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A primer on the group of self-homotopy equivalences: a rational homotopy theory approach

Cristina Costoya and Antonio Viruel*

Abstract

This manuscript is a revised version of the lecture notes of a course given in the 5th GeToPhyMa Summer School on "Rational Homotopy Theory and its Interactions" (July 11–21, 2016, Rabat, Morocco). It intends to be a gentle, while non trivial, introduction to the study of the group of self-homotopy equivalences from a Rational Homotopy Theory perspective. It also makes accessible some material and proofs that are not readily available in the literature.

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Introduction

When working in a category $\mathcal{C}$, understanding the group of automorphisms of an object X in $\mathcal{C}$, $\mathrm{aut}_{\mathcal{C}}(X)$, is of central interest, and indeed it is considered the historical source of Group Theory [59]. One may try to understand the object X by analyzing the algebraic structure of $\mathrm{aut}_{\mathcal{C}}(X)$. This is the key idea underlying the seminal work of Galois. Objects with a large group of automorphisms are thought of as *regular* ones, while objects with no automorphism at all (the so called *rigid objects*) are usually thought of as building blocks in $\mathcal{C}$.

One may also think that groups involved in the algebraic structure of $\mathrm{aut}_{\mathcal{C}}(X)$ should inherit the special features, or properties, the object X holds. Within this framework one wants to understand a group G by analyzing the objects $X \in \mathcal{C}$ such that $G \leq \mathrm{aut}_{\mathcal{C}}(X)$. These are the ideas behind Frobenius' work, laying the foundation of what is nowadays Representation Theory.

In these notes, we work in the homotopy category of (pointed) topological spaces. Therefore our objects are topological spaces, and the group of automorphisms of an object in our category is the group of self-homotopy equivalences of a space.

The study of groups of self-homotopy equivalences has been an active and relevant subject in Algebraic Topology during the last decades, and deserved its own code in the 2010 Mathematics Subject Classification: 55P10. Let us mention two important applications of the group of self-homotopy equivalences. First the set of CW-complexes with the same n-type for all n is determined by an inverse limit involving groups of self-homotopy equivalences [57]. Second, the study of self-homotopy equivalences of p-completed classifying spaces of Lie groups has led to what is nowadays called Homotopical Group Theory, with strong connections with Representation Theory [6].

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The paper is organized as follows. In Section 1 the notation is fixed, and the main objects of interest are introduced. That is, for a given space X, the space of self-homotopy equivalences of X, $\mathrm{aut}(X)$, and the group of self-homotopy equivalences of X, $\mathcal{E}(X)$, are defined. Free (or unpointed) versions of these concepts are also defined, $\mathrm{aut}_{free}(X)$ and $\mathcal{E}_{free}(X)$. Particulary, in Section 2 the spaces of (pointed and free) self-homotopy equivalences of Eilenberg-MacLane spaces are described. In Section 3 we consider finite type simply connected spaces, and review the work of Sullivan and Wilkerson showing that under these assumptions, the group of self-homotopy equivalences has nice algebraic properties. In Section 4 normal subgroups of the group of self-homotopy equivalences are constructed, and we show that some of these are nilpotent when considering finite type simply connected spaces. Section 5 is devoted to show that the group of self-homotopy equivalences that induce the identity morphism in homotopy groups, be-

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haves nicely with respect to localization functors. We then use Rational Homotopy Theory tools to describe the nilpotency degree of this group. Finally, Section 6 is devoted to the so called Realizability Problem.

1 Notation and basics in localization and minimal models

Given a space X, we denote by $\mathrm{aut}(X)$ (respectively by $\mathrm{aut}_{free}(X)$) the topological space of pointed (resp. free) self-homotopy equivalences of X.

Both $\mathrm{aut}(X)$ and $\mathrm{aut}_{free}(X)$ are group-like topological monoids with the composition of maps. We define the group of self-homotopy equivalences, denoted by $\mathcal{E}(X)$, to be the group of pointed homotopy classes of elements in $\mathrm{aut}(X)$, i.e. $\mathcal{E}(X) = \pi_0(\mathrm{aut}(X))$, and in a similar way we define $\mathcal{E}_{free}(X) = \pi_0(\mathrm{aut}_{free}(X))$ to be the group of free self-homotopy equivalences.

Given two spaces X and Y, we denote by $\mathrm{map}_*(X,Y)$ (resp. $\mathrm{map}(X,Y)$) the topological space of pointed (resp. free) continuous maps between X and Y. The set of pointed homotopy classes of continuous maps between X and Y is denoted by $[X,Y]$. Given a pointed (resp. free) map $f\colon X \to Y$, $\mathrm{map}_*(X,Y)_f$ (resp. $\mathrm{map}(X,Y)_f$) denotes the path component of $\mathrm{map}_*(X,Y)$ (resp. $\mathrm{map}(X,Y)$) containing f.

Given a group G, we denote by $\mathrm{Aut}(G)$ the automorphism group of G, while $\mathrm{Out}(G)$ is the group of outer-automorphisms of G [6, p. 4]. Therefore, if $Z(G)$ denotes the center G, there is a long exact sequence

$$1 \to Z(G) \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1,$$

where the image of G in $\mathrm{Aut}(G)$ is identified with the automorphisms of G induced by inner conjugation. We use multiplicative notation for G so the neutral element is 1. If the group G is nilpotent, we denote by $\mathrm{nil}(G)$ its nilpotency class [37, Chapter I, Section 1]. Given a group G and a positive integer n, $K(G,n)$ denotes the Eilenberg-MacLane space with a single non trivial homotopy group in dimension n which is isomorphic to G [36, §4.2].

We understand P-localization as in [37]. So, fix a collection of prime numbers $P \subset \mathbb{N}$, and define

$$P' = \{n \in \mathbb{N}\colon\ \gcd(p,n) = 1 \text{ for every } p \in P\}.$$

Then a multiplicative group G is said to be P-local [37, Definition I.1.1] if the map $g \mapsto g^n$, $g \in G$, is bijective for all $n \in P'$. When $P = \{p\}$ (resp. $P = \emptyset$), we shall say that G is p-local (resp. 0-local, or rational) instead of P-local.

According to the Fundamental Theorem on the P-localization of Nilpotent Groups [37, p. 7], there exists a P-localization theory on $\mathcal{N}il_c$, the category of nilpotent group of class at most c. That is, for every nilpotent group G there exists a group morphism $e\colon G \to G_{(P)}$, called the P-localization, such that

1. $G_{(P)}$ is P-local and $\mathrm{nil}(G_{(P)}) \leq \mathrm{nil}(G)$, and
2. for every group morphism $f\colon G \to K$ where K is a P-local nilpotent group, there exists a unique $f_{(P)}\colon G_{(P)} \to K$ such that $f = f_{(P)} \circ e$.

Then, group theoretical P-localization gives rise to topological P-localization as follows. A nilpotent pointed topological space X is said to be P-local [37, Definition II.3.1] if $\pi_n(X)$ is a P-local group for every $n > 0$, and there exists a P-localization theory on nilpotent pointed spaces [37, Theorems II.3A and II.3B]. That is, for every nilpotent pointed topological space X, there exists a continuous map $e\colon X \to X_{(P)}$, called the P-localization, such that

1. $X_{(P)}$ is P-local space and the group morphisms
$$\pi_n(e)\colon \pi_n(X) \to \pi_n(X_{(P)})$$
and
$$H_n(e)\colon H_n(X;\mathbb{Z}) \to H_n(X_{(P)};\mathbb{Z})$$
are group theoretical P-localizations for every $n \geq 1$, and
2. for every continuous map $f\colon X \to Y$, where Y is a P-local nilpotent pointed space, there exists a unique homotopy class of continuous maps
$$f_{(P)}\colon X_{(P)} \to Y$$
such that $f \simeq f_{(P)} \circ e$.

