Graduate J. Math. **5** (2020), 1 – 8

A nonstandard invariant of coarse spaces

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Abstract

We construct a set-valued invariant $\iota(X,\xi)$ of pointed coarse spaces (X,ξ) by using nonstandard analysis. The invariance under coarse equivalence is established. A sufficient condition for the invariant to be of cardinality ≤ 1 is provided. Miller *et al.* [15] and subsequent researchers have introduced a similar but standard set-valued coarse invariant $\sigma(X,\xi)$ of pointed metric spaces (X,ξ). In order to compare these two invariants, we construct a natural transformation $\omega_{(X,\xi)}$ from $\sigma(X,\xi)$ to $\iota(X,\xi)$. The surjectivity of $\omega_{(X,\xi)}$ is proved for all proper geodesic spaces (X,ξ).

MSC 2010. Primary: 54J05; Secondary: 40A05.

1 Introduction

Small-scale topology focuses on *fine* structures of spaces and maps, such as openness, continuity, and convergence. In contrast, large-scale (coarse) topology does not care about fine structures, and instead focuses on *coarse* or *asymptotic* structures of spaces and maps, such as boundedness, bornologicity, and divergence. Large-scale concepts appear in many fields of mathematics, e.g., functional analysis [1, 8], geometric group theory [2, 19], and infinite combinatorics [16, 17]. Roe [19] developed a general and unified framework for large-scale topology, called coarse spaces. A *coarse structure* on a set X is a family $\mathcal{C}_X$ of binary relations on X which satisfies the following conditions:

1. $\Delta_X := \{ (x,x) \mid x \in X \} \in \mathcal{C}_X$;
2. $E \subseteq F \in \mathcal{C}_X \implies E \in \mathcal{C}_X$;
3. $E, F \in \mathcal{C}_X \implies E \cup F \in \mathcal{C}_X$;
4. $E, F \in \mathcal{C}_X \implies E \circ F \in \mathcal{C}_X$;
5. $E \in \mathcal{C}_X \implies E^{-1} \in \mathcal{C}_X$.

A set equipped with a coarse structure is called a *coarse space*. For example, given a pseudometric d_X on a set X, the family of all binary relations E on X with $\sup_{(x,y)\in E} d_X(x,y) < \infty$ is a coarse structure on X, called the *bounded coarse structure*.

Miller *et al.* [15] introduced a set-valued invariant $\sigma(X,\xi)$ of pointed metric spaces (X,ξ) with a certain property (called σ-stability). Fox *et al.* [7] proved that $\sigma(X,\xi)$ is invariant under coarse equivalence. DeLyser *et al.* [4] extended it to general pointed metric spaces via the direct limit construction. The definition of $\sigma(X,\xi)$ can be obviously extended to general pointed coarse spaces. Roughly speaking, $\sigma(X,\xi)$ counts *the ways to tend to infinity from* ξ up to a certain equivalence. Analogously, it is natural to consider the number of *ideal points infinitely far away from* ξ up to some equivalence.

In this paper, we employ Robinson's NonStandard Analysis (NSA) to realise the above idea. NSA is a powerful framework for finding and proving theorems, refactoring known theories, constructing mathematical objects, and crystallising intuitive ideas, based on the existence of saturated models of set theory. The basic strategy of NSA is to enrich the mathematical world by adding ideal entities such as infinitesimals (infinitely small quantities). We refer to [5, 13, 18] for NSA and [3, 12] for the foundations of NSA.

This paper is organised as follows. In Section 2, we review the treatment of coarse spaces by means of nonstandard analysis. In Section 3, we construct a set-valued coarse invariant $\iota(X,\xi)$ for each pointed coarse space (X,ξ). We consider an asymptotic property of pointed coarse spaces, called "non-scattering at infinity", and show that it is a sufficient condition for $\iota(X,\xi)$ to be of cardinality ≤ 1. In Section 4, we discuss the relationship between $\sigma(X,\xi)$ and $\iota(X,\xi)$ by considering a natural transformation $\omega\colon \sigma \to \iota$. The surjectivity of $\omega_{(X,\xi)}\colon \sigma(X,\xi) \to \iota(X,\xi)$ is proved for all proper geodesic spaces (X,ξ). Finally, in Section 5, we pose some open problems. For instance, it is open whether the map $\omega_{(X,\xi)}$ can be non-injective.

2 Preliminaries

In this section, we recall basic definitions and results in nonstandard large-scale topology following [11].

Let $(X,\mathcal{C}_X)$ be a coarse space. Consider the nonstandard extension $({}^*X, {}^*\mathcal{C}_X)$. We say that two points $x,y \in {}^*X$ are *finitely close* (and denote by $x \sim_X y$) if $(x,y) \in {}^*E$ for some $E \in \mathcal{C}_X$. In other words, the *finite closeness relation* $\sim_X$ is a binary relation on ${}^*X$ defined as the union $\bigcup_{E\in\mathcal{C}_X} {}^*E$. For $\xi \in X$ let $\mathrm{FIN}(X,\xi) := \{ x \in {}^*X \mid x \sim_X \xi \}$. For example, if $\mathcal{C}_X$ is induced by a pseudometric d_X, then $x \sim_X y$ if and only if ${}^*d_X(x,y)$ is finite (in the sense of nonstandard analysis).

We then obtain the following nonstandard characterisations.

Proposition 2.1 ([11, Proposition 3.4])**.** *A binary relation E on a coarse space X is controlled (i.e. $\in \mathcal{C}_X$) if and only if ${}^*E \subseteq \sim_X$.*

Proof. The "only if" part is trivial. Suppose ${}^*E \subseteq \sim_X$. By the saturation principle, there exists an $F \in {}^*\mathcal{C}_X$ such that $\sim_X \subseteq F$ (see also [11, Lemma 3.3]), so ${}^*E \subseteq F$. By the transfer principle, ${}^*E \in {}^*\mathcal{C}_X$. Again by transfer, we have $E \in \mathcal{C}_X$. □

Proposition 2.2 ([11, Proposition 3.10])**.** *A subset B of a coarse space X is bounded if and only if $x \sim_X y$ for all $x, y \in {}^*B$.*

Proof. Only if: B is bounded, i.e. $B \times B \in \mathcal{C}_X$. Then ${}^*B \times {}^*B = {}^*(B \times B) \subseteq \sim_X$, where the equality follows from transfer. Hence $x \sim_X y$ holds for all $x, y \in {}^*B$.

