(x^i,.) & \text{in } \mathcal{C}^{4,\alpha}_{loc}(\Omega-\{x^1,...,x^p\}) \\ \lim\limits_{\rho\longrightarrow 0} u_2^\rho &=& \sum\limits_{i=p+1}^{m} G(x^i,.) & \text{in } \mathcal{C}^{4,\alpha}_{loc}(\Omega-\{x^{p+1},...,x^m\}). \end{cases}$$

To facilitate the presentation, we will look at the special case where we have only two singular points.

Theorem 1.2. *Let Ω be a regular domain of $\mathbb{R}^4$ and $x^1,\ x^2\in\Omega$ be given disjoint points. We suppose that (x^1,x^2) is a nondegenerate critical point of the function*

$$\mathcal{F}(x^1,x^2) = \frac{1}{2\gamma_1}H(x^1,x^1) + \frac{1}{2\gamma_2}H(x^2,x^2) + G(x^1,x^2).$$

Then there exist γ_0 in $(0,1)$ such that for $\gamma_1,\ \gamma_2 \in (0,\gamma_0)$, there exist $\rho_0>0$ and $(u_i^\rho)_{0<\rho\le\rho_0}$ a one parameter family of solutions of (1.2), such that

$$\lim_{\rho\longrightarrow 0} u_i^\rho = G(x^i,.) \quad \text{in } \mathcal{C}^{4,\alpha}_{loc}(\Omega-\{x^i\}) \qquad \text{for } i\in\{1,2\}.$$

Following similar arguments as in the proof of Theorem 1.1, we prove also

Theorem 1.3. *Let Ω be a regular domain of $\mathbb{R}^4$ and $x^1,...,x^p$ be given points in Ω. We suppose that $(x^1,...,x^p)$ is a nondegenerate critical point of the function*

$$\mathcal{F}(x^1,...,x^p) = \sum_{j=1}^{p} H(x^j,x^j).$$

Then there exist γ_0 in $(0,1)$ such that for $\gamma_1,\ \gamma_2 \in (0,\gamma_0)$, there exist $\rho_0>0$ and $(u_i^\rho)_{0<\rho\le\rho_0}$ a one parameter family of solutions of (1.2), such that

$$\lim_{\rho\longrightarrow 0} u_1^\rho = \sum_{j=1}^{p} G(x^j,.) \quad \text{in } \mathcal{C}^{4,\alpha}_{loc}(\Omega-\{x^1,...,x^p\}) \quad \text{and} \quad \lim_{\rho\longrightarrow 0} u_2^\rho = 0 \quad \text{in } \mathcal{C}^{4,\alpha}_{loc}(\Omega).$$

We recall that Theorem 1.1 is a generalisation of Theorem 1.2. We will only prove Theorem 1.2, where we have only two singular points.

We now briefly describe the plan of the paper : we start by proving Theorem 1.2, motivated by the techniques of Baraket and al [4]. We introduce and recall the linearized operators and the harmonic extensions which are crucial for subsequent sections, and we discuss rotationally symmetric solutions of (1.2). We also recall some known results about the analysis of the Bi-Laplace operator in weighted spaces. In section 2.2, we study a nonlinear interior problem where the existence of an infinite dimensional family of solutions of (1.2), which are defined on a large ball and which are close to the rotationally symmetric solution, is proven. Then, we prove the existence of an infinite dimensional family of solutions of (1.2) which are defined on Ω with small ball removed. Finally, we show how elements of these infinite dimensional families can be connected to produce solutions of (1.2) described in Theorem 1.2. In fact, we patch these pieces together via a nonlinear version of the Cauchy data matching. We conclude the paper by giving a simple sketch of the proof of Theorem 1.3.

2 Proof of Theorem 1.2

2.1 Construction of the approximate solution

We denote by ε the smallest positive parameter satisfying $\rho^4 = \dfrac{384\,\varepsilon^4}{(1+\varepsilon^2)^4}$.

Let

$$u_\varepsilon(x) := 4\,\ln \frac{1+\varepsilon^2}{\varepsilon^2+|x|^2} \tag{2.1}$$

which is a solution of

$$\Delta^2 u = \rho^4 e^u \quad \text{in} \quad \mathbb{R}^4. \tag{2.2}$$

Hence for all $\tau > 0$ the function

$$u_{\varepsilon,\tau}(x) := 4\,\ln \frac{\tau(1+\varepsilon^2)}{\varepsilon^2+|\tau x|^2} \tag{2.3}$$

is also solution to (2.2).

2.1.1 A linearized operator

First we introduce some definitions and notations.

Definition 2.1. *Given $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\mu \in \mathbb{R}$ and $|x| = r$, let $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ be the space of functions in $\mathcal{C}^{k,\alpha}_{loc}(\mathbb{R}^4)$ for which the following norm*

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)} = \|u\|_{\mathcal{C}^{k,\alpha}(\bar B_1(0))} + \sup_{r\geq 1}\left((1+r^2)^{-\frac{\mu}{2}}\,\|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar B_1(0)-B_{\frac12}(0))}\right)$$

is finite. Similarly, for given $\bar r \geq 1$, let $\mathcal{C}^{k,\alpha}_\mu(B_{\bar r}(0))$ be the space of functions in $\mathcal{C}^{k,\alpha}(B_{\bar r}(0))$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(B_{\bar r(0)})} = \|u\|_{\mathcal{C}^{k,\alpha}(B_1(0))} + \sup_{1\leq r\leq \bar r}\left(r^{-\mu}\,\|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar B_1(0)-B_{\frac12}(0))}\right)$$

is finite. Finally, set $B_1^*(x^i) = B_1(x^i) - \{x^i\}$, let $\mathcal{C}^{k,\alpha}_\mu(\bar B_1^*(0))$ be the space of functions in $\mathcal{C}^{k,\alpha}_{loc}(\bar B_1^*(0))$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(\bar B_1^*(0))} = \sup_{r\leq \frac12}\left(r^{-\mu}\,\|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar B_2(0)-B_1(0))}\right)$$

is finite.

We define the linear elliptic operator $\mathbb{L}$ by $\mathbb{L} := \Delta^2 - \frac{384}{(1+r^2)^4}$, which is the linearized operator of $\Delta^2 u - \rho^4 e^u = 0$ about the radial symmetric solution $u_{\varepsilon=1,\tau=1}$ defined by (2.3). When $k \geq 2$, we let $[\mathcal{C}^{k,\alpha}_\mu(\bar\Omega)]_0$ to be the subspace of functions $w \in \mathcal{C}^{k,\alpha}_\mu(\bar\Omega)$ satisfying $\Delta w = w = 0$ on $\partial\Omega$.

For all $\varepsilon, \tau > 0$ and $\gamma_1, \gamma_2 \in (0,1)$, we define

$$r_\varepsilon := \max(\varepsilon^{\frac12},\, \varepsilon^{(1-\gamma_1)},\, \varepsilon^{(1-\gamma_2)}) \qquad \text{and} \qquad R_\varepsilon := \tau\,\frac{r_\varepsilon}{\varepsilon}. \tag{2.4}$$

Proposition 2.1. *[4] All bounded solution of* $\mathbb{L}w = 0$ *on* $\mathbb{R}^4$ *are linear combination of*

$$\phi_0(x) = 4\,\frac{1-|x|^2}{1+|x|^2} \quad \text{and} \quad \phi_i(x) = \frac{8\ x_i}{1+|x|^2} \quad \text{for}\ \ i = 1, ..., 4.$$

Moreover, for $\mu > 1$, $\mu \notin \mathbb{Z}$, *the operator* $\mathbb{L} : \mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4) \longrightarrow \mathcal{C}^{0,\alpha}_{\mu-4}(\mathbb{R}^4)$ *is surjective.*

In the following, we denote by $\mathcal{G}_\mu$ to be a right inverse of $\mathbb{L}$. Similarly, using the fact that any bounded bi-harmonic solution on $\mathbb{R}^4$ is constant, we claim

Proposition 2.2. *Let* $\delta > 0$, $\delta \notin \mathbb{Z}$ *then* Δ^2 *is surjective from* $\mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)$ *to* $\mathcal{C}^{0,\alpha}_{\delta-4}(\mathbb{R}^4)$.

We denote by $\mathcal{K}_\delta : \mathcal{C}^{0,\alpha}_{\delta-4}(\mathbb{R}^4) \longrightarrow \mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)$ a right inverse of Δ^2 for $\delta > 0$, $\delta \notin \mathbb{Z}$.

Finally, we consider punctured domains. Given $\tilde{x}^1 \neq \tilde{x}^2 \in \Omega$, we define $\tilde{\mathbf{x}} := (\tilde{x}^1, \tilde{x}^2)$ and $\bar{\Omega}^*(\tilde{\mathbf{x}}) := \bar{\Omega} - \{\tilde{x}^1, \tilde{x}^2\}$. Let $r_0 > 0$ be small such that $\overline{B}_{r_0}(\tilde{x}^i)$ are disjoint and included in Ω. For all $r \in (0, r_0)$, we define

$$\bar{\Omega}_r(\tilde{\mathbf{x}}) := \bar{\Omega} - \cup_{i=1}^2 B_r(\tilde{x}^i).$$

Definition 2.2. *Let* $k \in \mathbb{R}, \alpha \in (0,1)$ *and* $\nu \in \mathbb{R}$, *let* $\mathcal{C}^{k,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}})) = \mathcal{C}^{k,\alpha}_{loc}(\bar{\Omega}^*(\tilde{\mathbf{x}})) \bigcap_{i=1,2} \mathcal{C}^{k,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{x}^i))$ *endowed the following norm*

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{\Omega}_{r_0/2}(\tilde{\mathbf{x}}))} + \sum_{i=1}^{2} \sup_{0<r\le r_0/2} \Big(r^{-\nu} \|w(\tilde{x}^i + r.)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2(0)-B_1(0))} \Big).$$

Furthermore, for $k \geq 2$, *let* $\left[\mathcal{C}^{k,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))\right]_0$ *to be the set of* $w \in \mathcal{C}^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$ *satisfying* $\Delta w = w = 0$ *on* $\partial\Omega$.

We recall the following result:

Proposition 2.3. *[10] Let* $\nu < 0$, $\nu \notin \mathbb{Z}$ *then* Δ^2 *is surjective from* $\left[\mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))\right]_0$ *to* $\mathcal{C}^{0,\alpha}_{\nu-4}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$

We denote by $\tilde{\mathcal{G}}_\nu : \mathcal{C}^{0,\alpha}_{\nu-4}(\bar{\Omega}^*(\tilde{\mathbf{x}})) \longrightarrow \left[\mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))\right]_0$ a right inverse of Δ^2 for $\nu < 0$, $\nu \notin \mathbb{Z}$.