Basic facts on the theory of minimal models are recalled now, we refer to [28] for more details. Let Z be a graded vector space. The free commutative graded algebra on Z, ΛZ, is by definition the tensor product of the symmetric algebra on Z^{even} with the exterior algebra on Z^{odd}.

A commutative differential graded algebra (A,d) is called a Sullivan algebra if as an algebra $A = \Lambda Z$ for some Z and Z admits a basis $\{x_\alpha\colon \alpha \in I\} \subset A$ indexed by a well-ordered set I such that $d(x_\alpha) \in \Lambda(x_\beta)_{\beta<\alpha}$. The Sullivan algebra $(\Lambda Z,d)$ is called minimal if $dZ \subset \Lambda^{\geq 2}Z$. For any commutative differential graded algebra $(A,\ d)$ whose cohomology is connected and finite type, there is a unique (up to isomorphism) minimal algebra $(\Lambda Z,d)$ provided with a quasi-isomorphism $\phi : (\Lambda Z,d) \to (A,d)$. The differential algebra is called a minimal model of (A,d). Henceforth, we will indistinctly talk about a minimal Sullivan algebra $\mathcal{M}$ or the rational homotopy type X uniquely determined by $\mathcal{M}$.

Finally, for a given space X and $m \in \mathbb{N}$ the m-fold cartesian product of X is denoted by $X^{\times m}$, the m^{th}-stage Postnikov piece of X is denoted by $X^{(m)}$, and the m-connected cover of X is denoted by $X\langle m\rangle$. If moreover X is a CW-complex, the m^{th}-dimensional skeleton of X is denoted by $X^{[m]}$.

2 Pointed versus free homotopy equivalences: the case of Eilenberg-MacLane spaces

In this section we compute the space of pointed (resp. free) self-homotopy equivalences of Eilenberg-MacLane spaces in order to illustrate how different pointed and free homotopy equivalences can be.

Notice that for a given space X, both $\operatorname{aut}(X)$ and $\operatorname{aut}_{free}(X)$ have the weak homotopy type of loop spaces. That is, there exist spaces $B\operatorname{aut}(X)$ and $B\operatorname{aut}_{free}(X)$ such that $\Omega B\operatorname{aut}(X) \underset{\text{w.e.}}{\simeq} \operatorname{aut}(X)$ and

$$\Omega B\operatorname{aut}_{free}(X) \underset{\text{w.e.}}{\simeq} \operatorname{aut}_{free}(X),$$

as topological monoids [45]. Therefore, the homotopy groups of $\operatorname{aut}(X)$ and $\operatorname{aut}_{free}(X)$ are completely determined by the spaces $B\operatorname{aut}(X)$ and $B\operatorname{aut}_{free}(X)$, and thus so are $\mathcal{E}(X)$ and $\mathcal{E}_{free}(X)$.

We first describe the homotopy groups of $B\operatorname{aut}\Big(K(G,n)\Big)$, originally calculated by Hansen [35, Theorem 1 and 2] and Gottlieb [34, Lemma 2].

Theorem 2.1. *Let G be a group, and n be a positive integer. Then*

$$\pi_i B\operatorname{aut}\Big(K(G,n)\Big) = \begin{cases} \operatorname{Aut}(G), & \text{for } i=1,\\ 1, & \text{otherwise.}\end{cases}$$

Therefore $\mathcal{E}\Big(K(G,n)\Big) \cong \operatorname{Aut}(G)$.

Proof. Recall that for any space X we have

$$\Omega B\operatorname{aut}(X) \underset{\text{w.e.}}{\simeq} \operatorname{aut}(X) = \bigcup_{f\in\mathcal{E}(X)} \operatorname{map}_*(X,X)_f.$$

Moreover, by the Brown Representability Theorem [36, Theorem 4.57]

$$\pi_0 \operatorname{map}_*\Big(X, K(G,r)\Big) \cong H^r(X;G),\ r>1, \tag{2.2}$$

while by [36, Proposition 1B.9]

$$\pi_0 \operatorname{map}_*\Big(X, K(G,1)\Big) \cong \operatorname{Hom}\Big(\pi_1 X, G\Big). \tag{2.3}$$

Assume first $X = K(G,n)$, $n>1$. By Hurewicz Theorem [36, Theorem 4.32], $H_n((K(G,n);\mathbb{Z}\Big) \cong G$, while the Universal Coefficient Theorem shows that

$$\begin{aligned} H^n\Big(K(G,n);G\Big) &\cong \operatorname{Hom}\Big(H_n(K(G,n);\mathbb{Z}),G\Big)\\ &= \operatorname{Hom}(G,G).\end{aligned}$$

Hence according to Equation (2.2),

$$\pi_0 \operatorname{map}_*\Big(K(G,n),K(G,n)\Big) \cong \operatorname{Hom}(G,G)$$

Assume now $X = K(G,1)$, hence G may not be abelian. Then according to Equation (2.3),

$$\pi_0 \operatorname{map}_*\Big(K(G,1),K(G,1)\Big) \cong \operatorname{Hom}(G,G)$$

In other words, for any $n>0$, there is a bijective correspondence between pointed homotopy classes of self-maps of $K(G,n)$ and endomorphisms of G such that $f\in\operatorname{map}_*\Big(K(G,n),K(G,n)\Big)$ is mapped to $\pi_n f\in \operatorname{Hom}(G,G)$. Since $f\in\operatorname{aut}\Big(K(G,n)\Big)$ if and only if $\pi_n f \in \operatorname{Aut}\Big(\pi_n K(G,n)\Big) = \operatorname{Aut}(G)$ by Whitehead Theorem [36, Theorem 4.5], then

$$\pi_1 B\operatorname{aut}\Big(K(G,n)\Big) = \pi_0\operatorname{aut}\Big(K(G,n)\Big) = \operatorname{Aut}(G).$$

We now calculate the higher homotopy groups of $B\operatorname{aut}\Big(K(G,n)\Big)$. Assume $i>1$, and notice that,

$$\begin{aligned}
&\pi_i B\operatorname{aut}\Big(K(G,n)\Big)\\
&= \pi_{i-1}\Big(\operatorname{aut}\Big(K(G,n)\Big);Id\Big), \text{(choose base point)}\\
&= \pi_{i-1}\operatorname{map}_*\Big(K(G,n),K(G,n)\Big)_{Id}\\
&\text{(by adjoint relation [36, p. 395] we obtain)}\\
&\cong \pi_0\Big(\bigcup_{\{g\,|\,ad(g)\simeq Id\}} \operatorname{map}_*\Big(\Sigma^{i-1}K(G,n),K(G,n)\Big)_g\Big)\\
&\leq \pi_0\operatorname{map}_*\Big(\Sigma^{i-1}K(G,n),K(G,n)\Big)\\
&\text{(by (2.3) for } n=1\text{, and (2.2) for } n>1\text{, we get)}\\
&= \begin{cases}\operatorname{Hom}\Big(\pi_1\Sigma^{i-1}K(G,1),G\Big), & \text{for } n=1,\\ H^n\Big(\Sigma^{i-1}K(G,n);G\Big), & \text{for } n>1.\end{cases}
\end{aligned}$$

Since the reduced suspension increases connectivity, and $i>1$, then $\pi_n\Sigma^{i-1}K(G,n)=1$. Therefore

$$\operatorname{Hom}\Big(\pi_1\Sigma^{i-1}K(G,1),G\Big)=1,$$

while the Universal Coefficient and Hurewizc Theorems show that $H^n\Big(\Sigma^{i-1}K(G,n);G\Big) = 1$ for $n>1$. In other words, for any $n>0$, $\pi_i B\operatorname{aut}\Big(K(G,n)\Big)=1$ for $i>1$. □

In the following theorem we describe the homotopy groups of $B\operatorname{aut}_{free}\Big(K(G,n)\Big)$.