If: $x \sim_X y$ holds for all $x, y \in {}^*B$, i.e. ${}^*B \times {}^*B \subseteq \sim_X$. By Proposition 2.1, $B \times B \in \mathcal{C}_X$, i.e. B is bounded. □

Proposition 2.3 ([11, Corollary 3.13])**.** *A coarse space X is coarsely connected if and only if $x \sim_X y$ for all $x, y \in X$.*

Proof. The coarse space X is coarsely connected if and only if $\{ x, y \}$ is bounded for all $x, y \in X$. Apply Proposition 2.2 to the latter condition. □

Proposition 2.4 ([11, Theorem 3.23])**.** *A map $f\colon X \to Y$ between coarse spaces is bornologous if and only if ${}^*f\colon {}^*X \to {}^*Y$ preserves finite closeness, i.e., $x \sim_X x'$ implies ${}^*f(x) \sim_Y {}^*f(x')$.*

Proof. Only if: let $x, x' \in {}^*X$ and suppose $x \sim_X x'$. Choose an $E \in \mathcal{C}_X$ such that $(x, x') \in {}^*E$. Since f is bornologous, $F := (f \times f)(E) \in \mathcal{C}_Y$. By transfer, we have $({}^*f(x), {}^*f(x')) \in {}^*F$. Hence ${}^*f(x) \sim_Y {}^*f(x')$.

If: let $E \in \mathcal{C}_X$. Then ${}^*((f \times f)(E)) = ({}^*f \times {}^*f)({}^*E) \subseteq ({}^*f \times {}^*f)(\sim_X) \subseteq \sim_Y$. By Proposition 2.1, $(f \times f)(E) \in \mathcal{C}_Y$. Hence f is bornologous, because E was arbitrary. □

Proposition 2.5 ([11, Theorem 2.26])**.** *A map $f\colon (X,\xi) \to (Y,\eta)$ between pointed coarse spaces is proper at the base point (i.e. the inverse image of each bounded set of Y containing η is bounded in X) if and only if ${}^*f^{-1}(\mathrm{FIN}(Y,\eta)) \subseteq \mathrm{FIN}(X,\xi)$.*

Proof. Only if:

$$\begin{aligned} {}^*f^{-1}(\mathrm{FIN}(Y,\eta)) &= \bigcup_{\eta\in B\subseteq_{\mathrm{bounded}} Y} {}^*f^{-1}({}^*B) \\ &\subseteq \bigcup_{\xi\in A\subseteq_{\mathrm{bounded}} X} {}^*A = \mathrm{FIN}(X,\xi). \end{aligned}$$

If: let B be a bounded set of Y that contains η. Then ${}^*B \subseteq \mathrm{FIN}(Y,\eta)$ by Proposition 2.2. By assumption, ${}^*(f^{-1}(B)) = {}^*f^{-1}({}^*B) \subseteq {}^*f^{-1}(\mathrm{FIN}(Y,\eta)) \subseteq \mathrm{FIN}(X,\xi)$. Hence $f^{-1}(B)$ is bounded by Proposition 2.2. □

Proposition 2.6. *Two maps $f, g\colon X \to Y$ between coarse spaces are bornotopic (or close) if and only if ${}^*f(x) \sim_Y {}^*g(x)$ for all $x \in {}^*X$.*

Proof. Recall that f and g are bornotopic if and only if $\{ (f(x), g(x)) \mid x \in X \} \in \mathcal{C}_Y$. Then apply Proposition 2.1. □

We say that a coarse space X is *coarsely Archimedean* if $X \times X = \bigcup_{n\in\mathbb{N}} E^n$ holds for some $E \in \mathcal{C}_X$, where E^n refers to the n-fold composition of E. Note that this notion is the large-scale counterpart of the Archimedean property of uniform spaces (Hu [9]). A (possibly external) subset B of ${}^*X$ is said to be *macrochain-connected* if for any $x, y \in B$, there exists an internal hyperfinite sequence $\{ s_i \}_{i\leq n}$ in B, where $n \in {}^*\mathbb{N}$, such that $s_0 = x$, $s_n = y$ and $s_i \sim_X s_{i+1}$ for all $i < n$. For standard sets, these two notions are equivalent.

Proposition 2.7. *X is coarsely Archimedean if and only if ${}^*X$ is macrochain-connected.*

Proof. Only if: choose an $E \in \mathcal{C}_X$ such that $X \times X = \bigcup_{n\in\mathbb{N}} E^n$. For any $x, y \in {}^*X$, by transfer, there exists an $n \in {}^*\mathbb{N}$ such that $(x, y) \in {}^*E^n \subseteq \sim_X{}^n$. Hence ${}^*X$ is macrochain-connected.

If: by saturation, we can find an $E \in {}^*\mathcal{C}_X$ such that $\sim_X \subseteq E$. Then we have that ${}^*X \times {}^*X = {}^*\bigcup_{n\in{}^*\mathbb{N}} E^n$. By transfer, $X \times X = \bigcup_{n\in\mathbb{N}} F^n$ holds for some standard $F \in \mathcal{C}_X$. Therefore, X is coarsely Archimedean. □

Example 2.8. Let X be an uncountable set equipped with the coarse structure

$$\mathcal{C}_X := \{ E \subseteq X^2 \mid E \setminus \varDelta_X \text{ is countable} \}.$$

This space is not coarsely Archimedean: for any $E \in \mathcal{C}_X$, since $\bigcup_{n\in\mathbb{N}} E^n \setminus \varDelta_X$ is countable, it follows that $\bigcup_{n\in\mathbb{N}} E^n \neq X \times X$.

3 Invariant ι

3.1 Construction

Let (X,ξ) be a pointed coarse space. We define the invariant $\iota(X,\xi)$ as follows. Denote the set ${}^*X \setminus \mathrm{FIN}(X,\xi)$ by $\mathrm{INF}(X,\xi)$. For $x, y \in \mathrm{INF}(X,\xi)$, we write $x \equiv^\iota_{X,\xi} y$ if they lie in the same macrochain-connected component of $\mathrm{INF}(X,\xi)$. It is clear that $\equiv^\iota_{X,\xi}$ is an equivalence relation on $\mathrm{INF}(X,\xi)$. For each $x \in \mathrm{INF}(X,\xi)$, let $[x]^\iota_{X,\xi}$ denote the $\equiv^\iota_{X,\xi}$-equivalence class of x. Finally, define

$$\iota(X,\xi) := \left\{ [x]^\iota_{X,\xi} \;\middle|\; x \in \mathrm{INF}(X,\xi) \right\}.$$

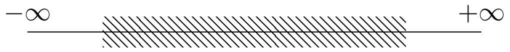


Fig. 3.1: The real line. The shaded region represents the finite part of the real line. The negative infinities and the positive infinities are separated from each other by the finite part.