2.1.2 Ansatz and first estimates

For all $\sigma \geq 1$, we denote by $\xi_{\mu,\sigma} : \mathcal{C}^{0,\alpha}_{\mu}(\bar{B}_\sigma(0)) \longrightarrow \mathcal{C}^{0,\alpha}_{\mu}(\mathbb{R}^4)$ the extension operator defined by

$$\begin{cases} \xi_{\mu,\sigma}(f) & \equiv f & \text{for } |x| \le \sigma \\ \xi_{\mu,\sigma}(f)(x) & = \chi\Big(\dfrac{|x|}{\sigma}\Big)\, f\Big(\sigma \dfrac{x}{|x|}\Big) & \text{for } |x| \ge \sigma. \end{cases} \tag{2.5}$$

Here χ is a cut-off function over $\mathbb{R}_+$, which is equal to 1 for $t \leq 1$ and equal to 0 for $t \geq 2$.
It is easy to check that there exists a constant $c = \bar{c}(\mu) > 0$, independent of σ such that

$$\|\xi_{\mu,\sigma}(w)\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathbb{R}^4)} \ \le\ \bar{c}\ \|w\|_{\mathcal{C}^{0,\alpha}_{\mu}(\bar{B}_\sigma(0))}. \tag{2.6}$$

Now we define an ansatz for solution of (1.2).

$$\widetilde{u}_1(x) = \begin{cases} u_{\varepsilon,\tau}(x - x^1) - \gamma_1 G(x, x^2) & x \in B_{r_\varepsilon}(x^1) \\ G(x, x^1) & x \in \Omega \backslash B_{r_\varepsilon}(x^1) \end{cases} \tag{2.7}$$

and

$$\widetilde{u}_2(x) = \begin{cases} u_{\varepsilon,\tau}(x - x^2) - \gamma_2 G(x, x^1) & x \in B_{r_\varepsilon}(x^2) \\ G(x, x^2) & x \in \Omega \backslash B_{r_\varepsilon}(x^2). \end{cases} \tag{2.8}$$

Therefore, in $B_{r_\varepsilon}(x^1)$, there holds

$$\begin{cases} \Delta^2 \widetilde{u}_1 - \rho^4\ e^{\widetilde{u}_1 + \gamma_1 \widetilde{u}_2} & = & 0 \\ \Delta^2 \widetilde{u}_2 - \rho^4\ e^{\widetilde{u}_2 + \gamma_2 \widetilde{u}_1} & = & \dfrac{-384\ \varepsilon^4\ \tau^{4\gamma_2}}{(1+\varepsilon^2)^{4(1-\gamma_2)}\ (\varepsilon^2 + \tau^2\ |x - x^1|^2)^{4\gamma_2}}\ e^{(1-\gamma_1\gamma_2)\ G(x,x^2)}\ . \end{cases} \tag{2.9}$$

Then for $r = |x - x^1|$ and $\delta < 8(1-\gamma_2)$, we have

$$\|\Delta^2 \widetilde{u}_2 - \rho^4 \, e^{\widetilde{u}_2 + \gamma_2 \widetilde{u}_1}\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{R_\varepsilon}(0))} = \sup_{r<R_\varepsilon} \frac{384 \, \varepsilon^{8-8\gamma_2-\delta} \, r^{4-\delta}}{(1+r^2)^{4\gamma_2}} \, e^{(1-\gamma_1\gamma_2) \, G(\varepsilon\frac{x}{\tau}, x^2)}$$

$$= \sup_{r<R_\varepsilon} 384 \, \varepsilon^{8-8\gamma_2-\delta} \, S(r) \, e^{(1-\gamma_1\gamma_2) \, G(\varepsilon\frac{x}{\tau}, x^2)},$$

where $S(r) = \dfrac{r^{4-\delta}}{(1+r^2)^{4\gamma_2}}$. If $4-\delta-8\gamma_2 \le 0$, then S is bounded on $\mathbb{R}_+$, hence

$$\left\|\Delta^2 \widetilde{u}_2 - \rho^4 \, e^{\widetilde{u}_2 + \gamma_2 \widetilde{u}_1}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{R_\varepsilon}(0))} \le C \varepsilon^{8-8\gamma_2-\delta}.$$

If $4-\delta-8\gamma_2 > 0$, $\sup_{[0,\frac{r_\varepsilon}{\varepsilon}[} S(r) = S(\frac{r_\varepsilon}{\varepsilon})$, then

$$\left\|\Delta^2 \widetilde{u}_2 - \rho^4 e^{\widetilde{u}_2 + \gamma_2 \widetilde{u}_1}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{R_\varepsilon}(0))} \le C r_\varepsilon^2, \quad \text{as} \quad \varepsilon^{8-8\gamma_2-\delta} \; S(\frac{r_\varepsilon}{\varepsilon}) \le C \; r_\varepsilon^2 \; \varepsilon^2. \tag{2.10}$$

2.1.3 Bi-harmonic extensions

Next, we will study the properties of interior and exterior bi-harmonic extensions. Given $(\varphi, \psi), (\tilde{\varphi}, \tilde{\psi}) \in \mathcal{C}^{4,\alpha}(S^3) \times \mathcal{C}^{2,\alpha}(S^3)$, we define respectively $H^{int} = H^{int}(\varphi, \psi \, ; \cdot) = H^{int}_{\varphi,\psi}$ and $H^{ext} = H^{ext}(\tilde{\varphi}, \tilde{\psi} \, ; \cdot) = H^{ext}_{\tilde{\varphi},\tilde{\psi}}$ to be the solutions

$$\begin{cases} \Delta^2 \, H^{int} &= 0 \quad \text{in} \quad B_1(0) \\ H^{int} &= \varphi \quad \text{on} \quad \partial B_1(0) \\ \Delta H^{int} &= \psi \quad \text{on} \quad \partial B_1(0) \end{cases} \tag{2.11}$$

and

$$\begin{cases} \Delta^2 \, H^{ext} &= 0 \quad \text{in} \quad \mathbb{R}^4 - B_1(0) \\ H^{ext} &= \tilde{\varphi} \quad \text{on} \quad \partial B_1(0) \\ \Delta H^{ext} &= \tilde{\psi} \quad \text{on} \quad \partial B_1(0) \end{cases} \tag{2.12}$$

which decay at infinity.

We will also use

Definition 2.3. *Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the space $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1(0))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{loc}(\mathbb{R}^4 - B_1(0))$ for which the norm below is finite $\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1(0))} = \sup_{r \ge 1} \left(r^{-\nu} \, \|w(r \, \cdot)\|_{\mathcal{C}^{k,\alpha}_\nu(\bar{B}_2(0) - B_1(0))} \right)$*

We denote by $e_1, ..., e_4$ the coordinate functions on S^3.

Lemma 2.1. *[2] Assume that*

$$\int_{S^3} (8\varphi - \psi) \, dv_{S^3} = 0 \quad \textit{and} \quad \int_{S^3} (12\varphi - \psi) \, e_\ell \, dv_{S^3} = 0 \quad \textit{for} \quad \ell = 1, ..., 4 \tag{2.13}$$

Then there exists $c > 0$ such that

$$\|H^{int}_{\varphi,\psi}\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}^*_1(0))} \le c \Big(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)} \Big).$$

Similarly, there exists $c > 0$ such that if

$$\int_{S^3} \tilde{\psi} \, dv_{S^3} = 0, \tag{2.14}$$

then

$$\|H^{ext}_{\tilde{\varphi},\tilde{\psi}}\|_{\mathcal{C}^{4,\alpha}_{-1}(\mathbb{R}^4 - B_1(0))} \le c \Big(\|\tilde{\varphi}\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\tilde{\psi}\|_{\mathcal{C}^{2,\alpha}(S^3)} \Big).$$

If $F \subset L^2(S^3)$ be a subspace S^3, we denote $F^\perp$ to be the subspace of F which are $L^2(S^3)$-orthogonal to the functions $1, e_1, ..., e_4$. We will need the following result :

Lemma 2.2. *[2] The following mapping is an isomorphism*

$$\begin{array}{rccc}\mathcal{P}: & \mathcal{C}^{4,\alpha}(S^3)^\perp \times \mathcal{C}^{2,\alpha}(S^3)^\perp & \longrightarrow & \mathcal{C}^{3,\alpha}(S^3)^\perp \times \mathcal{C}^{1,\alpha}(S^3)^\perp \\ & (\varphi,\psi) & \longmapsto & \Big(\partial_r(H^{int}_{\varphi,\psi} - H^{ext}_{\varphi,\psi}), \partial_r(\Delta H^{int}_{\varphi,\psi} - \Delta H^{ext}_{\varphi,\psi})\Big)\end{array}$$

2.2 The nonlinear interior problem

Here we are interested to study the system in $B_{r_\varepsilon}(x^1)$

$$\left\{\begin{array}{rcl}\Delta^2 u_1 &=& \rho^4 e^{u_1+\gamma_1 u_2}\\ \Delta^2 u_2 &=& \rho^4 e^{u_2+\gamma_2 u_1}.\end{array}\right. \tag{2.15}$$

Using the following transformation

$$\left\{\begin{array}{rcl} v_1(x) &=& u_1(\frac{\varepsilon}{\tau}x) + 8\ \ln\varepsilon - 4\ \ln(\frac{\tau(1+\varepsilon^2)}{2})\\ v_2(x) &=& u_2(\frac{\varepsilon}{\tau}x),\end{array}\right. \tag{2.16}$$

the previous system can be written, in $B_{R_\varepsilon}(x^1)$, as

$$\left\{\begin{array}{rcl}\Delta^2 v_1 &=& 24\ e^{v_1+\gamma_1 v_2}\\ \Delta^2 v_2 &=& 24\ \dfrac{2^{4(1-\gamma_2)}\ \varepsilon^{8(1-\gamma_2)}}{\tau^{4(1-\gamma_2)}\ (1+\varepsilon^2)^{4(1-\gamma_2)}}\ e^{v_2+\gamma_2 v_1}.\end{array}\right.$$

Here $\tau > 0$ is a constant which will be fixed later.