Theorem 2.4. *Let G be a group, and n be a positive integer. Then*

$$\pi_i B\operatorname{aut}_{free}\Big(K(G,n)\Big) = \begin{cases}\operatorname{Out}(G), & \text{for } i=1,\\ Z(G), & \text{for } i=n+1,\\ 1, & \text{otherwise.}\end{cases}$$

Proof. Recall (e.g. [50, Proposition 1.2]) that the universal Hurewicz fibration with fibre X is

$$X \xrightarrow{\ i\ } B\operatorname{aut}(X) \to B\operatorname{aut}_{free}(X). \tag{2.5}$$

When considering $X = K(G,1)$, we can apply Theorem 2.1 to reduce the long exact sequence of homotopy groups associated to (2.5), to the following exact sequence:

$$1 \to \pi_2\Big(B\,\mathrm{aut}_{free}(X)\Big) \to G \xrightarrow{\pi_1 i} \pi_1\Big(B\,\mathrm{aut}(X)\Big) \to \pi_1\Big(B\,\mathrm{aut}_{free}(X)\Big) \to 1.$$

The image of G by $\pi_1 i : G \to \pi_1\Big(B\,\mathrm{aut}(X)\Big) = \mathrm{Aut}(G)$ can be identified with the group of inner automorphisms [6, p. 4]. Therefore $\pi_2\Big(B\,\mathrm{aut}_{free}(X)\Big)$ must be isomorphic to $Z(G)$, the center of G, while $\pi_1\Big(B\,\mathrm{aut}_{free}(X)\Big)$ must be isomorphic to $\mathrm{Out}(G)$. This completely describes the homotopy groups of $B\,\mathrm{aut}\Big(K(G,1)\Big)$.

Similar reasoning can be made for $X = K(G,n)$, $n > 1$. In this case G must be abelian, thus $\mathrm{Aut}(G) = \mathrm{Out}(G)$ and $Z(G) = G$, while the long exact sequence of homotopy groups associated to (2.5) becomes

$$1 \to \pi_{n+1}\Big(B\,\mathrm{aut}_{free}(X)\Big) \to \pi_n K(G,n) = G \to 1 \to \cdots \to 1 \to \begin{matrix}\pi_1\Big(B\,\mathrm{aut}(X)\Big) \\ = \mathrm{Aut}(G)\end{matrix} \to \pi_1\Big(B\,\mathrm{aut}_{free}(X)\Big) \to 1,$$

and the result follows. □

Arguments above show how self-homotopy equivalences of Eilenberg-MacLane spaces reduce to Group Theory.

The following result illustrates how different the pointed and the free cases are. One of the classical problems in the study of the group of *pointed* self-homotopy equivalences of *connected* spaces, is the so-called Realizability Problem (see Section 6). Though relevant progress has been made in the case of finite groups [14, 15, 16, 17], and of algebraic linear groups [18], the general question still remains open. In contrast, if we consider the group of *free* self-homotopy equivalences we can prove the following:

Proposition 2.6. *Let G be a group. Then there is a space X such that $\mathcal{E}_{free}(X) \cong G$.*

Proof. Given G, there exists a simple group H such that $\mathrm{Out}(H) \cong G$ [24, Theorem 1.1]. Define $X = K(H,1)$. Since H is simple $Z(H) = 1$, and therefore $\pi_1 B\,\mathrm{aut}_{free}(X) \cong G$ by Theorem 2.4. Hence $\mathcal{E}_{free}(X) \cong G$. □

The technique used in the proof above can be applied in order to construct interesting examples. For any space X, let $\mathrm{cat}(X)$, $\mathrm{cocat}(X)$, and $\mathrm{TC}_k(X)$, denote the Lusternik-Schnirelmann category [13, Definition 1.1], cocategory [47, Definition 3.4], and the k-higher topological complexity [49, Definition 3.1] of X respectively. Then we can prove (compare with [52, Problem 2.7]):

Proposition 2.7. *Let $n \in \mathbb{N}$. Then:*

1. *There is a space X_1 such that*
$$\mathrm{cat}\Big(B\,\mathrm{aut}_{free}(X_1)\Big) = n.$$
2. *There is a space X_2 such that*
$$\mathrm{cocat}\Big(B\,\mathrm{aut}_{free}(X_2)\Big) = n.$$
3. *There is a space X_3 such that for every $k \in \mathbb{N}$,*
$$\mathrm{TC}_k\Big(B\,\mathrm{aut}_{free}(X_3)\Big) = (k-1)n$$

Proof. Consider $G_1 = \mathbb{Z}^{\oplus n}$, the direct sum of n copies of the integers, and $G_2 = D_{2^{n+1}}$, the dihedral group of order 2^{n+1}. For $i = 1,2$, G_i is countable, hence there exists a 2-generated group N_i such that $\mathrm{Out}(H_i) \cong G_i$ [11, Theorem 11]. Moreover, N_i is not cyclic [11, Lemma 10], and therefore $Z(H_i) = 1$ [11, Corollary 4].

Define $X_i = K(H_i,1)$. Since $Z(H_i) = 1$, then

$$B\,\mathrm{aut}_{free}(X_i) \underset{\text{w.e.}}{\simeq} K(G_i,1) \tag{2.8}$$

by Theorem 2.4. Recall H_i is 2-generated, hence we may assume $X_i^{[1]} = S^1 \vee S^1$, and therefore $\mathrm{map}(X_i,X_i)$ has the homotopy type of a CW-complex [40, Theorem 1.1]. Since $\mathrm{aut}_{free}(X_i)$ consists on path components of $\mathrm{map}(X_i,X_i)$, then $\mathrm{aut}_{free}(X_i)$ has the homotopy type of a CW-complex, and so $B\,\mathrm{aut}_{free}(X_i)$ has [45, Proposition 7.2]. Therefore, the weak equivalence given in Equation (2.8) is an actual homotopy equivalence by Whitehead Theorem [36, Theorem 4.5].

Now, $K(G_1,1) = (S^1)^{\times n}$ and therefore

$$\mathrm{cat}\Big(B\,\mathrm{aut}_{free}(X_1)\Big) = \mathrm{cat}\Big((S^1)^{\times n}\Big) = n$$

[13, Example 1.8]. On the other hand,

$$\begin{aligned}\mathrm{cocat}\Big(B\,\mathrm{aut}_{free}(X_2)\Big) &= \mathrm{cocat}\Big(K(D_{2^{n+1}},1)\Big) \\ &= \mathrm{nil}(D_{2^{n+1}}) = n\end{aligned}$$

see [47, Section 4] and [33, Theorem 2.5]. Finally, define $X_3 = X_1$, and notice that $B\,\mathrm{aut}_{free}(X_3) = (S^1)^{\times n}$ is indeed an H-space. Therefore, according to [41, Theorem 1] we get that

$$\begin{aligned}\mathrm{TC}_k\Big(B\,\mathrm{aut}_{free}(X_3)\Big) &= \mathrm{cat}\Big(B\,\mathrm{aut}_{free}(X_3)^{k-1}\Big) \\ &= \mathrm{cat}\Big((S^1)^{\times(k-1)n}\Big) \\ &= (k-1)n.\end{aligned}$$

□

Observe that the groups provided by [24, Theorem 1.1] and [11, Theorem 11] are not finitely presented, and therefore, the related Eilenberg-MacLane spaces are not of finite type. Hence, spaces given by Propositions 2.6 and 2.7 may be somehow considered as "non-natural" ones.

We now show how one can get around base point issues while loosing the connectivity of our spaces. Given a space X, let X_+ denote the disjoint union of X and a singleton $\{\star\}$. So $X_+ = X \cup \{\star\}$ is thought of as a non connected pointed topological space with base point $\star$. Then, the inclusion $X \subset X_+$ induces a homeomorphism of topological monoids $\mathrm{aut}_{free}(X) \cong \mathrm{aut}(X_+)$ that gives rise to the pointed version of Propositions 2.6 and 2.7.