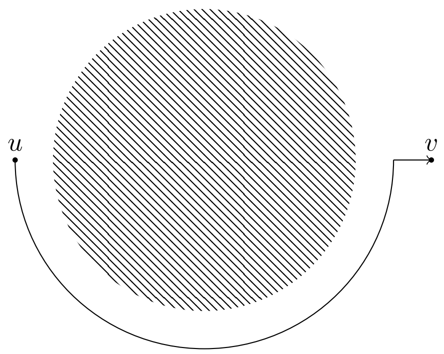


Fig. 3.2: The Euclidean plane. The set of all infinite points of the plane looks like the plane with a single hole.

In short, $\iota(X, \xi)$ is the set of all macrochain-connected components of $\mathrm{INF}(X, \xi)$.

Proposition 3.1 (Changing the base point)**.** *Let X be a coarse space. Suppose that two points $\xi, \eta \in X$ lie on the same coarsely connected component. Then $\iota(X, \xi) = \iota(X, \eta)$.*

Proof. By Proposition 2.3, we have that $\xi \sim_X \eta$. For every $x \in {}^*X$, if $x \sim_X \xi$, then $x \sim_X \eta$, and vice versa. Hence $\mathrm{INF}(X, \xi) = \mathrm{INF}(X, \eta)$, and thus $\iota(X, \xi) = \iota(X, \eta)$. □

Because of this, if X is coarsely connected, one can simply write $\iota(X)$ for $\iota(X, \xi)$ dropping the base point ξ with no ambiguity. The same applies to $\mathrm{INF}(X, \xi)$ and $\equiv^{\iota}_{X,\xi}$. In particular, one can use this notation for all metrisable spaces.

Example 3.2. Consider the real line $\mathbb{R}$ (Figure 3.1 on page 3). The set $\mathrm{INF}(\mathbb{R})$ consists of positive and negative infinite hyperreals. Two infinite hyperreals are $\equiv^{\iota}_{\mathbb{R}}$-related if and only if they have the same sign. Hence we have that $\iota(\mathbb{R}) = \{\pm\infty\}$, where $\pm\infty$ denote the equivalence classes of positive and negative infinite hyperreals, respectively. See also [15, Corollary 15].

Example 3.3. Consider the plane $\mathbb{R}^2$ (Figure 3.2 on page 3). It is easy to see that any two points u and v of $\mathrm{INF}(\mathbb{R}^2)$ are $\equiv^{\iota}_{\mathbb{R}^2}$-related. Hence $\iota(\mathbb{R}^2) = \{\infty\}$. The same equation holds for all dimensions greater than 2. This is contrasted to the case of $\mathbb{R}$.

Example 3.4. [cf. Miller *et al.* [15, Example 3]] Let $X := \{(\pm 1, y) \mid 1 \leq y\} \cup \{(x, 1) \mid -1 \leq x \leq 1\}$ (Figure 3.3 on page 3). Any two points of $\mathrm{INF}(X)$ are $\equiv^{\iota}_X$-related. Hence $\iota(X) = \{\infty_{\upuparrows}\}$. Note that the "collapsing" map $X \ni (x, y) \mapsto y \in \mathbb{R}_{\geq 1}$ gives a coarse equivalence. See also [15, Corollary 17].

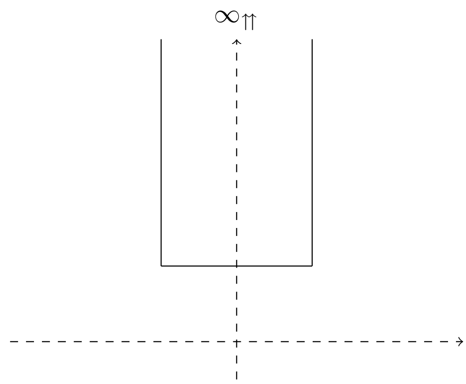


Fig. 3.3: The straight vase.

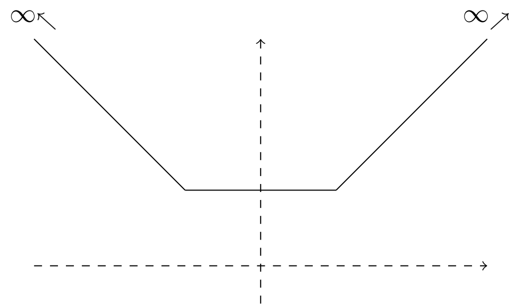


Fig. 3.4: The flared vase.

Example 3.5. Let

$$X := \{(\pm y, y) \mid 1 \leq y\} \cup \{(x, 1) \mid -1 \leq x \leq 1\}$$

(Figure 3.4 on page 3). Any two infinite points on the left and the right "verges" are not $\equiv^{\iota}_X$-related. Hence $\iota(X) = \{\infty_{\nwarrow}, \infty_{\nearrow}\}$. Note that the projection $X \ni (x, y) \mapsto x \in \mathbb{R}$ gives an isomorphism of coarse spaces.

Example 3.6. [cf. DeLyser *et al.* [6, Example 4.5]] Let $X = \{(0, y) \mid y \in \mathbb{R}_{\geq 0}\} \cup \{(x, 2^n) \mid x \in \mathbb{R}, n \in \mathbb{N}\}$ (Figure 3.5 on page 3). Let us show that each infinite point (x, y) of ${}^*X$ is $\equiv^{\iota}_X$-related to one of $(-\omega, 1)$, $(0, \omega)$ and $(\omega, 1)$ for a fixed $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. There are three cases.

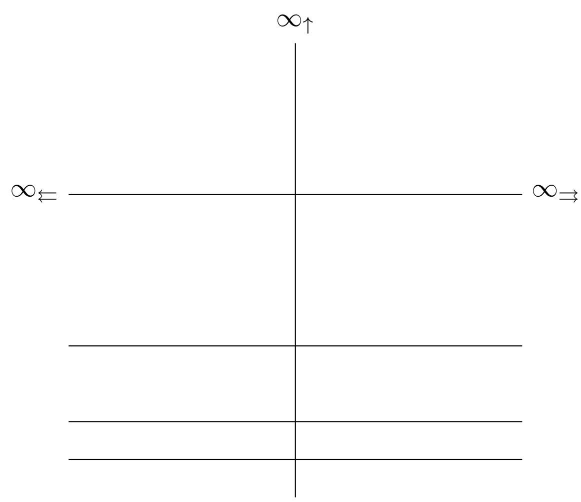


Fig. 3.5: The antenna.