Given $\varphi^1 := (\varphi^1_1, \varphi^1_2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^1 := (\psi^1_1, \psi^1_2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ^1_1, ψ^1_1) and (φ^1_2, ψ^1_2) are satisfying (2.13). We denote by $\bar u = u_{\varepsilon=1,\tau=1}$, we write for $x \in B_{R_\varepsilon}(x^1)$ the following system

$$\left\{\begin{array}{rcl} v_1(x) &=& \bar u(x-x^1) + h^1_1(x) + H^{int}(\varphi^1_1, \psi^1_1, \frac{x-x^1}{R_\varepsilon}) - \gamma_1\, G(\frac{\varepsilon x}{\tau},\ x^2)\\ v_2(x) &=& G(\frac{\varepsilon x}{\tau}, x^2) + h^1_2(x) + H^{int}(\varphi^1_2, \psi^1_2, \frac{x-x^1}{R_\varepsilon}).\end{array}\right.$$

Using the fact that H^{int} is bi-harmonic and that $e^{\bar u(x-x^1)} = \dfrac{16}{(1+|x-x^1|^2)^4}$, we see that this amounts to solve the system

$$\left\{\begin{array}{rcl} \mathbb{L}h^1_1(x) &=& \dfrac{384}{(1+|x-x^1|^2)^4}\Big(e^{h^1_1(x)+H^{int}_{\varphi^1_1,\psi^1_1}(\frac{x-x^1}{R_\varepsilon})+\gamma_1\,(h^1_2(x)+H^{int}_{\varphi^1_2,\psi^1_2}(\frac{x-x^1}{R_\varepsilon}))} - h^1_1(x) - 1\Big)\\ \Delta^2 h^1_2(x) &=& \dfrac{384 C_\varepsilon\, \varepsilon^{8(1-\gamma_2)}}{(1+|x-x^1|^2)^{4\gamma_2}}\ e^{(1-\gamma_1\gamma_2)G(\frac{\varepsilon}{\tau}x,x^2)+h^1_2(x)+H^{int}_{\varphi^1_2,\psi^1_2}(\frac{x-x^1}{R_\varepsilon})+\gamma_2(h^1_1(x)+H^{int}_{\varphi^1_1,\psi^1_1}(\frac{x-x^1}{R_\varepsilon}))}\end{array}\right. \tag{2.17}$$

where $C_\varepsilon = \left(\dfrac{1}{\tau\ (1+\varepsilon^2)}\right)^{4(1-\gamma_2)}$.

Fix $\mu \in (1,\ 2)$ and $\delta \in (0, \min\{1,\ 8(1-\gamma_1),\ 8(1-\gamma_2)\})$. To find a solution of (2.17), it is enough to find a fixed point (h^1_1, h^1_2) in a small ball of $\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ solutions of

$$\left\{\begin{array}{rcl} h^1_1 &=& \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon} \circ \mathcal{T}_1(h^1_1, h^1_2)\\ h^1_2 &=& \mathcal{K}_\delta \circ \xi_{\delta,R_\varepsilon} \circ \mathcal{T}_2(h^1_1, h^1_2).\end{array}\right. \tag{2.18}$$

Here $\xi_{\sigma,R_\varepsilon}$ is defined in (2.5), $\mathcal{K}_\delta$, $\mathcal{G}_\mu$ are defined after Propositions 2.2, 2.3 and

$$\left\{\begin{array}{rcl} \mathcal{T}_1(h^1_1, h^1_2)(x) &=& \dfrac{384}{(1+|x-x^1|^2)^4}\Big(e^{h^1_1(x)+H^{int}_{\varphi^1_1,\psi^1_1}(\frac{x-x^1}{R_\varepsilon})+\gamma_1\,(h^1_2(x)+H^{int}_{\varphi^1_2,\psi^1_2}(\frac{x-x^1}{R_\varepsilon}))} - h^1_1(x) - 1\Big)\\ \mathcal{T}_2(h^1_1, h^1_2)(x) &=& \dfrac{384\ C_\varepsilon\ \varepsilon^{8(1-\gamma_2)}}{(1+|x-x^1|^2)^{4\gamma_2}}\ e^{(1-\gamma_1\gamma_2)\ G(\frac{\varepsilon}{\tau}x,x^2)\ +(h^1_2(x)+H^{int}_{\varphi^1_2,\psi^1_2}(\frac{x-x^1}{R_\varepsilon}))\ +\gamma_2(h^1_1(x)+H^{int}_{\varphi^1_1,\psi^1_1}(\frac{x-x^1}{R_\varepsilon})}.\end{array}\right. \tag{2.19}$$

We denote by $\mathcal{N}(=\mathcal{N}_{\varepsilon,\tau,\varphi_j^1,\psi_j^1})$ and by $\mathcal{M}(=\mathcal{M}_{\varepsilon,\tau,\varphi_j^1,\psi_j^1})$ the nonlinear operators appearing on the right hand side of the two equations in (2.18).

Given $\kappa>0$ (whose value will be fixed later on), we further assume that the functions φ_j^1 and ψ_j^1 satisfy

$$\|\varphi_j^1\|_{\mathcal{C}^{4,\alpha}(S^3)} \le \kappa\, r_\varepsilon^2 \quad \text{and} \quad \|\psi_j^1\|_{\mathcal{C}^{2,\alpha}(S^3)} \le \kappa\, r_\varepsilon^2, \;\; \text{for } j=1,\,2. \tag{2.20}$$

Then we have the following result:

Lemma 2.3. *Let $\varphi^1 := (\varphi_1^1,\varphi_2^1)\in(\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^1 := (\psi_1^1,\psi_2^1)\in(\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ_1^1,ψ_1^1) and (φ_2^1,ψ_2^1) are satisfying (2.13) and (2.20). Given $\kappa>0$, there exists $\varepsilon_\kappa>0$, $c_\kappa>0$ and γ_0 in $(0,1)$, such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\gamma_1\in(0,\gamma_0)$, $\mu\in(1,2)$ and $\delta\in(0,\min\{1,\,8(1-\gamma_1),\,8(1-\gamma_2)\})$*

$$\|\mathcal{N}(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \;\le\; c_\kappa\, r_\varepsilon^2, \qquad \|\mathcal{M}(0,0)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \;\le\; c_\kappa\, r_\varepsilon^2,$$

$$\|\mathcal{N}(h_1^1,h_2^1)-\mathcal{N}(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \;\le\; c_\kappa\, r_\varepsilon^2\,\|h_1^1-k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \;+\; c_\kappa\gamma_1\|h_2^1-k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}$$

and

$$\|\mathcal{M}(h_1^1,h_2^1)-\mathcal{M}(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \;\le\; c_\kappa\, r_\varepsilon^2\,\|(h_1^1,h_2^1)-(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}$$

provided $(h_1^1,h_2^1),(k_1^1,k_2^1)\in\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ satisfying

$$\|(h_1^1,h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \;\le\; 2\;c_\kappa\, r_\varepsilon^2, \qquad \|(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \;\le\; 2\;c_\kappa\, r_\varepsilon^2. \tag{2.21}$$

Proof. The first estimate follows from the result of Lemma 2.1 together with the assumption on the norms of φ_j and ψ_j, we have

$$\|H^{int}_{\varphi,\psi}(\frac{r}{R_\varepsilon}\cdot)\|_{\mathcal{C}^{4,\alpha}(\bar{B}_2(0)-B_1(0))} \le C r^2 R_\varepsilon^{-2}\Big(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)}+\|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}\Big),$$

for all $r\le\frac{R_\varepsilon}{2}$. Then by (2.20), we get $\|H^{int}_{\varphi,\psi}(\frac{r}{R_\varepsilon}\cdot)\|_{\mathcal{C}^{4,\alpha}(\bar{B}_2(0)-B_1(0))}\le c_\kappa\varepsilon^2 r^2$.

On the other hand,

$$\begin{aligned}
\sup_{r\le R_\varepsilon} r^{4-\mu}\,|\,\mathcal{N}(0,0)\,| \;&\le\; \sup_{r\le R_\varepsilon}\frac{384\;r^{4-\mu}}{(1+r^2)^4}\left(e^{H^{int}_{\varphi_1^1,\psi_1^1}+\gamma_1 H^{int}_{\varphi_2^1,\psi_2^1}}-1\right)\\
&\le\; \sup_{r\le R_\varepsilon}\frac{384\;r^{4-\mu}}{(1+r^2)^4}\,(|H^{int}_{\varphi_1^1,\psi_1^1}|+\gamma_1|H^{int}_{\varphi_2^1,\psi_2^1}|)\\
&\le\; c\,\sup_{r\le R_\varepsilon}\frac{r^{4-\mu}}{(1+r^2)^4}(r^2\|H^{int}_{\varphi_1^1,\psi_1^1}\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_1^*(0))}+r^2\gamma_1\|H^{int}_{\varphi_2^1,\psi_2^1}\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_1^*(0))})\\
&\le\; c\,\sup_{r\le R_\varepsilon}\frac{r^{4-\mu}\,r^2\,\varepsilon^2}{(1+r^2)^4}.
\end{aligned}$$

Making use of Proposition 2.1 together with (2.6) we get for $\mu\in(1,2)$, that there exist $\bar{c}_\kappa$, such that

$$\|\mathcal{N}(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \;\le\; c_\kappa\, r_\varepsilon^2. \tag{2.22}$$

For the second estimate, we have

$$\mathcal{M}(0,0)=\frac{384\;C_\varepsilon\,\varepsilon^{8(1-\gamma_2)}}{(1+r^2)^{4\gamma_2}}\;e^{(1-\gamma_1\gamma_2)\;G(\frac{\varepsilon}{\tau}x,x^2)+H^{int}_{\varphi_2^1,\psi_2^1}(\frac{x-x^1}{R_\varepsilon})+\gamma_2\,H^{int}_{\varphi_1^1,\psi_1^1}(\frac{x-x^1}{R_\varepsilon})}.$$

Then,

$$\sup_{r\le R_\varepsilon} r^{4-\delta}\,|\,\mathcal{M}(0,0)\,| \;\;\le\;\; c_\kappa\sup_{r\le R_\varepsilon}\frac{384\;C_\varepsilon\;\varepsilon^{8(1-\gamma_2)}}{(1+r^2)^{4\gamma_2}}r^{4-\delta}.$$

Using the same argument as below, we get $\|\mathcal{M}(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \;\le\; c_\kappa\, r_\varepsilon^2$.

To derive the third estimate, for h_i, k_i verifying (2.21), we have

$$\begin{aligned}
&\sup_{r\le R_\varepsilon} r^{4-\mu}\left|\mathcal{N}(h_1^1,h_2^1)-\mathcal{N}(k_1^1,k_2^1)\right| \\
\le\ & c\ \sup_{r\le R_\varepsilon}\frac{r^{4-\mu}}{(1+r^2)^4}\left|(e^{h_1^1+H^{int}_{\varphi_1^1,\psi_1^1}+\gamma_1\,(h_2^1+H^{int}_{\varphi_2^1,\psi_2^1})}-h_1^1-1)-(e^{k_1^1+H^{int}_{\varphi_1^1,\psi_1^1}+\gamma_1\,(k_2^1+H^{int}_{\varphi_2^1,\psi_2^1})}-k_1^1-1)\right| \\
\le\ & c\ \sup_{r\le R_\varepsilon}\frac{r^{4-\mu}}{(1+r^2)^4}\Big(|h_1^1-k_1^1|\ |h_1^1+k_1^1|+\gamma_1|h_2^1-k_2^1|\Big) \\
\le\ & c\ \sup_{r\le R_\varepsilon}\frac{r^{4-\mu}}{(1+r^2)^4}\Big(r^{2\mu}(\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}+\|k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu})\|h_1^1-k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}+\gamma_1 r^\delta\|h_2^1-k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta}\Big).
\end{aligned}$$

We conclude that

$$\|\mathcal{N}(h_1^1,h_2^1)-\mathcal{N}(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\ \le\ c_\kappa\, r_\varepsilon^2\,\|h_1^1-k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\ +\ c_\kappa\gamma_1\|h_2^1-k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}. \tag{2.23}$$