Proposition 2.9. *Let G be a group. Then there is a (non connected, non finite type) space X such that $B\,\mathrm{aut}(X) \simeq K(G,1)$, and therefore $\mathcal{E}(X) \cong G$.*

Proposition 2.10. *Let $n \in \mathbb{N}$. Then:*

1. *There is a (non connected, non finite type) space X_1 such that $\mathrm{cat}\Big(B\,\mathrm{aut}(X_1)\Big) = n$.*
2. *There is a (non connected, non finite type) space X_2 such that $\mathrm{cocat}\Big(B\,\mathrm{aut}(X_2)\Big) = n$.*
3. *There is a (non connected, non finite type) space X_3 such that $\mathrm{TC}_k\Big(B\,\mathrm{aut}(X_3)\Big) = (k-1)n$ for every $k \in \mathbb{N}$.*

3 Spaces with finiteness conditions

The previous section indicates that we have to consider spaces X which are simply connected, so $\mathcal{E}(X) = \mathcal{E}_{free}(X)$ [50, Theorem 1.3] while keeping finite type. Under these hypothesis, we leave behind pure group theoretical arguments to introduce Rational Homotopy Theory arguments. To that purpose, we review the work of Wilkerson [58] and Sullivan [54].

We say that two groups G and H are commensurable [58, Notation 0.5] if there exists a finite string of group morphisms

$$G \to G_1 \leftarrow G_2 \dots G_{i-1} \to G_i \leftarrow G_{i+1} \dots \leftarrow H$$

such that each map has finite kernel and the image has finite index. Notice that the notion of commensurability here is what in Geometric Group Theory is called commensurability up to finite kernels [20, Definition IV.27.(ii)].

If G and H are finitely generated nilpotent groups, G and H are commensurable if and only if the rationalizations of G and H are isomorphic, $G_{(0)} \cong H_{(0)}$ [37, Theorem 3.3]. Finite presentation is preserved by this commensurability relation [20, Remark IV.29.(i)].

A matrix group $G \leq GL(n,\mathbb{C})$ is a linear algebraic group defined over $\mathbb{Q}$ if it consists of all invertible matrices whose coefficients annihilate some set of polynomials with rational coefficients $\{P_\mu[X_{1,1},\dots,X_{n,n}]\}$ in n^2 indeterminates. Given a subring $R \subset \mathbb{C}$, let G_R be the subgroup of elements in G that have coefficients in R and whose determinant is a unit of R. Then we say that $G_{\mathbb{Z}}$ is an arithmetically defined subgroup of G_R, or more briefly, an arithmetic subgroup of G_R [8]. Arithmetic subgroups of G_R are finitely presented groups, and their finite subgroups form a finite number of conjugacy classes [7, Section 5].

We can now state Sullivan-Wilkerson's theorem [58, Theorem B], [54, Theorem (10.3)]:

Theorem 3.1 (Sullivan-Wilkerson)**.** *Let X and Y be simply connected finite CW-complexes. Then*

(1) $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are commensurable groups if $X_{(0)} \simeq Y_{(0)}$.

(2) $\mathcal{E}(X_{(0)}) \cong G_{\mathbb{Q}}$ for a linear algebraic group G defined over $\mathbb{Q}$, and $\mathcal{E}(X)$ is commensurable with an arithmetic subgroup of $\mathcal{E}(X_{(0)})$. Hence $\mathcal{E}(X)$ is a finitely presented group.

The techniques used by Wilkerson and Sullivan, respectively, have different flavors: while Wilkerson uses simplicial nilpotent group approximations for spaces, Sullivan's proof relies on Postnikov systems and Sullivan minimal models. We sketch Sullivan's approach since it also covers the case of finite type CW-complexes with finitely many homotopy groups, and the arguments will be useful in this exposition.

The first step is to describe the homotopy groups of simply connected finite CW-complexes. Recall that given $\mathcal{S}$, a Serre class of abelian groups, see [51, Chapitre I] or [53, Section 9.6], a group morphism $f\colon A \to B$ is called a $\mathcal{S}$-isomorphism if $\ker f \in \mathcal{S}$ and $\operatorname{coker} f \in \mathcal{S}$ [51, p. 260], [53, p. 505].

Proposition 3.2. *Let X be a simply connected finite CW-complex. Then for any $m \in \mathbb{N}$, $\pi_m(X)$ is a finitely generated abelian group.*

Proof. Let $\mathcal{S}$ be the Serre class of finitely generated abelian groups, and consider the constant map $f\colon X \to *$. Since both X and $*$ are simply connected spaces and $\pi_2(f)$ is surjective, according to [51, Chapitre III, Théorème 3], $H_m(f)$ is a $\mathcal{S}$-isomorphism for every $m \in \mathbb{N}$, if and only if $\pi_m(f)$ is so. Since X is finite, $H_m(X)$ is finitely generated, and therefore $H_m(f)$ is a $\mathcal{S}$-isomorphism for every $m \in \mathbb{N}$. Then $\pi_m(f)$ is also a $\mathcal{S}$-isomorphism, which immediately implies $\pi_m(X) \in \mathcal{S}$. □

We now show that computing pointed self-homotopy equivalences of simply connected finite dimensional CW-complexes is equivalent to computing pointed self-homotopy equivalences of finite stage Postnikov pieces. Given an space X and $n \in \mathbb{N}$, the n^{th}-stage Postnikov piece of X, $X^{(n)}$, and the n-connected cover of X, $X\langle n\rangle$, fit in a fibration sequence

$$X\langle n\rangle \to X \xrightarrow{p_n} X^{(n)}. \tag{3.3}$$

Proposition 3.4. *Let X be a finite dimensional CW-complex, and let $n = \dim X$. Then $[X, X] \cong [X^{(n)}, X^{(n)}]$ as monoids, thus $\mathcal{E}(X) \cong \mathcal{E}(X^{(n)})$ as groups.*

Proof. Recall that the construction of the n^{th}-stage Postnikov piece of a space can be done functorially [21, Example 1.E.1], such that the map p_n in (3.3) is homotopy universal with respect to maps $X \to Y^{(n)}$ for any space Y [21, Definition A.2]. Therefore, we do have a well defined monoid morphism

$$\begin{array}{rcl} \psi\colon [X,X] & \to & [X^{(n)}, X^{(n)}] \\ f & \mapsto & \psi(f) = f^{(n)}. \end{array}$$

Moreover, $X^{(n)}$ can be constructed from X by adding m-cells for $m > n$ [36, Example 4.17], hence we may assume that X is the n-skeleton of $X^{(n)}$ and p_n is the inclusion $X \subset X^{(n)}$. Now, by the cellular approximation theorem [36, Theorem 4.8], any map $g\colon X^{(n)} \to X^{(n)}$ can be thought cellular and therefore induces a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p_n} \\ X^{(n)} & \xrightarrow{g} & X^{(n)} \end{array}$$

where $f = g|_X$. Since p_n is homotopy universal with respect to maps $X \to X^{(n)}$, then $g = f^{(n)}$, that is $\psi(f) = g$. Hence ψ is surjective.