Case 1: y is finite and x is negative infinite. Then $y = 2^n$ for some (standard) $n \in \mathbb{N}$. Firstly, the internal hyperfinite sequence

$$\underbrace{(x, y), \left(x, 2^{-1} y\right), \left(x, 2^{-2} y\right), \ldots, (x, 1)}_{n+1}$$

connects (x, y) and $(x, 1)$. Any two adjacent points are of distance less than or equal to 2^{n-1}. Secondly, the internal hyperfinite sequence

$$\underbrace{(x, 1), \left(x - \frac{x + \omega}{\lceil |x + \omega| \rceil}, 1\right), \left(x - 2 \cdot \frac{x + \omega}{\lceil |x + \omega| \rceil}, 1\right), \ldots, (-\omega, 1)}_{\lceil |x+\omega| \rceil + 1}$$

connects $(x, 1)$ and $(-\omega, 1)$, where $\lceil \cdot \rceil$ is the ceiling function. Any two adjacent points are of distance less than or equal to $|x + \omega| / \lceil |x + \omega| \rceil \le 1$. Hence $(x, y) \equiv^\iota_X (x, 1) \equiv^\iota_X (-\omega, 1)$.

Case 2: y is finite and x is positive infinite. Similarly to the first case, we have that $(x, y) \equiv^\iota_X (\omega, 1)$.

Case 3: y is infinite. Firstly, the internal hyperfinite sequence

$$\underbrace{(x, y), \left(x - \frac{x}{\lceil |x| \rceil}, y\right), \left(x - 2 \cdot \frac{x}{\lceil |x| \rceil}, y\right), \ldots, (0, y)}_{\lceil |x| \rceil + 1}$$

connects (x, y) and $(0, y)$. Any two adjacent points are of distance less than or equal to $|x| / \lceil |x| \rceil \le 1$. Secondly, the internal hyperfinite sequence

$$\underbrace{(0, y), \left(0, y + \frac{\omega - y}{\lceil |\omega - y| \rceil}\right), \left(0, y + 2 \cdot \frac{\omega - y}{\lceil |\omega - y| \rceil}\right), \ldots, (0, \omega)}_{\lceil |\omega - y| \rceil + 1}$$

connects $(0, y)$ and $(0, \omega)$. Any two adjacent points are of distance less than or equal to $|\omega - y| / \lceil |\omega - y| \rceil \le 1$. Hence $(x, y) \equiv^\iota_X (0, y) \equiv^\iota_X (0, \omega)$.

On the other hand, any two of $(-\omega, 1)$, $(0, \omega)$ and $(\omega, 1)$ are not $\equiv^\iota_X$-related.

1. To prove $(-\omega, 1) \not\equiv^\iota_X (0, \omega)$, suppose on the contrary that $(-\omega, 1) \equiv^\iota_X (\omega, 1)$, i.e., there is an internal hyperfinite sequence $\{ (x_i, y_i) \}_{i \le n}$ in $\mathrm{INF}(X)$ such that $(x_0, y_0) = (-\omega, 1)$, $(x_n, y_n) = (0, \omega)$ and $(x_i, y_i) \sim_X (x_{i+1}, y_{i+1})$ for all $i < n$. Let $i_0 > 0$ be the smallest hyperinteger such that $x_{i_0} = 0$. Since $(x_{i_0}, y_{i_0}) = (0, y_{i_0})$ is infinite, y_{i_0} must be infinite. Since $(x_{i_0-1}, y_{i_0-1}) \sim_X (x_{i_0}, y_{i_0})$, we have that $y_{i_0-1} \sim_\mathbb{R} y_{i_0}$, so y_{i_0-1} is infinite too. For each $i < i_0$, since $x_i \ne 0$ by the choice of i_0, it follows that $y_i = 2^{k_i}$ for some $k_i \in {}^*\mathbb{N}$. Let us show that $y_{i_0-j} = y_{i_0-1}$ for all $1 \le j \le i_0$ by (internal) induction on j. The case $j = 1$ is trivial. Suppose $j > 1$. By the induction hypothesis, $2^{k_{i_0-j}} = y_{i_0-j} \sim_\mathbb{R} y_{i_0-j+1} = y_{i_0-1} = 2^{k_{i_0-1}}$, where k_{i_0-j} and k_{i_0-1} are both infinite. Hence $k_{i_0-j} = k_{i_0-1}$ and $y_{i_0-j} = y_{i_0-1}$. (Otherwise, let $k = \min \{ k_{i_0-j}, k_{i_0-1} \}$, then $|y_{i_0-j} - y_{i_0-1}| \ge 2^k =$ infinite, which contradicts with $y_{i_0-j} \sim_\mathbb{R} y_{i_0-1}$.) In particular, we have $0 = y_0 = y_{i_0-1} \ne 0$, a contradiction.
2. With a similar argument, we can prove that $(\omega, 1) \not\equiv^\iota_X (0, \omega)$.
3. To prove $(-\omega, 1) \not\equiv^\iota_X (\omega, 1)$, suppose on the contrary that $(-\omega, 1) \equiv^\iota_X (\omega, 1)$, i.e., there is an internal hyperfinite sequence $\{ (x_i, y_i) \}_{i \le n}$ in $\mathrm{INF}(X)$ such that $(x_0, y_0) = (-\omega, 1)$, $(x_n, y_n) = (\omega, 1)$ and $(x_i, y_i) \sim_X (x_{i+1}, y_{i+1})$ for all $i < n$. Then $\{ x_i \}_{i \le n}$ is an internal hyperfinite sequence in ${}^*\mathbb{R}$ such that $x_0 = -\omega$, $x_n = \omega$ and $x_i \sim_\mathbb{R} x_{i+1}$. However, since $|\iota(\mathbb{R})| = 1$, x_{i_0} must be finite for some $i_0 \le n$. On the other hand, since (x_{i_0}, y_{i_0}) is infinite, y_{i_0} must be infinite. Hence $(-\omega, 1) \equiv^\iota_X (x_{i_0}, y_{i_0}) \equiv^\iota_X (0, \omega)$, a contradiction.

Consequently, $\iota(X) = \{ \infty_{\leftleftarrows}, \infty_\uparrow, \infty_{\rightrightarrows} \}$, where $\infty_{\leftleftarrows}$, $\infty_\uparrow$ and $\infty_{\rightrightarrows}$ denote the equivalence classes of $(-\omega, 1)$, $(0, \omega)$ and $(\omega, 1)$, respectively.