Similarly we get

$$\|\mathcal{M}(h_1^1,h_2^1)-\mathcal{M}(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\ \le\ c_\kappa\, r_\varepsilon^2\,\|(h_1^1,h_2^1)-(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}. \tag{2.24}$$

□

Reducing ε_κ if necessary, we can assume that $\bar c_\kappa\, r_\varepsilon^2\ \le\ \dfrac{1}{2}$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. There exists $\gamma_0\in(0,1)$ such that $c_\kappa\,\gamma_1\le\frac12$ for all $\gamma_1\in(0,\gamma_0)$. Therefore, (2.23) and (2.24) are enough to show that

$$(h_1^1,h_2^1)\mapsto\Big(\mathcal{N}(h_1^1,h_2^1),\mathcal{M}(h_1^1,h_2^1)\Big)$$

is a contraction from the ball

$$\Big\{(h_1^1,h_2^1)\in\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4):\|(h_1^1,h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}\ \le\ 2\ c_\kappa\, r_\varepsilon^2\Big\}$$

into itself and hence a unique fixed point (h_1^1,h_2^1) exists in this set. This fixed point is a solution of (2.18). Then we have:

Proposition 2.4. *Given $\kappa>0$, there exists $\varepsilon_\kappa>0$, $c_\kappa>0$ and $\gamma_0\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\gamma_1\in(0,\gamma_0)$, for all τ in some fixed compact subset of $[\tau^-,\tau^+]\subset(0,\infty)$ and for φ and ψ satisfying* (2.13) *and* (2.20)*, there exists a unique $(h_1^1,h_2^1)(:=(h_{1,\varepsilon,\tau,\varphi,\psi},h_{2,\varepsilon,\tau,\varphi,\psi}))$ solution of* (2.18) *such that*

$$\|(h_1^1,h_2^1)\|_{C^{4,\alpha}_\mu(\mathbb{R}^4)\times C^{4,\alpha}_\delta(\mathbb{R}^4)}\le 2c_\kappa\ r_\varepsilon^2$$

Hence

$$\begin{cases} v_1(x) &:= \bar u(x-x^1) &+h_1^1(x) &+H^{int}(\varphi_1^1,\psi_1^1,\frac{x-x^1}{R_\varepsilon}) &-\gamma_1\,G(\frac{\varepsilon x}{\tau},\,x^2)\\ v_2(x) &:= G(\frac{\varepsilon x}{\tau},x^2) &+h_2^1(x) &+H^{int}(\varphi_2^1,\psi_2^1,\frac{x-x^1}{R_\varepsilon}) \end{cases}$$

solves (2.15) *in $B_{r_\varepsilon}(x^1)$.*

Similarly, we prove

Proposition 2.5. *Given $\kappa>0$, there exists $\varepsilon_\kappa>0$, $c_\kappa>0$ and $\gamma_0\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\gamma_2\in(0,\gamma_0)$, for all τ in some fixed compact subset of $[\tau^-,\tau^+]\subset(0,\infty)$ and for φ and ψ satisfying* (2.13) *and* (2.20)*, there exists a unique $(h_1^2,h_2^2)(:=(h_{1,\varepsilon,\tau,\varphi,\psi},h_{2,\varepsilon,\tau,\varphi,\psi}))$ solution of* (2.18) *such that*

$$\|(h_1^2,h_2^2)\|_{C^{4,\alpha}_\delta(\mathbb{R}^4)\times C^{4,\alpha}_\mu(\mathbb{R}^4)}\le 2c_\kappa\ r_\varepsilon^2$$

Hence

$$\begin{cases} v_1(x) &:= G(\frac{\varepsilon x}{\tau},x^1) &+h_1^2(x) &+H^{int}(\varphi_1^2,\psi_1^2,\frac{x-x^2}{R_\varepsilon})\\ v_2(x) &:= \bar u(x-x^2) &+h_2^2(x) &+H^{int}(\varphi_2^2,\psi_2^2,\frac{x-x^2}{R_\varepsilon}) &-\gamma_2\,G(\frac{\varepsilon x}{\tau},\,x^1) \end{cases}$$

solves (2.15) *in $B_{r_\varepsilon}(x^2)$.*

Remark also that the functions $(h_1^i,h_2^i)(:=(h^i_{1,\varepsilon,\tau,\varphi,\psi},h^i_{2,\varepsilon,\tau,\varphi,\psi}))$, for $i\in\{1,2\}$, obtained in the above Propositions, depend continuously on the parameter τ.

2.3 The nonlinear exterior problem

Given $\tilde{\mathbf{x}} := (\tilde{x}^1, \tilde{x}^2) \in \Omega^2$ close to $\mathbf{x} := (x^1, x^2)$, $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$ close to 0, $\tilde{\varphi}_1 := (\tilde{\varphi}_1^1, \tilde{\varphi}_1^2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$, $\tilde{\varphi}_2 := (\tilde{\varphi}_2^1, \tilde{\varphi}_2^2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$, $\tilde{\psi}_1 := (\tilde{\psi}_1^1, \tilde{\psi}_1^2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ and $\tilde{\psi}_2 := (\tilde{\psi}_2^1, \tilde{\psi}_2^2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ satisfying (2.14). Define for $k = 1, 2$

$$\tilde{\mathbf{w}}_{\mathbf{k}}(x) := (1 + \lambda_k)\ G(x, \tilde{x}^k) + \sum_{i=1}^{2} \chi_{r_0}(x - \tilde{x}^i)\, H^{ext}(\tilde{\varphi}_k^i, \tilde{\psi}_k^i; \frac{x - \tilde{x}^i}{r_\varepsilon}). \tag{2.25}$$

Here χ_{r_0} is a cut-off function identically equal to 1 in $B_{\frac{r_0}{2}}(0)$ and identically equal to 0 outside $B_{r_0}(0)$. We would like to find a solution of the system

$$\Delta^2 u_1 = \rho^4\ e^{u_1 + \gamma_1 u_2} \quad \text{and} \quad \Delta^2 u_2 = \rho^4\ e^{u_2 + \gamma_2 u_1} \tag{2.26}$$

in the domain $\overline{\Omega}_{r_\varepsilon}(\tilde{x})$ with $u_k = \tilde{\mathbf{w}}_{\mathbf{k}} + \tilde{v}_k$ a perturbation of $\tilde{\mathbf{w}}_{\mathbf{k}}$.

This amounts to solve in $\overline{\Omega}_{r_\varepsilon}(\tilde{x})$

$$\Delta^2\, \tilde{v}_1 = \rho^4\ e^{\tilde{\mathbf{w}}_1 + \tilde{v}_1 + \gamma_1(\tilde{\mathbf{w}}_2 + \tilde{v}_2)} - \Delta^2 \tilde{\mathbf{w}}_1 \quad \text{and} \quad \Delta^2\, \tilde{v}_2 = \rho^4\ e^{\tilde{\mathbf{w}}_2 + \tilde{v}_2 + \gamma_2(\tilde{\mathbf{w}}_1 + \tilde{v}_1)} - \Delta^2 \tilde{\mathbf{w}}_2. \tag{2.27}$$

For all $\sigma \in (0, \frac{r_0}{2})$ and all $\tilde{\mathbf{x}} = (\tilde{x}^1, \tilde{x}^2) \in \Omega^2$ such that $\|\mathbf{x} - \tilde{\mathbf{x}}\| \leq \frac{r_0}{2}$, where $\mathbf{x} = (x^1, x^2)$, we denote by $\tilde{\xi}_{\sigma,\tilde{\mathbf{x}}} : \mathcal{C}_\nu^{0,\alpha}(\bar{\Omega}_\sigma(\tilde{\mathbf{x}})) \longrightarrow \mathcal{C}_\nu^{0,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$ the extension operator defined by

$$\begin{cases} \tilde{\xi}_{\sigma,\tilde{\mathbf{x}}}(f) \equiv & f & \text{in } \bar{\Omega}(\tilde{\mathbf{x}}) \\ \tilde{\xi}_{\sigma,\tilde{\mathbf{x}}}(f)(\tilde{x}^j + x) = & \tilde{\chi}\Big(\dfrac{|x|}{\sigma}\Big)\, f\Big(\tilde{x}^j + \sigma \dfrac{x}{|x|}\Big) & \text{in } B_\sigma(\tilde{x}^j) - B_{\frac{\sigma}{2}}(\tilde{x}^j) \quad \forall\, 1 \leq j \leq 2 \\ \tilde{\xi}_{\sigma,\tilde{\mathbf{x}}}(f) \equiv & 0 & \text{in } B_{\frac{\sigma}{2}}(\tilde{x}^1) \cup B_{\frac{\sigma}{2}}(\tilde{x}^2). \end{cases}$$

Here $\tilde{\chi}$ is a cut-off function over $\mathbb{R}_+$ which is equal to 1 for $t \geq 1$ and equal to 0 for $t \leq \frac{1}{2}$. Obviously, there exists a constant $\bar{c} = \bar{c}(\nu) > 0$ only depending on ν such that

$$\left\|\tilde{\xi}_{\sigma,\tilde{\mathbf{x}}}(w)\right\|_{\mathcal{C}_\nu^{0,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \leq\ \bar{c}\ \|w\|_{\mathcal{C}_\nu^{0,\alpha}(\bar{\Omega}_\sigma(\tilde{\mathbf{x}}))} \tag{2.28}$$

We fix $\nu \in (-1, 0)$, to solve (2.27), it is enough to find $(\tilde{v}_1, \tilde{v}_2) \in (\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\tilde{\mathbf{x}})))^2$ solution of

$$\tilde{v}_1 = \tilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_\varepsilon,\tilde{\mathbf{x}}} \circ \tilde{S}_1(\tilde{v}_1, \tilde{v}_2) \quad \text{and} \quad \tilde{v}_2 = \tilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_\varepsilon,\tilde{\mathbf{x}}} \circ \tilde{S}_2(\tilde{v}_1, \tilde{v}_2) \tag{2.29}$$

where

$$\tilde{S}_1(\tilde{v}_1,\, \tilde{v}_2) = \rho^4\ e^{\tilde{\mathbf{w}}_1 + \tilde{v}_1 + \gamma_1(\tilde{\mathbf{w}}_2 + \tilde{v}_2)} - \Delta^2\, \tilde{\mathbf{w}}_1 \quad \text{and} \quad \tilde{S}_2(\tilde{v}_1,\, \tilde{v}_2) = \rho^4\ e^{\tilde{\mathbf{w}}_2 + \tilde{v}_2 + \gamma_2(\tilde{\mathbf{w}}_1 + \tilde{v}_1)} - \Delta^2\, \tilde{\mathbf{w}}_2.$$