To show that ψ is injective we use classical obstruction theory associated to the lifting problem for the fibration (3.3): given $f \in [X, X]$, the obstruction to f being the unique lifting g of $f^{(n)} p_n$, as in the diagram

$$\begin{array}{ccccc} & & & \overset{g}{\dashrightarrow} & X \\ & & & & \downarrow{\scriptstyle p_n} \\ X & \xrightarrow{p_n} & X^{(n)} & \xrightarrow{f^{(n)}} & X^{(n)}, \end{array}$$

is a class that lives in $H^m\Big(X, \pi_m(X\langle n\rangle)\Big)$ (e.g. see [36, Problem 24, p. 420]). Since X is n-dimensional while $X\langle n\rangle$ is n-connected, $H^m\Big(X, \pi_m(X\langle n\rangle)\Big) = 0$ for every $m > 0$. Thus $g = f$ and ψ is injective. □

If X is a finite stage Postnikov piece, namely $X = X^{(n)}$, whose homotopy groups are finitely generated, then $\mathcal{M}$, the minimal Sullivan model of $X_{(0)}$, is a finitely generated DGA. We now describe $\mathcal{E}(X_{(0)})$:

Lemma 3.5. *Let $X_{(0)}$ be a rational 1-connected space with finitely generated minimal Sullivan model. Then $\mathcal{E}(X_{(0)}) = G_{\mathbb{Q}}$ for a linear algebraic group G defined over $\mathbb{Q}$.*

Proof. Let $\mathcal{M}$ be the minimal Sullivan model of $X_{(0)}$. Then $\mathcal{E}(X_{(0)}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Aut}_1(\mathcal{M})$ where $\mathrm{Aut}_1(\mathcal{M})$ is the normal subgroup of automorphisms which are homotopy equivalent to the identity. Now, $\mathrm{Aut}(\mathcal{M})$ is the group of rational points of an algebraic linear group defined over $\mathbb{Q}$ since every automorphism of $\mathcal{M}$ is a $\mathbb{Q}$-linear map subject to the multiplicative and differential structure of $\mathcal{M}$ (which can be codified by polynomials with rational coefficients). Moreover, $\mathrm{Aut}_1(\mathcal{M})$ is the group of unipotent elements $f \in \mathrm{Aut}(\mathcal{M})$ of the form $f = Idx_{\mathcal{M}} + e^{iod+doi}$ where i is a derivation of degree -1. Hence $\mathcal{E}(X_{(0)})$ is a quotient of rational points of algebraic groups defined over $\mathbb{Q}$ and therefore, it is so. □

We have all the ingredients for:

Sketch of proof of Theorem 3.1. According to Proposition 3.4, we may assume that X is a finite stage Postnikov piece, namely $X = X^{(n)}$. We proceed by induction on n.

For $n = 2$, since X is simply connected, $X = K(\pi, 2)$ and $X_{(0)} = K(\pi \otimes \mathbb{Q}, 2)$. Let $Tor(\pi)$ denote the normal subgroup of π generated by torsion elements, and define the quotient $Free(\pi) = \pi/Tor(\pi)$. Notice that since π is finitely generated, $Free(\pi)$ is a finite direct sum of copies of $\mathbb{Z}$, and $\pi \otimes \mathbb{Q} \cong Free(\pi) \otimes \mathbb{Q}$. Therefore, $\mathcal{E}(X) = \mathrm{Aut}(\pi)$ is commensurable with $\mathrm{Aut}(Free(\pi))$, which is an arithmetic subgroup of

$$\mathrm{Aut}(Free(\pi) \otimes \mathbb{Q}) = \mathcal{E}\Big(K(\pi \otimes \mathbb{Q}, 2)\Big) = \mathcal{E}(X_{(0)}).$$

Assume now the result holds for $(n-1)^{th}$-stage Postnikov pieces. Then, following the ideas in the proof of Proposition 3.4, obstruction theory gives rise to a diagram of exact sequences:

$$\begin{array}{ccccc} H^n(X; \pi_n X) & \longrightarrow & \mathcal{E}(X) & \longrightarrow & \mathcal{E}(X^{(n-1)}) \\ \downarrow & & \downarrow & & \downarrow \\ H^n(X_{(0)}; \pi_n X \otimes \mathbb{Q}) & \longrightarrow & \mathcal{E}(X_{(0)}) & \longrightarrow & \mathcal{E}(X_{(0)}^{(n-1)}), \end{array} \tag{3.6}$$

where $H^n(X; \pi_n X)$ is commensurable with an arithmetic subgroup of $H^n(X_{(0)}; \pi_n X \otimes \mathbb{Q})$, $\mathcal{E}(X^{(n-1)})$ is commensurable with an arithmetic subgroup of $\mathcal{E}(X_0^{(n-1)})$ by induction, and therefore $\mathcal{E}(X)$ is commensurable with an arithmetic subgroup of $\mathcal{E}(X_{(0)})$ by [56, Proposition (3.3)]. □

Along the proof of Theorem 3.1 we have shown that the commensurability of $\mathcal{E}(X)$ is obtained via the group morphism $\mathcal{E}(X) \xrightarrow{\vartheta_0} \mathcal{E}(X_{(0)})$ induced by rationalization. Thus,

Corollary 3.7. *Let X be a simply connected finite CW-complex. Then the kernel of the group morphism*

$$\mathcal{E}(X) \xrightarrow{\vartheta_0} \mathcal{E}(X_{(0)})$$

is finite. Therefore if $\mathcal{E}(X_{(0)})$ is finite then $\mathcal{E}(X)$ is so.

Notice that the converse of the second statement of Corollary 3.7 does not hold in general: taking $X = S^n$ we obtain that $\mathcal{E}(X) \cong \mathbb{Z}/2$ is finite while $\mathcal{E}(X_{(0)}) \cong \mathbb{Q}^*$, the multiplicative group of non zero rational numbers, is not.

We finish this section by pointing out that although $\mathcal{E}(X)$ is finitely presented for X a virtually nilpotent finite CW-complex [22, Theorem 1.1], the thesis in Theorem 3.1.(2) cannot be extended to finite CW-complexes in general. We illustrate this fact with an easy example coming from Group Theory [10, Example 1]:

Given an integer $n > 1$, let $G(n)$ be the 1-relator group with presentation

$$G(n) = \langle a, t \colon t^{-1}a^{-1}ta^nt^{-1}ata^{-1} = a\rangle,$$

and consider $X = K(G(n), 1)$. Therefore $\mathcal{E}(X) \cong \operatorname{Aut}\big(G(n)\big)$, and $\mathcal{E}_{free}(X) \cong \operatorname{Out}\big(G(n)\big)$ by Section 2. Moreover, X is a finite dimensional CW-complex. Indeed, since the relator $t^{-1}a^{-1}ta^nt^{-1}ata^{-2}$ is not a proper power, then X is the presentation complex $X \simeq (S^1 \vee S^1) \cup e^2$ [25, Theorem 2.1], so X is a finite CW-complex of dimension 2. Since $G(n) = G(n, 1; 1)$ in [9], then $\operatorname{Out}\big(G(n)\big) = \mathbb{Z}[\frac{1}{n}]$, the additive group of rational numbers with denominator a non negative power of n [9, Theorem 3.3]. The group $\mathbb{Z}[\frac{1}{n}]$ is infinitely generated, since it is locally cyclic (every finitely generated subgroup is cyclic), but not cyclic. Therefore $\mathcal{E}_{free}(X)$ is infinitely generated and so is $\mathcal{E}(X)$, although X is a finite dimensional CW-complex.

4 Generating normal subgroups in $\mathcal{E}(X)$

It is commonly accepted that the study of a group G should be reduced to the study of its composition series (if it exists!), that is, to the study of a finite chain of subnormal subgroups

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

such that each factor G_i/G_{i-1} is a simple group. The idea is that, by an inductive process, the group theoretical properties of G_i should be related to those of G_{i-1} and G_i/G_{i-1}, which are assumed to be structurally less complicated groups.