3.2 Coarse invariance

Lemma 3.7. *Let $f\colon (X, \xi) \to (Y, \eta)$ be a proper map. Then ${}^*f\colon {}^*(X, \xi) \longrightarrow {}^*(Y, \eta)$ maps $\mathrm{INF}(X, \xi)$ into $\mathrm{INF}(Y, \eta)$.*

Proof. By Proposition 2.5, ${}^*f^{-1}({}^*Y \setminus \mathrm{INF}(Y, \eta)) \subseteq {}^*X \setminus \mathrm{INF}(X, \xi)$ holds. It follows that ${}^*f(\mathrm{INF}(X, \xi)) \subseteq \mathrm{INF}(Y, \eta)$. □

Lemma 3.8. *Let $f\colon (X, \xi) \to (Y, \eta)$ be a coarse (i.e. proper bornologous) map. Then ${}^*f\colon {}^*(X, \xi) \to {}^*(Y, \eta)$ sends $\equiv^\iota_{X,\xi}$-related pairs to $\equiv^\iota_{Y,\eta}$-related pairs.*

Proof. Let $x, y \in \mathrm{INF}(X, \xi)$ be $\equiv^\iota_{X,\xi}$-related. Choose an internal hyperfinite sequence $\{ s_i \}_{i \le n}$ in $\mathrm{INF}(X, \xi)$ such that $s_0 = x$, $s_n = y$ and $s_i \sim_X s_{i+1}$ for all $i < n$. Then the internal hyperfinite sequence $\{ {}^*f(s_i) \}_{i \le n}$ satisfies that ${}^*f(s_0) = {}^*f(x)$, ${}^*f(s_n) = {}^*f(y)$, and ${}^*f(s_i) \sim_Y {}^*f(s_{i+1})$ for all $i < n$ by Proposition 2.4. Hence ${}^*f(x) \equiv^\iota_{Y,\eta} {}^*f(y)$. □

Theorem 3.9 (Functoriality)**.** *Every coarse map $f\colon (X, \xi) \to (Y, \eta)$ functorially induces a map $\iota f\colon \iota(X, \xi) \to \iota(Y, \eta)$.*

Proof. Define $\iota f\colon \iota(X, \xi) \to \iota(Y, \eta)$ by letting

$$\iota f\, [x]^\iota_{X,\xi} := [{}^*f(x)]^\iota_{Y,\eta}.$$

By the previous lemmas, ιf is well-defined. Thus $\iota(-)$ can be extended to a functor from (an arbitrary small full subcategory of) the category of coarse spaces to the category of sets. □

Corollary 3.10. *ι is invariant under isomorphisms of coarse spaces.*

Theorem 3.11 (Coarse invariance)**.** *If two coarse maps $f, g\colon (X,\xi) \to (Y,\eta)$ are bornotopic, then $\iota f = \iota g$.*

Proof. Let $[x]^{\iota}_{X,\xi} \in \mathrm{INF}(X,\xi)$. By Proposition 2.6, ${}^*f(x) \sim_Y {}^*g(x)$, so ${}^*f(x) \equiv_{Y,\eta} {}^*g(x)$. Hence $\iota f[x]^{\iota}_{X,\xi} = \iota g[x]^{\iota}_{X,\xi}$. □

Corollary 3.12. *ι is invariant under coarse equivalences.*

3.3 Wedge sums

Let (X,ξ) and (Y,η) be pointed sets. The wedge sum $(X \vee Y, p)$ is the quotient set $(X \sqcup Y)/E$, where E is the equivalence closure of $\{(\xi,\eta)\}$, with the base point $p = [\xi] = [\eta]$. Suppose X and Y are coarse spaces. The wedge sum $X \vee Y$ is then equipped with the coarse structure whose finite closeness relation is described as

$$u \sim_{X\vee Y} v \iff \begin{cases} u \sim_X v, & u,v \in {}^*X,\\ u \sim_Y v, & u,v \in {}^*Y,\\ u \sim_X \xi \text{ and } \eta \sim_Y v, & u \in {}^*X, v \in {}^*Y,\\ u \sim_Y \eta \text{ and } \xi \sim_X v, & u \in {}^*Y, v \in {}^*X.\end{cases}$$

For instance, if X and Y are metrisable, then the coarse structure of $X \vee Y$ coincides with the coarse structure induced by the following metric:

$$d_{X\vee Y}(u,v) := \begin{cases} d_X(u,v), & u,v \in X,\\ d_Y(u,v), & u,v \in Y,\\ d_X(u,\xi) + d_Y(\eta,v), & u \in X, v \in Y,\\ d_Y(u,\eta) + d_X(\xi,v), & u \in Y, v \in X.\end{cases}$$

We consider $(X \vee Y, p)$ as a pointed coarse space equipped with the coarse structure above.

Lemma 3.13. *Let (X,ξ) and (Y,η) be pointed coarse spaces. Then $\mathrm{INF}(X \vee Y, p) = \mathrm{INF}(X,\xi) \sqcup \mathrm{INF}(Y,\eta)$.*

Proof. For every $[u] \in {}^*(X \vee Y)$, we have that

$$\begin{aligned} & u \in \mathrm{FIN}(X \vee Y, p)\\ \iff & [u] \sim_{X\vee Y} p\\ \iff & (u \in {}^*X \text{ and } u \sim_X \xi) \text{ or } (u \in {}^*Y \text{ and } u \sim_Y \eta)\\ \iff & u \in \mathrm{FIN}(X,\xi) \text{ or } u \in \mathrm{FIN}(Y,\eta),\end{aligned}$$

thereby $\mathrm{INF}(X \vee Y, p) = \mathrm{INF}(X,\xi) \sqcup \mathrm{INF}(Y,\eta)$. □

Proposition 3.14. *Let (X,ξ) and (Y,η) be pointed coarse spaces. Then $\iota(X \vee Y, p) = \iota(X,\xi) \sqcup \iota(Y,\eta)$.*

Proof. Immediate from the above lemma. □

Example 3.15. The pointed real line $(\mathbb{R},0)$ is isomorphic to the wedge sum $(\mathbb{R}_{\le 0},0) \vee (\mathbb{R}_{\ge 0},0)$. So $\iota(\mathbb{R},0) = \iota(\mathbb{R}_{\le 0},0) \sqcup \iota(\mathbb{R}_{\ge 0},0) = \{-\infty\} \sqcup \{+\infty\}$.

3.4 Non-scattering property

Let (X,ξ) be a pointed coarse space. Denote the set of bounded sets of X containing ξ by $\mathcal{B}_X(\xi)$ and consider it as a partially ordered set with respect to the inclusion $\subseteq$. A subset A of X is said to be *E-connected* for $E \in \mathcal{C}_X$ if $X \times X = \bigcup_{n\in\mathbb{N}} E^n$. We say that (X,ξ) is *non-scattering at infinity* if there exists an $E \in \mathcal{C}_X$ such that the set $\{B \in \mathcal{B}_X(\xi) \mid X \setminus B \text{ is } E\text{-connected}\}$ is cofinal in $\mathcal{B}_X(\xi)$, i.e. if for any $A \in \mathcal{B}_X(\xi)$ there is a $B \supseteq A$ such that $X \setminus B$ is E-connected.