We denote by

$$\tilde{\mathcal{N}}(\tilde{v}_1, \tilde{v}_2) = \tilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_\varepsilon,\tilde{\mathbf{x}}} \circ \tilde{S}_1(\tilde{v}_1, \tilde{v}_2) \quad \text{and} \quad \tilde{\mathcal{M}}(\tilde{v}_1, \tilde{v}_2) = \tilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_\varepsilon,\tilde{\mathbf{x}}} \circ \tilde{S}_2(\tilde{v}_1, \tilde{v}_2).$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that for $i, j \in \{1,\, 2\}$ the functions $\tilde{\varphi}_j^i$, $\tilde{\psi}_j^i$, the parameters λ_i and the point $\tilde{\mathbf{x}} = (\tilde{x}^1, \tilde{x}^2)$ satisfy

$$\|\tilde{\varphi}_j^i\|_{\mathcal{C}^{4,\alpha}(S^3)}\ \leq\ \kappa\, r_\varepsilon^2\ , \qquad \|\tilde{\psi}_j^i\|_{\mathcal{C}^{2,\alpha}(S^3)}\ \leq\ \kappa\, r_\varepsilon^2, \tag{2.30}$$

$$|\lambda_i|\ \leq\ \kappa\ r_\varepsilon^2\ , \qquad |\tilde{x}^i - x^i|\ \leq\ \kappa\, r_\varepsilon. \tag{2.31}$$

Then the following result holds

Lemma 2.4. *Under the above assumptions, there exists a constant $c_\kappa > 0$ such that*

$$\|\tilde{\mathcal{N}}(0,0)\|_{\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))}\ \leq\ c_\kappa\, r_\varepsilon^2, \|\tilde{\mathcal{M}}(0,0)\|_{\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))}\ \leq\ c_\kappa\, r_\varepsilon^2,$$

$$\|\tilde{\mathcal{N}}(\tilde{v}_1,\, \tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}_1',\, \tilde{v}_2')\|_{\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))}\ \leq\ c\, r_\varepsilon^2\, \|(\tilde{v}_1,\, \tilde{v}_2) - (\tilde{v}_1',\, \tilde{v}_2')\|_{(\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}})))^2}$$

and

$$\|\tilde{\mathcal{M}}(\tilde{v}_1,\,\tilde{v}_2)-\tilde{\mathcal{M}}(\tilde{v}_1',\,\tilde{v}_2')\|_{\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{x}}))}\;\le\; c\,r_\varepsilon^2\,\|(\tilde{v}_1,\,\tilde{v}_2)-(\tilde{v}_1',\,\tilde{v}_2')\|_{(\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{x}})))^2}$$

provided $\left(\tilde{v}_1,\tilde{v}_2,\tilde{v}_1',\tilde{v}_2'\right)\in\left(\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{x}}))\right)^4$ *satisfy*

$$\|(\tilde{v}_1,\tilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{x}})))^2}\;\le\;2\;c_\kappa\,r_\varepsilon^2\quad\textit{and}\quad\|(\tilde{v}_1',\,\tilde{v}_2')\|_{(\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{x}})))^2}\;\le\;2\;c_\kappa\,r_\varepsilon^2. \tag{2.32}$$

Proof. As for the interior problem, the proof of the two first estimates follows from the asymptotic behavior of H^{ext} together with the assumption on the norm of boundary data $\tilde{\varphi}_j^i$ and $\tilde{\psi}_j^i$ given by (2.30). Indeed, let c_κ be a constant depending only on κ, by Lemma 2.1,

$$|H^{ext}(\tilde{\varphi},\tilde{\psi})|\;\le\;c_\kappa\,r_\varepsilon^3\;r^{-1}. \tag{2.33}$$

On the other hand,

$$\tilde{S}_1(0,0)=\rho^4\;e^{\tilde{\mathbf{w}}_\mathbf{1}+\gamma_1\,\tilde{\mathbf{w}}_\mathbf{2}}\;-\;\Delta^2\,\tilde{\mathbf{w}}_\mathbf{1}\quad\text{and}\quad\tilde{S}_2(0,0)=\rho^4\;e^{\tilde{\mathbf{w}}_\mathbf{2}+\gamma_\mathbf{2}\,\tilde{\mathbf{w}}_\mathbf{1}}\;-\;\Delta^2\,\tilde{\mathbf{w}}_\mathbf{2}.$$

We will estimate $\tilde{S}_1(0,0)$ in different subregions of $\bar{\Omega}^*(\tilde{\mathbf{x}})$.

- In $B_{\frac{r_0}{2}}(\tilde{x}^1)-B_{r_\varepsilon}(\tilde{x}^1)$, we have $\chi_{r_0}(x-\tilde{x}^1)=1$, $\chi_{r_0}(x-\tilde{x}^2)=0$ and $\Delta^2\,\tilde{\mathbf{w}}_\mathbf{1}=0$, so that

$$|\tilde{S}_1(0,0)|=\rho^4\;e^{\tilde{w}_1+\gamma_1\tilde{w}_2}.$$

$$\begin{array}{rcl}|\tilde{S}_1(0,0)| & \le & c_\kappa\;\varepsilon^4\;|x-\tilde{x}^1|^{-8(1+\lambda_1)}\;|x-\tilde{x}^2|^{-8\gamma_1(1+\lambda_2)}\\ & \le & c_\kappa\;\varepsilon^4\;|x-\tilde{x}^1|^{-8(1+\lambda_1)}\\ & \le & c_\kappa\;\varepsilon^4 r^{-8(1+\lambda_1)}.\end{array}$$

Hence, for $\nu\in(-1,0)$ and λ_1 small enough, we get

$$\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{\frac{r_0}{2}}(\tilde{x}^1))}\;\le\;\sup_{r_\varepsilon\;\le\;r\le\frac{r_0}{2}}\;r^{4-\nu}|\tilde{S}_1(0,0)|\;\le\;c_\kappa\;\varepsilon^4\;r_\varepsilon^{-4}.$$

- In $B_{r_0}(\tilde{x}^1)-B_{\frac{r_0}{2}}(\tilde{x}^1)$, using the estimate (2.33), then we have

$$\begin{array}{rcl}|\tilde{S}_1(0,0)| & \le & c_\kappa\;\varepsilon^4\;r^{-8(1+\lambda_1)}+\displaystyle\sum_{i=1}^{2}|[\Delta^2,\chi_{r_0}(x-\tilde{x}^i)]H^{ext}(\tilde{\varphi},\tilde{\psi})|\\ & \le & c_\kappa\;(\varepsilon^4\;r^{-8(1+\lambda_1)}\;+\;r^{-1}\;r_\varepsilon^3),\end{array}$$

where

$$[\Delta^2,\chi_{r_0}]\;w=w\;\Delta^2\chi_{r_0}+2\;\Delta w\;\Delta\chi_{r_0}+4\;\nabla(\Delta w)\cdot\;\nabla\chi_{r_0}+4\;\nabla w\;\cdot\nabla(\Delta\chi_{r_0})+4\sum_{i,j=1}^{4}\frac{\partial^2\chi_{r_0}}{\partial x_i\partial x_j}\frac{\partial^2 w}{\partial x_i\partial x_j}.$$

Hence, for $\nu\in(-1,0)$ and λ_1 small enough, we get

$$\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\tilde{x}^1)-B_{\frac{r_0}{2}}(\tilde{x}^1))}\;\le\;\sup_{\frac{r_0}{2}\;\le r\;\le r_0}\;r^{4-\nu}|\tilde{S}_1(0,0)|\;\le\;c_\kappa\;\varepsilon^4\;r_\varepsilon^2.$$

Similarly, for $\nu\in(-1,0)$ and λ_2 small enough, we can prove the same result for x^2.

- In $\Omega-\Big(B_{r_0}(\tilde{x}^1)\cup B_{r_0}(\tilde{x}^2)\Big)$, we have $\chi_{r_0}(x-\tilde{x}^1)=0$, $\chi_{r_0}(x-\tilde{x}^2)=0$ and $\Delta^2\,\tilde{\mathbf{w}}_\mathbf{1}=0$. Thus

$$|\tilde{S}_1(0,0)|\le c_\kappa\;\varepsilon^4 e^{(1+\lambda_1)\;G(x,\tilde{x}^1)}\;e^{\gamma_1(1+\lambda_2)\;G(x,\tilde{x}^2)}.$$

So for $\nu\in(-1,0)$, we have

$$\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}-\,\cup_{i=1}^2 B_{r_0}(\tilde{x}^i))}\;\le\;\sup_{r\ge r_0}r^{4-\nu}|\tilde{S}_1(0,0)|\;\le\;c_\kappa\;\varepsilon^4.$$

Then with the previous three steps, we can conclude that

$$\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\bar{\Omega}_{r_0}(\tilde{\mathbf{x}}))} \;\leq\; c_\kappa\; r_\varepsilon^4. \tag{2.34}$$

Making use of Proposition 2.3 together with (2.28) we conclude that

$$\|\tilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \;\leq\; c_\kappa\; r_\varepsilon^4 \qquad \text{and} \qquad \|\tilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \;\leq\; c_\kappa\; r_\varepsilon^4. \tag{2.35}$$

For the proof of the third estimate, let $\tilde{v}_1, \tilde{v}_2, \tilde{v}'_1$ and $\tilde{v}'_2 \in \mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*)$ satisfy (2.32), we have

$$\begin{aligned} \left|\tilde{S}_1(\tilde{v}_1,\tilde{v}_2) - \tilde{S}_1(\tilde{v}'_1,\tilde{v}'_2)\right| \;&\leq\; c_\kappa\; \varepsilon^4\; e^{\tilde{w}_1+\gamma_1\tilde{w}_2}\; |e^{\tilde{v}_1+\gamma_1\tilde{v}_2} - e^{\tilde{v}'_1+\gamma_1\tilde{v}'_2}| \\ &\leq\; c_\kappa\; \varepsilon^4 r^{-8(1+\lambda_1)}\Big(|\tilde{v}_1-\tilde{v}'_1| + |\tilde{v}_2-\tilde{v}'_2|\Big). \end{aligned}$$

So, for λ_1 small enough and using the estimate (2.28), there exist $\bar{c}_\kappa$ (depending on κ) such that

$$\|\tilde{\mathcal{N}}(\tilde{v}_1,\tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}'_1,\tilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \;\leq\; \bar{c}_\kappa\; r_\varepsilon^2\left(\|\tilde{v}_1-\tilde{v}'_1\|_{C^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} + \|\tilde{v}_2-\tilde{v}'_2\|_{C^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))}\right). \tag{2.36}$$

Similarly we can use the same argument to prove,

$$\|\tilde{\mathcal{M}}(\tilde{v}_1,\tilde{v}_2) - \tilde{\mathcal{M}}(\tilde{v}'_1,\tilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \leq \bar{c}_\kappa\; r_\varepsilon^2\left(\|\tilde{v}_1-\tilde{v}'_1\|_{C^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} + \|\tilde{v}_2-\tilde{v}'_2\|_{C^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))}\right). \tag{2.37}$$