Therefore, we are interested in finding normal subgroups of $\mathcal{E}(X)$. The standard technique goes as follows: given a category $\mathcal{C}$, and a functor $F \colon \mathcal{H}oTop_* \rightsquigarrow \mathcal{C}$, for every space X we obtain a well defined group morphism

$$F_\sharp \colon \mathcal{E}(X) \to \operatorname{Aut}_{\mathcal{C}}(F(X)).$$

Then, the kernel of $F_\sharp$, denoted by $\mathcal{E}_F(X)$, is a normal subgroup of $\mathcal{E}(X)$. In this way, we define the following classical normal subgroups of $\mathcal{E}(X)$:

1. For $\mathcal{C} = \mathcal{G}ps$ and $F(X) = \bigoplus_{i=1}^m \pi_i(X)$, then
$$\mathcal{E}_\sharp^{(m)}(X) = \mathcal{E}_F(X).$$
When X is finite dimensional, and $m = \dim(X)$, we simply write $\mathcal{E}_\sharp(X) = \mathcal{E}_\sharp^{(m)}(X)$.
2. For $\mathcal{C} = \mathcal{G}ps$ and $F(X) = \bigoplus_{i=1}^\infty H_i(X; A)$ where A is an abelian group, then $\mathcal{E}_*^A(X) = \mathcal{E}_F(X)$. When $G = \mathbb{Z}$, we simply write $\mathcal{E}_*(X) = \mathcal{E}_*^{\mathbb{Z}}(X)$.
3. For $\mathcal{C} = \mathcal{H}oTop_*$ and $F(X) = \Omega X$, then $\mathcal{E}_\Omega(X) = \mathcal{E}_F(X)$.
4. For $\mathcal{C} = \mathcal{H}oTop_*$ and $F(X) = \Sigma X$, then $\mathcal{E}_\Sigma(X) = \mathcal{E}_F(X)$.

We focus now on the groups $\mathcal{E}_\Omega(X)$ and $\mathcal{E}_\sharp^\infty(X)$. In general, $\mathcal{E}_\Omega(X) \leq \mathcal{E}_\sharp^\infty(X)$, and the inclusion may be strict [30, Remarks.(a)]. Nevertheless, in our setting they behave nicely:

Proposition 4.1. *Let X be a rational space with finitely generated Sullivan minimal model. Then*

$$\mathcal{E}_\Omega(X) = \mathcal{E}_\sharp^\infty(X),$$

and moreover it is a nilpotent group.

Proof. Let $\mathcal{M} = \big(\Lambda(x_1, \ldots, x_n), d\big)$ be a Sullivan minimal model for X, where $|x_i| \leq |x_{i+1}|$. Then a Sullivan minimal model for ΩX is $\overline{\mathcal{M}} = \big(\Lambda(y_1, \ldots, y_n), 0\big)$, where y_i is the looping of the class x_i (hence $|y_i| = |x_i| - 1$).

Define

$$\operatorname{Aut}_\sharp(\mathcal{M}) = \{g \in \operatorname{Aut}(\mathcal{M}) \colon g(x_i) = x_i + \text{decomposables}, i = 1, \ldots, n\},$$

that is, $\operatorname{Aut}_\sharp(\mathcal{M})$ is the subgroup of elements in $\operatorname{Aut}(\mathcal{M})$ that induce the identity on the module of indecomposable elements of $\mathcal{M}$ (the homotopy groups of X). Therefore $\mathcal{E}_\sharp^\infty(X)$ is the quotient of homotopy classes of elements in $\operatorname{Aut}_\sharp(\mathcal{M})$.

We now show $\mathcal{E}_\sharp^\infty(X) \leq \mathcal{E}_\Omega(X)$. Indeed, given $f \in \mathcal{E}_\sharp^\infty(X)$, with algebraic model $g \in \operatorname{Aut}_\sharp(\mathcal{M})$, the algebraic model for Ωf, must be defined by $\overline{g}(y_i) = y_i$. Hence $f \in \mathcal{E}_\Omega(X)$.

Finally, we prove that $\operatorname{Aut}_\sharp(\mathcal{M})$ is a nilpotent group, and therefore $\mathcal{E}_\sharp^\infty(X)$ is so. Let $m = |x_n|$, the highest degree among generators, and let $B = \{b_1, \ldots, b_s\}$ be a base of $\mathcal{M}^{\leq m}$ consisting of monomials written in standard form (i.e. $b_j = \prod_{i=1}^n x_i^{a_i(j)}$) and lexicographically ordered. Notice that the elements in B may not be ordered by degree, since every monomial containing x_1 will appear earlier in B than anyone without an x_1. Nevertheless, every decomposable element in B having the same degree as x_i must show up earlier than x_i since it has to contain some x_j for $j < i$.

Then every element $g \in \mathrm{Aut}_\sharp(\mathcal{M})$ is completely determined by the linear map induced by the restriction $g|_{\mathcal{M}^{\leq m}}$, and the matrix associated to $g|_{\mathcal{M}^{\leq m}}$ in base B is upper triangular. Therefore we can identify $\mathrm{Aut}_\sharp(\mathcal{M})$ with a subgroup of upper triangular matrices, and then $\mathrm{Aut}_\sharp(\mathcal{M})$ is nilpotent. □

The second part in the conclusion of Proposition 4.1 is a particular case of the celebrated Dror-Zabrodsky's Theorem [23, Theorem A].

Theorem 4.2 (Dror-Zabrodsky). *Let X be a finite dimensional CW-complex, $n = \dim(X)$. Then $\mathcal{E}_\sharp^{(n)}(X)$ is nilpotent.*

Proof. Recall that since X is a finite dimensional CW-complex, $n = \dim(X)$, then $\mathcal{E}(X) \cong \mathcal{E}(X^{(n)})$ (Proposition 3.4), and therefore we may assume that X is an n-stage Postnikov piece.

We proceed by induction on n. Applying obstruction theory and following the ideas in the proof of Proposition 3.4, the fibration

$$K(\pi_n X, n) \longrightarrow X \longrightarrow X^{(n-1)}$$

gives rise to an exact sequence

$$H^n(X;\pi_n X) \xrightarrow{i} \mathcal{E}_\sharp^{(n)}(X) \xrightarrow{p} \mathcal{E}_\sharp^{(n-1)}(X^{(n-1)}) \quad (4.3)$$

where $\mathcal{E}_\sharp^{(n-1)}(X^{(n-1)})$ is nilpotent by induction. Therefore p is an $\mathcal{E}_\sharp^{(n)}(X)$-equivariant map since $\mathcal{E}_\sharp^{(n)}(X)$ acts on $\mathcal{E}_\sharp^{(n-1)}(X^{(n-1)})$ nilpotently by inner conjugations.

Since $\mathcal{E}_\sharp^{(n)}(X)$ acts trivially on $\pi_n X$, an inductive argument on the Serre spectral sequence associated to the Postnikov decomposition of X shows that $\mathcal{E}_\sharp^{(n)}(X)$ acts nilpotently on $H^*(X;\pi_n X)$ (e.g. [19, p. 360]), and the map i is an $\mathcal{E}_\sharp^{(n)}(X)$-equivariant map. So, (4.3) is a sequence of $\mathcal{E}_\sharp^{(n)}(X)$-groups, where kernel and cokernel are $\mathcal{E}_\sharp^{(n)}(X)$-nilpotent. Therefore, $\mathcal{E}_\sharp^{(n)}(X)$ is $\mathcal{E}_\sharp^{(n)}(X)$-nilpotent (that is, nilpotent). □

Indeed, Dror and Zabrodsky proved that if X is a nilpotent finite dimensional CW-complex, and $G \leq \mathcal{E}(X)$ acts nilpotently on $H_j(X;\mathbb{Z})$ for $0 \leq j \leq n = \dim(X)$, then G is nilpotent [23, Theorem D]. So if X is a nilpotent finite dimensional CW-complex, then $\mathcal{E}_*^A(X)$ and $\mathcal{E}_*(X)$ are nilpotent groups [5, Proposition 4.9].