Lemma 3.16. *If a pointed coarse space (X,ξ) is non-scattering at infinity, then $\mathrm{INF}(X,\xi)$ is macrochain-connected.*

Proof. Let E be a witness of the non-scattering property of (X,ξ). Let $x,y \in \mathrm{INF}(X,\xi)$. For each $A \in \mathcal{B}_X(\xi)$ consider the set $\mathcal{F}_A$ of all $B \in {}^*\mathcal{B}_X(\xi)$ such that ${}^*A \subseteq B$, $x,y \notin B$, and ${}^*X \setminus B$ is internally ${}^*E$-connected. Then the family $\{\mathcal{F}_A \mid A \in \mathcal{B}_X(\xi)\}$ has the finite intersection property, i.e. any finite intersection from that family is non-empty. Hence, by saturation, we can find a $B \in {}^*\mathcal{B}_X(\xi)$ such that $\mathrm{FIN}(X,\xi) \subseteq B$, $x,y \notin B$, and ${}^*X\setminus B$ is internally ${}^*E$-connected. (It can be taken as an element of the intersection $\bigcap_{A\in\mathcal{B}_X} \mathcal{F}_A$.) There exists an internal hyperfinite sequence $\{s_i\}_{i\le n}$ in ${}^*X \setminus B$ such that $s_0 = x$, $s_n = y$ and $(s_i, s_{i+1}) \in {}^*E$ for all $i < n$. Since ${}^*E \subseteq \sim_X$ and ${}^*X \setminus B \subseteq \mathrm{INF}(X,\xi)$, the sequence $\{s_i\}_{i\le n}$ is included in $\mathrm{INF}(X,\xi)$, and satisfies that $s_i \sim_X s_{i+1}$ for all $i < n$. Hence $\mathrm{INF}(X,\xi)$ is macrochain-connected. □

Proposition 3.17. *If a pointed coarse space (X,ξ) is non-scattering at infinity, then $\iota(X,\xi)$ is an empty set or a singleton.*

Proof. Immediate from the above lemma. □

4 Comparison with the invariant σ

4.1 Natural transformation $\omega\colon \sigma \to \iota$

We first recall the invariant $\sigma(X,\xi)$ of a pointed metric space (X,ξ) defined in [4]. We adopt a simplified but equivalent definition given in [6].

A *coarse sequence* on X based at ξ is a coarse map $s\colon (\mathbb{N},0) \to (X,\xi)$. Obviously every coarse sequence tends from ξ to infinity. We denote by $S(X,\xi)$ the set of all coarse sequences on X based at ξ. Given two coarse sequences $s,t \in S(X,\xi)$, we write $s \sqsubseteq t$ if s is a subsequence of t, i.e., there is a strictly monotone function $\kappa\colon \mathbb{N} \to \mathbb{N}$ such that $s = t \circ \kappa$. We denote by $\equiv^{\sigma}_{X,\xi}$ the smallest equivalence relation on $S(X,\xi)$ that contains $\sqsubseteq$. In other words, $s \equiv^{\sigma}_{X,\xi} t$ if and only if s and t can be obtained from each other by taking (coarse) subsequences and supersequences repeatedly. Finally define

$$\sigma(X,\xi) := \{[s]^{\sigma}_{X,\xi} \mid s \in S(X,\xi)\},$$

where $[s]^{\sigma}_{X,\xi}$ denotes the $\equiv^{\sigma}_{X,\xi}$-equivalence class of s. The definition can be extended to arbitrary pointed coarse spaces by simply replacing 'metric' with 'coarse'.

The definitions of σ and ι look similar in the following sense. The standard invariant σ was constructed from (bornologous) sequences tending to infinity. These sequences could be thought of as *dynamic* infinities in the sense that such infinities are captured through the limit process. On the other hand, the nonstandard invariant ι was constructed from nonstandardly infinite points. These points could be thought of as *static* infinities. In spite of their similarity, $\sigma(X)$ and $\iota(X)$ do not coincide for some metric spaces as we shall see later.

In order to compare those two invariants, let us construct a natural transformation from σ to ι. We first prove the following lemma.

Lemma 4.1. *Let (X,ξ) be a coarse space, $s,t \in S(X,\xi)$ and $i,j \in {}^*\mathbb{N}\setminus\mathbb{N}$. Then ${}^*s_i, {}^*t_j$ belong to $\mathrm{INF}(X,\xi)$. If $s \equiv^{\sigma}_{X,\xi} t$, then ${}^*s_i \equiv^{\iota}_{X,\xi} {}^*t_j$.*

Proof. It is not difficult to prove that a sequence u in X tends (from ξ) to infinity if and only if ${}^*u_k \in \mathrm{INF}(X,\xi)$ holds for all $k \in {}^*\mathbb{N}\setminus\mathbb{N} = \mathrm{INF}(\mathbb{N},0)$. Hence ${}^*s_i, {}^*t_j$ belong to $\mathrm{INF}(X,\xi)$.

To prove the latter part, we only need to prove the case $s \sqsubseteq t$, because the general case can be reduced to that case by induction. Choose a strictly monotone function $\kappa\colon \mathbb{N}\to\mathbb{N}$ such that $s = t\circ\kappa$. By transfer, this holds also at i, i.e. ${}^*s_i = {}^*t_{{}^*\kappa(i)}$. So ${}^*s_i$ and ${}^*t_j$ can be connected by an internal hyperfinite sequence ${}^*t_{{}^*\kappa(i)},\dots,{}^*t_j$. Since t is coarse, this sequence lies in $\mathrm{INF}(X,\xi)$ and any two adjacent points are finitely close. Hence ${}^*s_i \equiv^{\iota}_{X,\xi} {}^*t_j$ holds. □

Thanks to this lemma, we can well-define a map $\omega_{(X,\xi)}\colon \sigma(X,\xi)\to\iota(X,\xi)$ by letting $\omega_{(X,\xi)}[s]^{\sigma}_{X,\xi} := [{}^*s_\omega]^{\iota}_{X,\xi}$, where $\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$. Clearly the map $\omega_{(X,\xi)}$ is natural in (X,ξ), i.e. it gives a natural transformation from σ to ι.

4.2 Surjectivity for proper geodesic spaces

We use the notation and terminology of nonstandard small-scale topology. Recall that a metric space is said to be *proper* if every closed bounded subset is compact. A metric space is proper if and only if every finite point is nearstandard [5, Theorem 5.6 of Chapter 3].