□

Reducing ε_κ if necessary, we can assume that $\bar{c}_\kappa\; r^2_{\varepsilon,\gamma_1,\gamma_2} \;\leq\; \dfrac{1}{2}$ for all $\varepsilon \in (0,\varepsilon_\kappa)$. Then, (2.36) and (2.37) are enough to show that

$$(\tilde{v}_1,\tilde{v}_2) \mapsto \Big(\tilde{\mathcal{N}}(\tilde{v}_1,\tilde{v}_2)\;,\;\tilde{\mathcal{M}}(\tilde{v}_1,\tilde{v}_2)\Big)$$

is a contraction from the ball

$$\left\{(\tilde{v}_1,\tilde{v}_2) \in \Big(C^{4,\alpha}_{\nu}(\mathbb{R}^4)\Big)^2 : \|(\tilde{v}_1,\tilde{v}_2)\|_{(C^{4,\alpha}_{\nu}(\mathbb{R}^4))^2} \;\leq\; 2\;\bar{c}_\kappa\; r^2_{\varepsilon,\gamma_1,\gamma_2}\right\}$$

into itself. Hence there exist a unique fixed point $(\tilde{v}_1,\tilde{v}_2)$ in this set, which is a solution of (2.29). Applying a fixed point Theorem for contraction mappings, we conclude that :

Proposition 2.6. *Given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ (depending on κ) such that for any $\varepsilon \in (0,\varepsilon_\kappa)$, λ_i and $\tilde{x}^i$ satisfying (2.31) and functions $\tilde{\varphi}^i_j$ and $\tilde{\psi}^i_j$ satisfying (2.14) and (2.30), there exists a unique $(\tilde{v}_1,\tilde{v}_2)(:= (\tilde{v}_{1,\varepsilon,\lambda,\tilde{\mathbf{x}},\tilde{\varphi},\tilde{\psi}}, \tilde{v}_{2,\varepsilon,\lambda,\tilde{\mathbf{x}},\tilde{\varphi},\tilde{\psi}}))$ solution of (2.29) so that for $v^1_k(k=1,2)$ defined by*

$$v_k(x) := (1+\lambda_k)\; G(x,\tilde{x}^k) + \sum_{i=1,2} \chi_{r_0}(x-\tilde{x}^i)\, H^{ext}(\tilde{\varphi}^i_k, \tilde{\psi}^i_k, x-\tilde{x}^k) + \tilde{v}_k(x)$$

solves (2.26) in $\overline{\Omega}_{r_\varepsilon}(\tilde{\mathbf{x}})$. In addition, we have

$$\|(\tilde{v}_1,\tilde{v}_2)\|_{C^{4,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \;\leq\; 2\;\bar{c}_\kappa\; r_\varepsilon^2.$$

2.4 The nonlinear Cauchy-data matching

We will gather the results of the previous sections. Using the previous notations, assume that $\tilde{\mathbf{x}} := (\tilde{x}^1,\, \tilde{x}^2) \in \Omega^2$ are given close to $\mathbf{x} := (x^1,\, x^2)$. Assume also that

$$\tau := (\tau_1,\, \tau_2) \in [\tau_1^-, \tau_1^+] \times [\tau_2^-, \tau_2^+] \subset (0,\infty)^2$$

are given (the values of τ_l^- and τ_l^+, for $l = 1,2$ will be fixed later). First, we consider some set of boundary data $\varphi^{\mathbf{i}} := (\varphi^i_1,\, \varphi^i_2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^{\mathbf{i}} := (\psi^i_1,\, \psi^i_2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$. According to the result of Proposition 2.5 and provided $\varepsilon \in (0, \varepsilon_\kappa)$, we can find, $u_{int} := (u_{int,1}, u_{int,2})$ a solution of (2.15) in $B_{r_\varepsilon}(\tilde{x}^1) \cup B_{r_\varepsilon}(\tilde{x}^2)$, which can be decomposed as

$$u_{int,1}(x) := \begin{cases} u_{\varepsilon,\tau_1}(x-\tilde{x}^1) + h_1^1(\frac{R_\varepsilon^1(x-\tilde{x}^1)}{r_\varepsilon}) + H^{int}(\varphi_1^1, \psi_1^1, \frac{x-\tilde{x}^1}{r_\varepsilon}) - \gamma_1 G(x,\tilde{x}^2) & \text{in } B_{r_\varepsilon}(\tilde{x}^1) \\ G(x,\tilde{x}^1) + h_1^2(\frac{R_\varepsilon^2(x-\tilde{x}^2)}{r_\varepsilon}) + H^{int}(\varphi_1^2, \psi_1^2, \frac{x-\tilde{x}^2}{r_\varepsilon}) & \text{in } B_{r_\varepsilon}(\tilde{x}^2) \end{cases}$$

and

$$u_{int,2}(x) := \begin{cases} G(x,\tilde{x}^2) + h_2^1(\frac{R_\varepsilon^1(x-\tilde{x}^1)}{r_\varepsilon}) + H^{int}(\varphi_2^1, \psi_2^1, \frac{x-\tilde{x}^1}{r_\varepsilon}) & \text{in } B_{r_\varepsilon}(\tilde{x}^1) \\ u_{\varepsilon,\tau_2}(x-\tilde{x}^2) + h_2^2(\frac{R_\varepsilon^2(x-\tilde{x}^2)}{r_\varepsilon}) + H^{int}(\varphi_2^2, \psi_2^2, \frac{x-\tilde{x}^2}{r_\varepsilon}) - \gamma_2 G(x,\tilde{x}^1) & \text{in } B_{r_\varepsilon}(\tilde{x}^2) \end{cases}$$

where for $i,j \in \{1,2\}$, $R_\varepsilon^i = \tau_i \frac{r_\varepsilon}{\varepsilon}$ and the functions h_j^i satisfy

$$\|(h_1^1, h_1^2)\|_{(\mathcal{C}_\mu^{2,\alpha}(\mathbb{R}^2))^2} \le 2c_\kappa\, r_\varepsilon^2 \quad \text{and} \quad \|(h_2^1, h_2^2)\|_{(\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4))^2} \le 2c_\kappa\, r_\varepsilon^2.$$

Similarly, given some boundary data $\tilde{\varphi}_j^i \in C^{4,\alpha}(S^3)$, $\tilde{\psi}_j^i \in C^{2,\alpha}(S^3)$ satisfying (2.14), $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ satisfying (2.31), provided $\varepsilon \in (0, \varepsilon_\kappa)$, by Proposition 2.6, we find a solution $u_{ext} := (u_{ext,1}, u_{ext,2})$ of (2.15) in $\bar{\Omega} \setminus (B_{r_\varepsilon}(\tilde{x}^1) \cup B_{r_\varepsilon}(\tilde{x}^2))$ which can be decomposed as

$$u_{ext,k}(x) := (1+\lambda_k) G(x,\tilde{x}^k) + \sum_{i=1,2} \chi_{r_0}(x-\tilde{x}^i)\, H^{ext}\left(\tilde{\varphi}_k^i, \tilde{\psi}_k^i, \frac{x-\tilde{x}^i}{r_\varepsilon}\right) + \tilde{v}_k$$

for $k = 1,2$ with $\tilde{v}_1, \tilde{v}_2 \in C_\nu^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$ satisfying $\|(\tilde{v}_1, \tilde{v}_2)\|_{(C_\nu^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}})))^2} \le 2\bar{c}_\kappa r_\varepsilon^2$.

It remains to determine the parameters and the boundary data in such a way that the function equal to u_{int} in $B_{r_\varepsilon}(\tilde{x}^1) \cup B_{r_\varepsilon}(\tilde{x}^2)$ and equal to u_{ext} in $\bar{\Omega}_{r_\varepsilon}(\tilde{\mathbf{x}})$ is a smooth function. This amounts to find the boundary data and the parameters so that, for each $i = 1$, 2

$$u_{int,i} = u_{ext,i}, \quad \partial_r u_{int,i} = \partial_r u_{ext,i}, \quad \Delta u_{int,i} = \Delta u_{ext,i} \quad \text{and} \quad \partial_r \Delta u_{int,i} = \partial_r \Delta u_{ext,i} \tag{2.38}$$

on $\partial B_{r_\varepsilon}(\tilde{x}^1)$ and $\partial B_{r_\varepsilon}(\tilde{x}^2)$.

Suppose that (2.38) is verified, this provides that for each ε small enough $u_\varepsilon \in \mathcal{C}^{4,\alpha}$ (which is obtained by patching together the functions u_{int} and the function u_{ext}), a weak solution of our system and elliptic regularity theory implies that this solution is in fact smooth. That will complete the proof since, as ε tends to 0, the sequence of solutions we have obtain satisfies the required singular limit behaviors, namely, u_ε converges to $G(\cdot, x^i)$.

Before we proceed, the following remarks are due. First it will be convenient to observe that the function u_{ε,τ_i} can be expanded as

$$u_{\varepsilon,\tau_i}(x) \;=\; -4\ln\tau_i - 8\ln|x| + \mathcal{O}\left(\frac{\varepsilon^2\tau_i^{-2}}{|x|^2}\right) \quad \text{on} \quad \partial B_{r_\varepsilon}(x^i). \tag{2.39}$$

Thus For x on $\partial B_{r_\varepsilon}(x^1)$, we have

$$\begin{aligned} (u_{int,1} - u_{ext,1})(x) &= -4\ln\tau_1 + 8\lambda_1 \ln|x-\tilde{x}^1| + h_1^1(R_\varepsilon^1 \tfrac{x-\tilde{x}^1}{r_\varepsilon}) \\ &+ H^{int}(\varphi_1^1, \psi_1^1, \tfrac{x-\tilde{x}^1}{r_\varepsilon}) - H^{ext}(\tilde{\varphi}_1^1, \tilde{\psi}_1^1, \tfrac{x-\tilde{x}^1}{r_\varepsilon}) \\ &-(1+\lambda_1) H(x,\tilde{x}^1) - \gamma_1 G(x,\tilde{x}^2) + \mathcal{O}(\tfrac{\varepsilon^2\tau_1^{-2}}{|x-\tilde{x}^1|^2}) + \mathcal{O}(r_\varepsilon^2). \end{aligned} \tag{2.40}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\tilde{x}^1)$ in (2.38), it will be more convenient to solve on S^3 the following set of equations

$$\begin{aligned} (u_{int,1} - u_{ext,1})(\tilde{x}^1 + r_\varepsilon\,\cdot) &= 0, & \partial_r(u_{int,1} - u_{ext,1})(\tilde{x}^1 + r_\varepsilon\,\cdot) &= 0, \\ \Delta(u_{int,1} - u_{ext,1})(\tilde{x}^1 + r_\varepsilon\,\cdot) &= 0 \quad \text{and} & \partial_r\Delta(u_{int,1} - u_{ext,1})(\tilde{x}^1 + r_\varepsilon\,\cdot) &= 0. \end{aligned} \tag{2.41}$$

Since the boundary data are chosen to satisfy (2.13) or (2.14). We decompose

$$\varphi_1^1 = \varphi_{1,0}^1 + \varphi_{1,1}^1 + \varphi_1^{1,\perp}, \psi_1^1 = 8\varphi_{1,0}^1 + 12\varphi_{1,1}^1 + \psi_1^{1,\perp},\ \tilde{\varphi}_1^1 = \tilde{\varphi}_{1,0}^1 + \tilde{\varphi}_{1,1}^1 + \tilde{\varphi}_1^{1,\perp} \ \text{ and } \ \tilde{\psi}_1^1 = \tilde{\psi}_{1,1}^1 + \tilde{\psi}_1^{1,\perp},$$

where $\varphi^1_{1,0}, \tilde\varphi^1_{1,0} \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi^1_{1,1}, \tilde\varphi^1_{1,1}, \tilde\psi^1_{1,1}$ belong to $\mathbb{E}_1 = Span\{e_1, e_2, e_3, e_4\}$ and $\varphi^{1,\perp}_1, \tilde\varphi^{1,\perp}_1, \psi^{1,\perp}_1, \tilde\psi^{1,\perp}_1$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

Using (2.40), we have for $x \in S^3$

$$\begin{aligned}(u_{int,1} - u_{ext,1})(\tilde x^1 + r_\varepsilon x) &= -4\ln\tau_1 + 8\lambda_1 \ln(r_\varepsilon|x|) + H^{int}(\varphi^1_1, \psi^1_1, x) - H^{ext}(\tilde\varphi^1_1, \tilde\psi^1_1, x)\\ &\quad -\gamma_1 \mathcal{E}_1(\tilde x^1 + r_\varepsilon x, \tilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2),\end{aligned}$$

where $\mathcal{E}_1(\cdot, \tilde{\mathbf{x}}) := \frac{1}{\gamma_1} H(\cdot, \tilde x^1) + G(\cdot, \tilde x^2)$.