5 Properties of $\mathcal{E}_\sharp(X)$

We have seen in Theorem 3.1 that the group morphism $\mathcal{E}(X) \xrightarrow{\vartheta_0} \mathcal{E}(X_{(0)})$ has finite kernel, and the image has finite index in an arithmetic subgroup of $\mathcal{E}(X_{(0)})$. In this section we give a better description that can be obtained when we consider the restriction of ϑ_0 to $\mathcal{E}_\sharp(X)$. As it was pointed out in Section 1, given P, an arbitrary collection of prime numbers, the P-localization of connected simple spaces can be thought as the group theoretical P-localization at the level of homotopy groups [37, Theorem II.3B]. Moreover, the group theoretical P-localization, which usually refers to abelian groups, has a natural extension to the class of nilpotent groups [37, Chapter I]. Then we can prove [42, Theorem 0.1]:

Theorem 5.1 (Maruyama). *Let X be a simple CW-complex and P be an arbitrary collection of prime numbers. Assume that $m \geq \dim(X)$. Then the group morphism*

$$\mathcal{E}_\sharp^{(m)}(X) \xrightarrow{\vartheta_P} \mathcal{E}_\sharp^{(m)}(X_{(P)})$$

induced by the P-localization of spaces, is the P-localization at the level of groups. In other words $\mathcal{E}_\sharp^{(m)}(X)_{(P)} = \mathcal{E}_\sharp^{(m)}(X_{(P)})$.

Proof. This proof follows the ideas we have already developed. Since X is a finite dimensional CW-complex, say $n = \dim(X)$, then $\mathcal{E}(X) \cong \mathcal{E}(X^{(n)})$ (Proposition 3.4), and therefore we may assume that X is an n-stage Postnikov piece.

We proceed by induction on n. For $n = 1$, $X = K(G, 1)$ where G is abelian, and therefore

$$X_{(P)} = K(G \otimes \mathbb{Z}_{(P)}, 1).$$

Then $\mathcal{E}_\sharp^{(m)}(X) = \mathcal{E}_\sharp^{(m)}(X_{(P)}) = \{1\}$ for $m \geq 1$ and the result holds.

Now, assume the result holds for $n - 1 \geq 1$. Combining diagrams (3.6) and (4.3), and considering P-localization of spaces instead of rationalization, we obtain

$$\begin{array}{ccccc} H^n(X;\pi_n X) & \longrightarrow & \mathcal{E}_\sharp^{(m)}(X) & \longrightarrow & \mathcal{E}_\sharp^{(m)}(X^{(n-1)}) \\ \downarrow & & \downarrow & & \downarrow \\ H^n(X_{(P)};\pi_n X \otimes \mathbb{Z}_{(P)}) & \longrightarrow & \mathcal{E}_\sharp^{(m)}(X_{(P)}) & \longrightarrow & \mathcal{E}_\sharp^{(m)}(X_{(P)}^{(n-1)}), \end{array}$$

where the first and third vertical arrows are, respectively, the P-localization at the level of groups by the Universal Coefficient Theorem (see also [37, Theorem I.1.12 and Corollary I.1.14]) and the induction hypothesis. Finally, since P-localization is an exact functor [37, Theorem I.2.4], the vertical arrow in the middle is as well a P-localization. □

Different generalizations of Theorem 5.1 are possible. So, Møller, in [46], discusses the behaviour of $H_*(-,\mathbb{Z}/p)$- localization of spaces and Ext $-p$-completion of nilpotent groups when considering the group $\mathcal{E}_\sharp^{(m)}(X)$. But Maruyama in [44], considers the effect of p-completion on the groups $\mathcal{E}_*(X)$ and $\mathcal{E}_\Sigma(X)$.

Notice that since $\mathcal{E}_\sharp(X)$ and $\mathcal{E}_\sharp(X_{(0)})$ are homotopy invariants of X, all the algebraic invariants of these groups become homotopy invariants of X. In particular, the nilpotency class of $\mathcal{E}_\sharp(X_{(0)})$ is an invariant of X. The following result of Félix-Murillo [29, Theorem 1] relates this invariant with classical Lusternik-Schnirelmann category:

Theorem 5.2 (Félix-Murillo)**.** *Let X be a simply connected finite CW-complex. Then*

$$\mathrm{nil}\left(\mathcal{E}_\sharp(X_{(0)})\right) \leq \mathrm{cat}(X_{(0)}) - 1$$

Proof. Let $\mathcal{M} = \left(\Lambda V, d\right)$ be the minimal Sullivan model of X, and let $\mathrm{Aut}_\sharp(\mathcal{M})$ the group of automorphisms of $\mathcal{M}$ which induce the identity on indecomposables (see also the proof of Proposition 4.1).

We first describe the commutator of elements in $\mathrm{Aut}_\sharp(\mathcal{M})$. Given $g \in \mathrm{Aut}_\sharp(\mathcal{M})$, define $l(g)$ to be the length of the shortest decomposable term obtained by applying g to each generator of $\mathcal{M}$. According to [3, Corollary 3.3], if $g_i \in \mathrm{Aut}_\sharp(\mathcal{M})$, for $i = 1, \ldots, r$, and $[g_1, \ldots, g_r]$ is the r-fold commutator, then

$$l([g_1, \ldots, g_r]) \geq l(g_1) + \ldots + l(g_r) - (r-1). \quad (5.3)$$

Notice that $\mathrm{nil}\left(\mathcal{E}_\sharp(X_{(0)})\right) \leq r-1$ if and only if every r-fold commutator in $\mathrm{Aut}_\sharp(\mathcal{M})$ is homotopy equivalent to the identity. Moreover, if $g \in \mathrm{Aut}_\sharp(\mathcal{M})$ is not the identity map, then $l(g) \geq 2$.

We now describe $\mathrm{cat}(X_{(0)})$. Given $r \in \mathbb{N}$, let

$$\Phi_r \colon \mathcal{M} \to \left(\Lambda V \otimes Z(r), d\right)$$

be the relative Sullivan model of the canonical projection $p_r \colon \mathcal{M} \to \left(\Lambda V / \Lambda^{>r} V, d\right)$. Then $\mathrm{cat}(X_{(0)}) \leq r$ if and only if Φ_r admits a section [28, Proposition 29.4]. Notice that every element in $\Lambda^{>r}V$ is a boundary in $\left(\Lambda V \otimes Z(r), d\right)$, hence if Φ_r admits a section, $\Lambda^{>r}V$ is a boundary in $\mathcal{M}$.

Finally, assume that $\mathrm{cat}(X_{(0)}) \leq r$. Given $g_i \in \mathrm{Aut}_\sharp(\mathcal{M})$, for $i = 1, \ldots, r$, define $f = [g_1, \ldots, g_r]$. Then we have to show that $f \sim Id_{\mathcal{M}}$.

We may assume that $g_i \neq Id_{\mathcal{M}}$, for $i = 1, \ldots, r$, since otherwise $f = Id_{\mathcal{M}}$. Therefore

$$\begin{aligned} l([g_1, \ldots, g_r]) &\geq l(g_1) + \ldots + l(g_r) - (r-1) \\ &\geq 2r - r + 1 = r + 1, \end{aligned}$$

and for each generator x_i of $\mathcal{M}$ there exists $w_i \in \Lambda^{>r}V$ such that $f(x_i) = x_i + w_i$. Since $\mathrm{cat}(X_{(0)}) \leq r$, there exits $z_i \in \mathcal{M}$ such that $w_i = d(z_i)$, and therefore for each generator x_i of $\mathcal{M}$, $f(x_i) = x_i + d(z_i)$. Then, a standard argument shows that $f \sim Id_{\mathcal{M}}$. □

The previous theorem has an integral version in [30], where techniques out of the scope of these notes are used. The authors there prove that for a finite X, $\mathrm{nil}\left(\mathcal{E}_\Omega(X)\right) \leq \mathrm{cat}(X) - 1$.