Theorem 4.2. *Let X be a proper geodesic metric space and $\xi \in X$. The map $\omega_{(X,\xi)}\colon \sigma(X,\xi)\to\iota(X,\xi)$ is surjective.*

Proof. Let $x \in \mathrm{INF}(X,\xi)$. Since X is geodesic, there is an internal isometry $\gamma\colon [0,{}^*d_X(\xi,x)] \to {}^*X$ such that $\gamma(0)=\xi$ and $\gamma({}^*d_X(\xi,x)) = x$ by transfer. For each $i\in\mathbb{N}$, since ${}^*d_X(\xi,\gamma(i)) = i$ is finite, $\gamma(i)\in\mathrm{FIN}(X,\xi)$, so $\gamma(i)$ is nearstandard by properness. We can take a sequence $\{s_i\}_{i\in\mathbb{N}}$ in X such that $s_i \approx_X \gamma(i)$, where $\approx$ denotes the infinitesimal closeness relation. Since $d_X(s_i,s_{i+1}) \approx_{\mathbb{R}} {}^*d_X(\gamma(i),\gamma(i+1)) = 1$, the sequence $\{s_i\}_{i\in\mathbb{N}}$ is bornologous. Moreover, it is proper (i.e. tends to infinity), because

$$d_X(\xi,s_i) \approx_{\mathbb{R}} {}^*d_X(\gamma(0),\gamma(i)) = i \to \infty$$

as $i\to\infty$. Therefore $s\in S(X,\xi)$. Furthermore, since ${}^*s_i \approx_X \gamma(i)$ holds for all $i\in\mathbb{N}$, it holds also for some $i\in{}^*\mathbb{N}\setminus\mathbb{N}$ by Robinson's lemma [18, Theorem 4.3.10]. For such $i\in{}^*\mathbb{N}\setminus\mathbb{N}$, the internal hyperfinite sequence ${}^*s_i \approx_X \gamma(i),\gamma(i+1),\dots,\gamma(\lfloor {}^*d_X(\xi,x)\rfloor)$, $\gamma({}^*d_X(\xi,x)) = x$ witnesses that $[{}^*s_i]^{\iota}_{X,\xi} = [x]^{\iota}_{X,\xi}$. Therefore $\omega_{(X,\xi)}[s]^{\sigma}_{X,\xi} = [x]^{\iota}_{X,\xi}$. □

4.3 Non-surjective examples

The map $\omega_{(X,\xi)}\colon \sigma(X,\xi)\to\iota(X,\xi)$ is bijective for *some* coarse spaces, e.g. the Euclidean spaces, the "vase" spaces and the "antenna" space which are presented in the previous section. It is however not bijective in general. The following are examples where $\omega_{(X,\xi)}$ is not surjective.

Example 4.3. Consider the subspace $X := \{n^2 \mid n\in\mathbb{N}\}$ of the real line $\mathbb{R}$ with an arbitrary base point ξ. This space has no coarse sequence. So $\sigma(X,\xi)$ is empty. On the other hand, since X is unbounded, $\iota(X,\xi)$ is nonempty. More precisely,

$$\iota(X,\xi) = \left\{ \left[n^2\right]^{\iota}_{X,\xi} \mid n\in{}^*\mathbb{N}\setminus\mathbb{N} \right\} \cong {}^*\mathbb{N}\setminus\mathbb{N} \cong {}^*\mathbb{N}.$$

Hence $\sigma(X,\xi)$ and $\iota(X,\xi)$ are not equipotent. In this case, the map $\omega_{(X,\xi)}\colon\sigma(X,\xi)\to\iota(X,\xi)$ is trivially injective but not surjective.

Example 4.4. Recall the coarse space X of Example 2.8. It is clear that every countable subset of X is bounded, and vice versa. Hence X has no divergent *sequence*, although it has a divergent *net*. Thus $\sigma(X,\xi)=\varnothing$ but $\iota(X,\xi)\neq\varnothing$ for any base point $\xi\in X$.

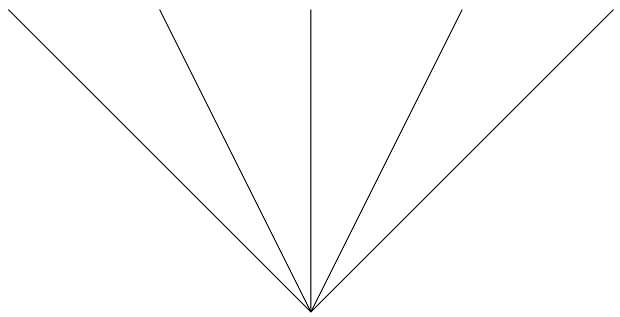
Example 4.5. [cf. DeLyser *et al.* [4, pp. 11–12]] Consider two metric spaces, called the open book and the discrete open book (Figure 4.1 on page 7). The open book B is the wedge sum of countable copies of the ray $\mathbb{R}_{\geq 0}$ at the origin 0. Let s^i be the coarse sequence $0,1,2,\dots$ on the i-th "page" $[0,+\infty)$ of B. Then,

$$\sigma(B,0) = \left\{ \left[s^i\right]^{\sigma}_{B,0} \;\middle|\; i=1,2,\dots \right\}.$$

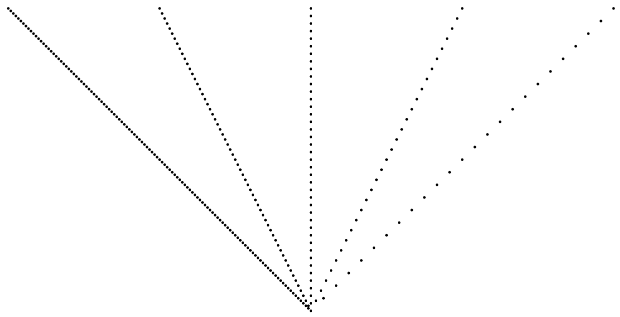
On the other hand, the discrete open book D is the wedge sum of $i\mathbb{N} := \{in \mid n\in\mathbb{N}\}$ $(i=1,2,\dots)$ at the origin 0, which is a subspace of B. Let t^i be the coarse sequence $0,i,2i,\dots$ on the i-th page $i\mathbb{N}$ of D. Then,

$$\sigma(D,0) = \left\{ \left[t^i\right]^{\sigma}_{D,0} \;\middle|\; i=1,2,\dots \right\}.$$

Thus $\sigma(B,0)\cong\sigma(D,0)\cong\mathbb{N}$.



(a) The open book.



(b) The discrete open book.

Fig. 4.1: The first 5 pages of the books.

Let us compute the nonstandard invariants ι of the two books B and D. In both books, any two infinite points on different pages are inequivalent, but any two infinite points on the same finite-numbered page are equivalent. In the open book B, the latter remains true on infinite-numbered pages. Hence

$$\iota(B,0) = \left\{ \left[{}^*s^i_\omega\right]^\iota_{B,0} \,\middle|\, i \in {}^*\mathbb{N}_{>0} \right\},$$

where $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. In the discrete open book D, any two different points on infinite-numbered pages are inequivalent even when they are on the same page. Hence

$$\begin{aligned} \iota(D,0) &= \left\{ \left[{}^*t^i_\omega\right]^\iota_{D,0} \,\middle|\, i \in \mathbb{N}_{>0} \right\} \\ &\quad \cup \left\{ \left[{}^*t^i_j\right]^\iota_{D,0} \,\middle|\, i \in {}^*\mathbb{N} \setminus \mathbb{N}, j \in {}^*\mathbb{N}_{>0} \right\}, \end{aligned}$$

where $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. Both $\iota(D,0)$ and $\iota(B,0)$ are equipotent to ${}^*\mathbb{N}$, i.e., $\iota(B,0) \cong \iota(D,0) \cong {}^*\mathbb{N}$.