Then, the projection of the set equations (2.41) over $\mathbb{E}_0$ will yield

$$\left\{\begin{aligned} -4\ln\tau_1 + 8\lambda_1 \ln r_\varepsilon + \varphi^1_{1,0} - \tilde\varphi^1_{1,0} - \gamma_1\mathcal{E}_1(\tilde x^1, \tilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0\\ 8\lambda_1 + 2\varphi^1_{1,0} + 2\tilde\varphi^1_{1,0} + \mathcal{O}(r_\varepsilon^2) &= 0\\ 16\lambda_1 + 8\varphi^1_{1,0} + \mathcal{O}(r_\varepsilon^2) &= 0\\ -32\lambda_1 + \mathcal{O}(r_\varepsilon^2) &= 0.\end{aligned}\right. \tag{2.42}$$

The system (2.42) can be simply written as

$$\lambda_1 = \mathcal{O}(r_\varepsilon^2), \quad \varphi^1_{1,0} = \mathcal{O}(r_\varepsilon^2), \quad \tilde\varphi^1_{1,0} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \frac{1}{\ln r_\varepsilon}\Big[4\ln\tau_1 + \gamma_1\mathcal{E}_1(\tilde x^1, \tilde{\mathbf{x}})\Big] = \mathcal{O}(r_\varepsilon^2).$$

We are now in a position to define τ_1^- and τ_1^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\tilde x^i$ will converge to x^i for $i \in \{1, 2\}$ and τ_1 will converge to τ_1^* satisfying

$$4\ln\tau_1^* = -\gamma_1\mathcal{E}_1(x^1, \mathbf{x}).$$

Hence it is enough to choose τ_1^- and τ_1^+ in such a way that $4\ln(\tau_1^-) < -\gamma_1\mathcal{E}_1(x^1, \mathbf{x}) < 4\ln(\tau_1^+)$.

Consider now the projection of (2.41) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, ..., \partial_{x_4} f)$ with the element of $\mathbb{E}_1$

$$\bar\nabla f = \sum_{i=1}^4 \partial_{x_i} f\, e_i.$$

With these notations in mind, we obtain the system of equations

$$\left\{\begin{aligned} \varphi^1_{1,1} - \tilde\varphi^1_{1,1} - \gamma_1\bar\nabla\mathcal{E}_1(\tilde x^1, \tilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0\\ 3\varphi^1_{1,1} + 3\tilde\varphi^1_{1,1} + \tfrac{1}{2}\tilde\psi^1_{1,1} - \gamma_1\bar\nabla\mathcal{E}_1(\tilde x^1, \tilde{\mathbf{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0\\ 15\varphi^1_{1,1} - 15\tilde\varphi^1_{1,1} + \mathcal{O}(r_\varepsilon^2) &= 0\\ 15\varphi^1_{1,1} + 69\tilde\varphi^1_{1,1} + \mathcal{O}(r_\varepsilon^2) &= 0.\end{aligned}\right. \tag{2.43}$$

Which can be simplified as follows

$$\varphi^1_{1,1} = \mathcal{O}(r_\varepsilon^2), \quad \psi^1_{1,1} = \mathcal{O}(r_\varepsilon^2), \quad \tilde\varphi^1_{1,1} = \mathcal{O}(r_\varepsilon^2), \quad \tilde\psi^1_{1,1} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \bar\nabla\mathcal{E}_1(\tilde x^1, \tilde{\mathbf{x}}) = \mathcal{O}(r_\varepsilon^2). \tag{2.44}$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This yields the system

$$\left\{\begin{aligned} \varphi^{1,\perp}_1 - \tilde\varphi^{1,\perp}_1 + \mathcal{O}(r_\varepsilon^2) &= 0\\ \partial_r\Big(H^{int}_{\varphi^{1,\perp}_1, \psi^{1,\perp}_1} - H^{ext}_{\tilde\varphi^{1,\perp}_1, \tilde\psi^{1,\perp}_1}\Big) + \mathcal{O}(r_\varepsilon^2) &= 0\\ \psi^{1,\perp}_1 - \tilde\psi^{1,\perp}_1 + \mathcal{O}(r_\varepsilon^2) &= 0\\ \partial_r\Delta\Big(H^{int}_{\varphi^{1,\perp}_1, \psi^{1,\perp}_1} - H^{ext}_{\tilde\varphi^{1,\perp}_1, \tilde\psi^{1,\perp}_1}\Big) + \mathcal{O}(r_\varepsilon^2) &= 0.\end{aligned}\right. \tag{2.45}$$

Thanks to the result of Lemma 2.2, this last system can be rewritten as

$$\varphi^{1,\perp}_1 = \mathcal{O}(r_\varepsilon^2), \quad \tilde\varphi^{1,\perp}_1 = \mathcal{O}(r_\varepsilon^2), \quad \psi^{1,\perp}_1 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \tilde\psi^{1,\perp}_1 = \mathcal{O}(r_\varepsilon^2).$$

If we define the parameter $t_1 \in \mathbb{R}$ by

$$t_1 = \frac{1}{\ln r_\varepsilon}\Big[4\ln \tau_1 + \gamma_1 \mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}})\Big],$$

then the systems found by projecting (2.41) gather in this equality

$$T^1_\varepsilon = \Big(t_1, \lambda_1, \varphi^1_{1,0}, \tilde{\varphi}^1_{1,0}, \varphi^1_{1,1}, \tilde{\varphi}^1_{1,1}, \bar{\nabla}\mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}}), \varphi^{1,\perp}_1, \tilde{\varphi}^{1,\perp}_1, \psi^{1,\perp}_1, \tilde{\psi}^{1,\perp}_1\Big) = \mathcal{O}_1(r^2_\varepsilon), \tag{2.46}$$

where as usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

On the other hand, on $\partial B_{r_\varepsilon}(\tilde{x}^2)$, we have

$$\begin{aligned}(u_{int,1} - u_{ext,1})(x) \; &= -\lambda_1 G(x, \tilde{x}^1) + h^2_1(R^1_\varepsilon \tfrac{x-\tilde{x}^2}{r_\varepsilon}) + H^{int}(\varphi^2_1, \psi^2_1, \tfrac{x-\tilde{x}^2}{r_\varepsilon}) \\ &\quad - H^{ext}(\tilde{\varphi}^2_1, \tilde{\psi}^2_1, \tfrac{x-\tilde{x}^2}{r_\varepsilon}) + \mathcal{O}(r^2_\varepsilon).\end{aligned} \tag{2.47}$$

In the same manner as above, we will solve on S^3 the following system

$$\begin{aligned} (u_{int,1} - u_{ext,1})(\tilde{x}^2 + r_\varepsilon\, \cdot) \; &= 0, & \partial_r(u_{int,1} - u_{ext,1})(\tilde{x}^2 + r_\varepsilon\, \cdot) \; &= 0,\\ \Delta(u_{int,1} - u_{ext,1})(\tilde{x}^2 + r_\varepsilon\, \cdot) \; &= 0 \quad \text{and} & \partial_r\Delta(u_{int,1} - u_{ext,1})(\tilde{x}^2 + r_\varepsilon\, \cdot) \; &= 0.\end{aligned} \tag{2.48}$$

We decompose

$$\varphi^2_1 = \varphi^2_{1,0} + \varphi^2_{1,1} + \varphi^{2,\perp}_1, \psi^2_1 = 8\varphi^2_{1,0} + 12\varphi^2_{1,1} + \psi^{2,\perp}_1, \;\; \tilde{\varphi}^2_1 = \tilde{\varphi}^2_{1,0} + \tilde{\varphi}^2_{1,1} + \tilde{\varphi}^{2,\perp}_1 \;\; \text{and} \;\; \tilde{\psi}^2_1 = \tilde{\psi}^2_{1,1} + \tilde{\psi}^{2,\perp}_1,$$

where $\varphi^2_{1,0}, \tilde{\varphi}^2_{1,0} \in \mathbb{E}_0$, $\varphi^2_{1,1}, \tilde{\varphi}^2_{1,1}, \tilde{\psi}^2_{1,1} \in \mathbb{E}_1$ and $\varphi^{2,\perp}_1, \tilde{\varphi}^{2,\perp}_1, \psi^{2,\perp}_1, \tilde{\psi}^{2,\perp}_1$ belong to $(L^2(S^3))^\perp$.