6 The realization problem for self-homotopy equivalences

In the previous sections we have described properties of the group of self-homotopy equivalences and some of its subgroups. Nevertheless, we have not shown which groups can appear as the group $\mathcal{E}(X)$ for some space X. This is the so called *Realizability Problem* for groups of self-homotopy equivalences that we introduce here below:

Problem 6.1. *Given a group G, find a simply connected space X such that $\mathcal{E}(X) \cong G$. In that case we say that G can be realized as the group of self-homotopy equivalences of a space.*

This problem, considered by many authors (see for example [2, 38, 39, 50]), frequently appears in lists of open problems on self-homotopy equivalences (see [1] and also [27]). Apart from the case of $G = \mathrm{Aut}(\pi)$ and $X = K(\pi, n)$, for which we have shown that $\mathrm{Aut}(\pi) \cong \mathcal{E}(K(\pi, n))$ in Section 2, no systematic procedure was known to tackle this problem until [17]. Ad-hoc techniques are developed in literature, for example the infinite cyclic group $G = \mathbb{Z}$ is realized by a non-finite space in [38] or by a finite space in [43]. Also, finite cyclic groups (excluding a few cases of 2-torsion) are realized by finite spaces in [48]. The group $G = \mathbb{Z}/2$ deserves special mention since it appears as the group of self-homotopy equivalences of a rational space [4, Example 5.2], pointing out the surprising appearance of a non trivial finite group in rational homotopy theory, and providing a counterexample to an old conjecture of Copeland-Shar [12, Conj. 5.8]. This fact motivated the authors in [4] to raise the following question:

Problem 6.2. *Which finite groups can be realized as the group of self-homotopy equivalences of a rational space?*

Of course, the realizability problem can be approached by studying distinguished subgroups, like those described in Section 4. Within this setting, Federinov-Félix have shown that every 2-solvable rational nilpotent group is realizable as $\mathcal{E}_\sharp(X)$ for some simply connected rational space X [26, Theorem 1].

We devote what remains of this section to overview the solution to Problem 6.2 [17]:

Theorem 6.3 (Costoya-Viruel)**.** *Every finite group G can be realized as the group of self-homotopy equivalences of infinitely many (non homotopy equivalent) rational elliptic spaces.*

The new insight in [17] is that, although we have presented the realization problem under a homotopy-theoretical point of view, we also provided a graph-theoretical formulation. In fact, in 1938 Frucht proved the following (see [31, 32]).

Theorem 6.4 (Frucht)**.** *Given a finite group G, there exist infinitely many non-isomorphic connected (finite)* *graphs $\mathcal{G}$ whose automorphism group is isomorphic to G.*

Then, Theorem 6.3 follows from the following

Theorem 6.5. *Let $\mathcal{G}$ be a finite connected graph with more than one vertex. Then there exist an elliptic space X such that the group of automorphisms of $\mathcal{G}$ is realizable by the group of self-homotopy equivalences of X, i.e.* $\mathrm{Aut}(\mathcal{G}) \cong \mathcal{E}(X)$.

Proof. Let $\mathcal{G} = (V, E)$ be a connected graph such that $\#V > 1$, and let X be a rational space whose minimal model $\mathcal{M}$ is

$$\Big(\Lambda(x_1, x_2, y_1, y_2, y_3, z) \otimes \Lambda(x_v, z_v | v \in V), d\Big)$$

where dimensions and differentials are

$$\begin{array}{ll} |x_1| = 8, & d(x_1) = 0 \\ |x_2| = 10, & d(x_2) = 0 \\ |y_1| = 33, & d(y_1) = x_1^3 x_2 \\ |y_2| = 35, & d(y_2) = x_1^2 x_2^2 \\ |y_3| = 37, & d(y_3) = x_1 x_2^3 \\ |x_v| = 40, & d(x_v) = 0 \\ |z| = 119, & d(z) = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + \\ & \qquad x_1^{15} + x_2^{12} \\ |z_v| = 119, & d(z_v) = x_v^3 + \sum_{(v,w)\in E} x_v x_w x_2^4. \end{array}$$

Recall that every $\sigma \in \mathrm{Aut}(\mathcal{G})$ is a permutation on V such that $(v, w) \in E$ if and only if $(\sigma(v), \sigma(w)) \in E$. Therefore, σ induces $f_\sigma \in \mathrm{Aut}(\mathcal{M})$ given by

$$\begin{array}{rl} f_\sigma(x_1) = & x_1 \\ f_\sigma(x_2) = & x_2 \\ f_\sigma(y_1) = & y_1 \\ f_\sigma(y_2) = & y_2 \\ f_\sigma(y_3) = & y_3 \\ f_\sigma(x_v) = & x_{\sigma(v)},\ v \in V \\ f_\sigma(z) = & z \\ f_\sigma(z_v) = & z_{\sigma(v)},\ v \in V. \end{array}$$

Notice that given $\sigma, \tau \in \mathrm{Aut}(\mathcal{G})$, $f_\sigma \circ f_\tau = f_{\sigma\circ\tau}$, and if moreover $\sigma \neq \tau$, then f_σ and f_τ induce different isomorphisms in the module of indecomposable elements of $\mathcal{M}$, hence $f_\sigma \not\simeq f_\tau$. Therefore, $\Upsilon\colon \mathrm{Aut}(\mathcal{G}) \to \mathcal{E}(X)$ given by $\Upsilon(\sigma) = [f_\sigma]$ is a well defined group monomorphism.

A more demanding task is showing that given $f \in \mathrm{Aut}(\mathcal{M})$, there exists $\sigma \in \mathrm{Aut}(\mathcal{G})$, such that

$$\begin{array}{rl} f(x_1) = & x_1 \\ f(x_2) = & x_2 \\ f(y_1) = & y_1 \\ f(y_2) = & y_2 \\ f(y_3) = & y_3 \\ f(x_v) = & x_{\sigma(v)},\ v \in V \\ f(z) = & z + d(m_z) \\ f(z_v) = & z_{\sigma(v)} + d(m_{z_v}),\ v \in V \end{array}$$

with m_z and m_{z_v} elements of degree 118 in $\mathcal{M}$. Notice that in this case, $f \sim f_\sigma$ what shows that Υ is surjective, and therefore an isomorphism.

To finish the proof, we have to show that $\mathcal{M}$ is elliptic. But $\mathcal{M}$ is elliptic if and only if $\widetilde{\mathcal{M}}$, its associated pure Sullivan algebra, is so. We get that

$$\widetilde{\mathcal{M}} = \Big(\Lambda(x_1, x_2, x_v | v \in V) \otimes \Lambda(y_1, y_2, y_3, z, z_v | v \in V), \tilde{d}\Big)$$

with

$$\begin{array}{rl} \tilde{d}(x_1) = & 0 \\ \tilde{d}(x_2) = & 0 \\ \tilde{d}(x_v) = & 0 \\ \tilde{d}(y_1) = & x_1^3 x_2 \\ \tilde{d}(y_2) = & x_1^2 x_2^2 \\ \tilde{d}(y_3) = & x_1 x_2^3 \\ \tilde{d}(z) = & x_1^{15} + x_2^{12} \\ \tilde{d}(z_v) = & x_v^3 + \sum_{(v,w)\in E} x_v x_w x_2^4. \end{array}$$

Since

$$\begin{array}{ll} x_1^{17} & = \tilde{d}(z x_1^2 - y_2 x_2^{10}) \\ x_2^{13} & = \tilde{d}(z x_2 - y_1 x_1^{12}) \\ [x_v^3]^4 & = \Big[-\sum_{(v,w)\in E} x_v x_w x_2^4\Big]^4 = 0 \end{array}$$

every element in $H^*(\widetilde{\mathcal{M}})$ is nilpotent, hence $\widetilde{\mathcal{M}}$ is elliptic and therefore $\mathcal{M}$ is so. □

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C. Costoya
CITIC Research Center,
Departamento de Computación,
Universidade da Coruña, A Coruña, Spain
E-mail address: `cristina.costoya@udc.es`

A. Viruel
Departamento de Álgebra, Geometría y Topología,
Universidad de Málaga,
Apdo correos 59, 29080 Málaga, Spain.
E-mail address: `viruel@uma.es`