Now consider the following diagram:

$$\begin{array}{ccc} \sigma(D,0) & \xrightarrow{\sigma(D\hookrightarrow B)} & \sigma(B,0) \\ \downarrow{\scriptstyle \omega_{(D,0)}} & & \downarrow{\scriptstyle \omega_{(B,0)}} \\ \iota(D,0) & \xrightarrow{\iota(D\hookrightarrow B)} & \iota(B,0) \end{array}$$

Both of the vertical maps are injective, but neither is surjective. The upper horizontal map is bijective, because it sends $\left[t^i\right]^\sigma_{D,0}$ to $\left[s^i\right]^\sigma_{B,0}$ for all $i \in \mathbb{N}$. On the other hand, the lower horizontal map is surjective but not injective, because it sends $\left[{}^*t^i_j\right]^\iota_{D,0}$ to $\left[{}^*s^i_\omega\right]^\iota_{B,0}$ for all $i \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $j \in {}^*\mathbb{N}_{>0}$.

5 Some open problems

It is easy to construct, for each $n \in \mathbb{N}$, a pointed coarse space (X_n, ξ_n) such that $\iota(X_n, \xi_n)$ is of cardinality n (e.g. the wedge sum of n copies of the ray $\mathbb{R}_{\geq 0}$ at 0). We also have a pointed coarse space (Y, η) such that $\iota(Y, \eta)$ is equipotent to the hypernatural numbers ${}^*\mathbb{N}$ (e.g. Example 4.3 and Example 4.5). The cardinality of ${}^*\mathbb{N}$ is uncountable, because each function $f\colon \mathbb{N} \to \{0,1\}$ can be coded by $\sum_{i=0}^{\omega} 2^i\, {}^*f(i) \in {}^*\mathbb{N}$, where $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$.

Remark 5.1. The cardinality of ${}^*\mathbb{N}$ is greater than that of any standard set by weak saturation (i.e. the enlargement property). Moreover, if the nonstandard universe is κ-saturated, the cardinality of ${}^*\mathbb{N}$ is at least κ. Thus the cardinality of ${}^*\mathbb{N}$ depends on the choice of the standard and nonstandard universes.

Problem 5.2. Does there exist a pointed coarse space (X, ξ) such that $\iota(X, \xi)$ is countably infinite?

We have compared two invariants $\sigma(X, \xi)$ and $\iota(X, \xi)$ through the natural transformation $\omega_{(X,\xi)}\colon \sigma(X,\xi) \to \iota(X,\xi)$. As already mentioned, this map is sometimes bijective, and is sometimes injective but not surjective. The following questions then arise.

Problem 5.3. *Does there exist a pointed coarse space (X, ξ) such that the map $\omega_{(X,\xi)}$ is not injective?*

Problem 5.4. *For what kind of pointed coarse spaces is the map $\omega_{(X,\xi)}$ injective, surjective, bijective?*

The invariant $\sigma(X, \xi)$ only deals with standard *bornologous* sequences, so it fails to capture infinite points that cannot be reached in a standard bornologous way. On the other hand, the invariant $\iota(X, \xi)$ deals with arbitrary infinite points. So $\iota(X, \xi)$ may have richer elements compared with $\sigma(X, \xi)$. Example 4.3 and Example 4.5 exemplify such circumstances. Thus those two invariants may focus on quite different properties of spaces. In fact, the surjectivity of $\omega_{(X,\xi)}$ can be recovered by reducing the elements of $\mathrm{INF}(X, \xi)$. Consider the following subset of $\mathrm{INF}(X, \xi)$:

$$\mathrm{INF}^b(X,\xi) := \\ \{ x \in \mathrm{INF}(X,\xi) \mid \exists s \in S(X,\xi)\, \exists \omega \in {}^*\mathbb{N}\, (x = {}^*s_\omega) \}.$$

Then define an invariant $\iota^b(X, \xi)$ as the set of all macrochain-connected components of $\mathrm{INF}^b(X, \xi)$. One can define a natural map $\omega^b_{(X,\xi)}\colon \sigma(X,\xi) \to \iota^b(X,\xi)$ in the same way as $\omega_{(X,\xi)}$. Obviously $\omega^b_{(X,\xi)}$ is surjective. If $\omega_{(X,\xi)}$ is injective, then so is $\omega^b_{(X,\xi)}$, because each macrochain-connected component of $\mathrm{INF}^b(X, \xi)$ is contained in some macrochain-connected component of $\mathrm{INF}(X, \xi)$. However, the injectivity of $\omega^b_{(X,\xi)}$ is still non-trivial.

Problem 5.5. *For what kind of pointed coarse spaces is the map $\omega^b_{(X,\xi)}$ injective (and hence bijective)?*

Because the invariant $\sigma(X,\xi)$ is made out of coarse *sequences*, it may be well-behaved only for coarse spaces with countable bases. The existence of countable bases is equivalent to metrisability provided that the metric function is allowed to take the value $+\infty$ (see [19, Theorem 2.55]).

Problem 5.6. *Find a more appropriate definition of the invariant $\sigma(X,\xi)$ for non-metrisable spaces.*

The author in [10] constructed a nonstandard homology theory for uniform spaces based on hyperfinite formal sums of *infinitesimally small* simplices.[1] Similarly, one can construct a nonstandard homology theory for coarse spaces based on hyperfinite formal sums of *finitely large* simplices. Then $\iota(X,\xi)$ can be regarded as a "basis" of the 0-th homology group of $\mathrm{INF}(X,\xi)$. Note that this is NOT a basis in the usual sense. Strictly speaking, the 0-th homology group of $\mathrm{INF}(X,\xi)$ consists of hyperfinite formal sums $\sum_i g_i\sigma_i$, where g_i are elements of the coefficient group and $\sigma_i = [x_i]^{\iota}_{X,\xi}$ are elements of $\iota(X,\xi)$ such that the sequence $\{(g_i,x_i)\}_i$ is internal.

Problem 5.7. *Can we interpret the invariant $\sigma(X,\xi)$ homology-theoretically?*

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[1] To be fair, this idea is due to McCord [14], who developed a nonstandard homology theory of topological spaces.

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