Projecting the set of equations (2.48) over $\mathbb{E}_0$, we get

$$\left\{\begin{aligned} \varphi^2_{1,0} - \tilde{\varphi}^2_{1,0} + \mathcal{O}(r^2_\varepsilon) \; &= 0\\ 2\varphi^2_{1,0} + 2\tilde{\varphi}^2_{1,0} + \mathcal{O}(r^2_\varepsilon) \; &= 0\\ 8\varphi^2_{1,0} + \mathcal{O}(r^2_\varepsilon) \; &= 0.\end{aligned}\right. \tag{2.49}$$

From the L^2-projection of (2.48) over $\mathbb{E}_1$, we obtain the system of equations

$$\left\{\begin{aligned} \varphi^2_{1,1} - \tilde{\varphi}^2_{1,1} + \mathcal{O}(r^2_\varepsilon) \; &= 0\\ 3\varphi^2_{1,1} + 3\tilde{\varphi}^2_{1,1} + \tfrac{1}{2}\tilde{\psi}^2_{1,1} + \mathcal{O}(r^2_\varepsilon) \; &= 0\\ 15\varphi^2_{1,1} - 15\tilde{\varphi}^2_{1,1} + \mathcal{O}(r^2_\varepsilon) \; &= 0\\ 15\varphi^2_{1,1} + 69\tilde{\varphi}^2_{1,1} + \mathcal{O}(r^2_\varepsilon) \; &= 0.\end{aligned}\right. \tag{2.50}$$

Finally, we consider the L^2-projection onto $(L^2(S^3))^\perp$. This yields the system

$$\left\{\begin{aligned} \varphi^{2,\perp}_1 - \tilde{\varphi}^{2,\perp}_1 + \mathcal{O}(r^2_\varepsilon) \; &= 0\\ \partial_r\Big(H^{int}_{\varphi^{2,\perp}_1, \psi^{2,\perp}_1} - H^{ext}_{\tilde{\varphi}^{2,\perp}_1, \tilde{\psi}^{2,\perp}_1}\Big) + \mathcal{O}(r^2_\varepsilon) \; &= 0\\ \psi^{2,\perp}_1 - \tilde{\psi}^{2,\perp}_1 + \mathcal{O}(r^2_\varepsilon) \; &= 0\\ \partial_r\Delta\Big(H^{int}_{\varphi^{2,\perp}_1, \psi^{2,\perp}_1} - H^{ext}_{\tilde{\varphi}^{2,\perp}_1, \tilde{\psi}^{2,\perp}_1}\Big) + \mathcal{O}(r^2_\varepsilon) \; &= 0.\end{aligned}\right. \tag{2.51}$$

Using again Lemma 2.2, the above system can be rewritten as

$$\varphi^{2,\perp}_1 = \mathcal{O}(r^2_\varepsilon), \quad \tilde{\varphi}^{2,\perp}_1 = \mathcal{O}(r^2_\varepsilon), \quad \psi^{2,\perp}_1 = \mathcal{O}(r^2_\varepsilon) \quad \text{and} \quad \tilde{\psi}^{2,\perp}_1 = \mathcal{O}(r^2_\varepsilon).$$

Then the systems found by projecting (2.48) gather in this equality

$$\Big(\varphi^2_{1,0}, \tilde{\varphi}^2_{1,0}, \varphi^2_{1,1}, \tilde{\varphi}^2_{1,1}, \psi^2_{1,1}, \tilde{\psi}^2_{1,1}, \varphi^{2,\perp}_1, \tilde{\varphi}^{2,\perp}_1, \tilde{\psi}^{2,\perp}_1, \tilde{\psi}^{2,\perp}_1\Big) = \mathcal{O}(r^2_\varepsilon). \tag{2.52}$$

By exactly the same arguments for (2.46), we can claim a solution of equation (2.52) in the ball of radius κr_ε^2 of the corresponding product space.

Similarly, in $\partial B_{r_\varepsilon}(\tilde{x}^1)$ using the fact that

$$\varphi_2^1 = \varphi_{2,0}^1 + \varphi_{2,1}^1 + \varphi_2^{1,\perp}, \psi_2^1 = 8\varphi_{2,0}^1 + 12\varphi_{2,1}^1 + \psi_2^{1,\perp},\ \tilde{\varphi}_2^1 = \tilde{\varphi}_{2,0}^1 + \tilde{\varphi}_{2,1}^1 + \tilde{\varphi}_2^{1,\perp} \text{ and } \tilde{\psi}_2^1 = \tilde{\psi}_{2,1}^1 + \tilde{\psi}_2^{1,\perp},$$

with $\varphi_{2,0}^1, \tilde{\varphi}_{2,0}^1 \in \mathbb{E}_0$, $\varphi_{2,1}^1, \tilde{\varphi}_{2,1}^1, \tilde{\psi}_{2,1}^1 \in \mathbb{E}_1 = ker(\Delta_{S^3} + 1) = Span\{e_1, e_2, e_3, e_4\}$ and $\varphi_2^{1,\perp}, \tilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \tilde{\psi}_2^{1,\perp} \in (L^2(S^3))^\perp$, we can prove that

$$\begin{array}{rlcrl} (u_{int,2} - u_{ext,2})(\tilde{x}^1 + r_\varepsilon\,\cdot) & = 0, & & \partial_r(u_{int,2} - u_{ext,2})(\tilde{x}^1 + r_\varepsilon\,\cdot) & = 0, \\ \Delta(u_{int,2} - u_{ext,2})(\tilde{x}^1 + r_\varepsilon\,\cdot) & = 0 & \text{and} & \partial_r\Delta(u_{int,2} - u_{ext,2})(\tilde{x}^1 + r_\varepsilon\,\cdot) & = 0, \end{array} \tag{2.53}$$

on S^3 yield to

$$\Big(\lambda_2, \varphi_{2,0}^1, \tilde{\varphi}_{2,0}^1, \varphi_{2,1}^1, \tilde{\varphi}_{2,1}^1, \psi_{2,1}^1, \tilde{\psi}_{2,1}^1, \varphi_2^{1,\perp}, \tilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \tilde{\psi}_2^{1,\perp}\Big) = \mathcal{O}(r_\varepsilon^2). \tag{2.54}$$

Similarly when ε tends to 0, we expect that $\tilde{x}^2$ converges to x^2 and τ_2 converges to τ_2^* satisfying

$$4\ln\tau_2^* = -\gamma_2\mathcal{E}_2(x^2, \mathbf{x}).$$

So we choose τ_2^- and τ_2^+ to satisfy

$$4\ln(\tau_2^-) < -\gamma_2\mathcal{E}_2(x^2, \mathbf{x}) < 4\ln(\tau_2^+)$$

where $\mathcal{E}_2(\cdot, \tilde{\mathbf{x}}) := \frac{1}{\gamma_2}H(\cdot, \tilde{x}^2) + G(\cdot, \tilde{x}^1)$.

Using the decomposition $\mathbb{E}_0 \oplus \mathbb{E}_1 \oplus (L^2(S^3))^\perp$

$$\varphi_2^2 = \varphi_{2,0}^2 + \varphi_{2,1}^2 + \varphi_2^{2,\perp}, \psi_2^2 = 8\varphi_{2,0}^2 + 12\varphi_{2,1}^2 + \psi_2^{2,\perp},\ \tilde{\varphi}_2^2 = \tilde{\varphi}_{2,0}^2 + \tilde{\varphi}_{2,1}^2 + \tilde{\varphi}_2^{2,\perp} \text{ and } \tilde{\psi}_2^2 = \tilde{\psi}_{2,1}^2 + \tilde{\psi}_2^{2,\perp}.$$

We can prove that

$$\begin{array}{rclcrcl} (u_{int,2} - u_{ext,2})(\tilde{x}^2 + r_\varepsilon\,\cdot) & = & 0, & & \partial_r(u_{int,2} - u_{ext,2})(\tilde{x}^2 + r_\varepsilon\,\cdot) & = & 0, \\ \Delta(u_{int,2} - u_{ext,2})(\tilde{x}^2 + r_\varepsilon\,\cdot) & = & 0 & \text{and} & \partial_r\Delta(u_{int,2} - u_{ext,2})(\tilde{x}^2 + r_\varepsilon\,\cdot) & = & 0, \end{array} \tag{2.55}$$

near S^3 yield to

$$T_\varepsilon^2 = \Big(t_2, \lambda_2, \varphi_{2,0}^2, \tilde{\varphi}_{2,0}^2, \varphi_{2,1}^2, \tilde{\varphi}_{2,1}^2, \psi_{2,1}^2, \tilde{\psi}_{2,1}^2, \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{x}^1), \varphi_2^{2,\perp}, \tilde{\varphi}_2^{2,\perp}, \psi_2^{2,\perp}, \tilde{\psi}_2^{2,\perp}\Big) = \mathcal{O}_2(r_\varepsilon^2), \tag{2.56}$$

where $t_2 = \frac{1}{\ln r_\varepsilon}\Big[4\ln\tau_2 + \gamma_2\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}})\Big]$.

We recall that $\mathbf{d} = r_\varepsilon(\tilde{\mathbf{x}} - \mathbf{x})$, in addition the previous systems can be written as:

$$\Big(\mathbf{d},\ t_i,\ \lambda_i,\ \varphi^i,\ \tilde{\varphi}^i,\ \psi^i,\ \tilde{\psi}^i,\ \bar{\nabla}\mathcal{E}_i\Big) = \mathcal{O}(r_\varepsilon^2).$$

Combining (2.46) and (2.56), we have

$$T_\varepsilon = (T_\varepsilon^1, T_\varepsilon^2) = \Big(\mathcal{O}_1(r_\varepsilon^2), \mathcal{O}_2(r_\varepsilon^2)\Big) = \mathcal{O}(r_\varepsilon^2) \tag{2.57}$$

Then the nonlinear mapping which appears on the right hand side of (2.57) is continuous, compact. In addition, reducing ε_κ if necessary, this nonlinear mapping sends the ball of radius $\kappa\, r_\varepsilon^2$ (for the natural product norm) into itself, provided κ is fixed large enough. Applying Schauder's fixed point Theorem in the ball of radius κr_ε^2 in the product space where the entries live, we obtain the existence of a solution of equation (2.57).

This completes the proof of Theorem 1.2.

2.5 Sketch of the proof of Theorem 1.3

We choose as an approximate solution $(\tilde{u}_1, \tilde{u}_2)$ of (1.2) such that

$$\begin{cases} \tilde{u}_{1,\varepsilon,\tau}(x) = u_{\varepsilon,\tau}(x - x^1), & x \in B_{r_\varepsilon}(x^1) \\ \tilde{u}_{1,\varepsilon,\tau}(x) = G(x, x^1), & x \in \Omega \backslash B_{r_\varepsilon}(x^1) \end{cases} \tag{2.58}$$

and $\tilde{u}_{2,\varepsilon,\tau}(x) \equiv 0$ in Ω. In $B_{r_\varepsilon}(x^1)$, we have

$$\begin{cases} \Delta^2 \widetilde{u}_{1,\varepsilon,\tau} - \rho^4 e^{\widetilde{u}_{1,\varepsilon,\tau} + \gamma_1 \widetilde{u}_{2,\varepsilon,\tau}} = 0 \\ \Delta^2 \widetilde{u}_{2,\varepsilon,\tau} - \rho^4 e^{\widetilde{u}_{2,\varepsilon,\tau} + \gamma_2 \widetilde{u}_{1,\varepsilon,\tau}} = \dfrac{-384\ \varepsilon^4\ \tau^{4\gamma_2}}{(1+\varepsilon^2)^{4(1-\gamma_2)}\ (\varepsilon^2 + \tau^2\ |x - x^1|^2)^{4\gamma_2}} \end{cases} \tag{2.59}$$

and we get the same estimation as in (2.10). We complete the proof of Theorem 1.3 by following the steps of the proof of Theorem 1.2. Note that the functional $\mathcal{F}$ will be replaced by the Robin function $R(x)$ and we need only to study the system near the point x^1 .

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