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Immersion theory for homotopy theorists

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Abstract

A set of lecture notes on immersion theory and related h-principles, based on the view that immersion theory is applied fibration theory.

1 Introduction

This article grew out of lecture notes for a course on immersion theory, submersion theory and related h-principles given to graduate students at the University of Aberdeen in 2004. The lecture notes languished in an unfinished state for many years, but were in circulation and found some friends. Kupers [18] refers to them in an important paper on the h-principle. So there are some reasons to have them published at last. Since I had occasions to use them again and make some revisions, their imperfections may have diminished over the years.

The viewpoint taken here is that immersion theory is applied fibration theory. A more specific goal which I set myself in 2004 was to explain the *corrugation method* in immersion theory, advertised in the film or video clip *Outside In* [26] about the eversion of the 2-sphere.

At this point, the reader may already have a few questions, like the following.

- What is immersion theory and what are its main results?
- Why is immersion theory related to the theory of fibrations?
- What is an h-principle?

These three will be answered right now, i.e., in the introduction.

Definition 1.1. Suppose that M^m and N^n are smooth manifolds, $\partial N = \emptyset$. A smooth map $f\colon M \to N$ is an *immersion* if for all $x \in M$, the differential $df(x)\colon T_xM \longrightarrow T_{f(x)}N$ is an *injective* linear map.

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Remark 1.2. Let $f\colon M \to N$ be an immersion, $x \in M$. By the implicit function theorem, there exist smooth local coordinates near $x \in M$ and $f(x) \in N$ such that, in these coordinates, f has the form $(z_1, z_2, \dots, z_m) \mapsto (z_1, z_2, \dots, z_m, 0, 0, 0, \dots, 0)$.

Let M^m, N^n be smooth manifolds, $m \le n$ and $\partial N = \emptyset$. In the case $m = n$, suppose that $M \smallsetminus \partial M$ has no compact components. Let $\mathrm{imm}(M, N)$ be the space of smooth immersions from M to N (details in section 5). Let $\mathrm{fimm}(M, N)$ be the space of *formal* immersions, i.e. the space of pairs $(f, \delta f)$ where $f\colon M \to N$ is continuous (not necessarily smooth) and δf is *some* vector bundle map $TM \to f^*TN$ (over M) which is injective on each fiber. (See remark 1.4 below.)

Theorem 1.3. (Main theorem of immersion theory, also known as the Smale-Hirsch theorem; [31], [14], [11], [12].) *In these circumstances the map*

$$\mathrm{imm}(M, N) \longrightarrow \mathrm{fimm}(M, N)$$

given by $f \mapsto (f, df)$ is a weak homotopy equivalence.

Remark 1.4. The vector bundle map $\delta f\colon TM \to f^*TN$ amounts to a choice of a linear injection $T_xM \to T_{f(x)}M$ for each $x \in M$, depending continuously on x. Think of δf as a *formal* total derivative for the continuous map f. (The notation δf may suggest that δf is fully determined by f, but this is not intended. In particular δf is not required to agree with the honest derivative df of f if f happens to be differentiable.) — The map from $\mathrm{imm}(M, N)$ to $\mathrm{fimm}(M, N)$ given by $f \mapsto (f, df)$ is also known as 1-*jet prolongation*.

Remark 1.5. If M is compact, then the weak homotopy equivalence of theorem 1.3 is a genuine homotopy equivalence. Indeed, both $\mathrm{imm}(M, N)$ and $\mathrm{fimm}(M, N)$ are then homotopy equivalent to CW-spaces. This follows from [23, Lem.4].

Notation 1.6. Following e.g. Atiyah [2], we will often commit the abuse of confusing a vector bundle with its total space. (Strictly speaking a vector bundle is a *map*.) So we may write *the tangent bundle* TM instead of *the tangent bundle* $TM \to M$, and the like.

Example 1.7. Take $M = S^1$ and $N = \mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, $\mathrm{fimm}(S^1, \mathbb{R}^2)$ is homotopy equivalent to the space of vector bundle monomorphisms (over S^1) from the tangent bundle TS^1 to a trivial vector bundle $S^1 \times \mathbb{R}^2$ on S^1. That in turn is homotopy equivalent to the space of maps from S^1 to S^1. So we have

$$\begin{aligned} \mathrm{imm}(S^1, \mathbb{R}^2) &\simeq \mathrm{fimm}(S^1, \mathbb{R}^2) \\ &\simeq \mathrm{map}(S^1, S^1) \simeq S^1 \times \mathbb{Z} . \end{aligned}$$

In particular $\pi_0(\mathrm{imm}(S^1, \mathbb{R}^2))$ is identified with $\mathbb{Z}$; this is the Whitney-Graustein theorem [39]. See also exercise 1.10 below.

Example 1.8. Take $M = S^{n-1}$ and $N = \mathbb{R}^n$, with $n \geq 2$. Since $\mathbb{R}^n$ is contractible, $\mathrm{fimm}(S^{n-1}, \mathbb{R}^n)$ is homotopy equivalent to the space of vector bundle embeddings (over S^{n-1}) from the tangent bundle TS^{n-1} to a trivial vector bundle $S^{n-1} \times \mathbb{R}^n$ on S^{n-1}. That in turn is homotopy equivalent to the space of orientation preserving vector bundle isomorphisms from $TS^{n-1} \times \mathbb{R}$ to a trivial vector bundle $S^{n-1} \times \mathbb{R}^n$; the homotopy equivalence is given by restriction from $TS^{n-1} \times \mathbb{R}$ to TS^{n-1}. But the vector bundle $TS^{n-1} \times \mathbb{R}$ is canonically trivialized; therefore $\mathrm{fimm}(S^{n-1}, \mathbb{R}^n)$ is homotopy equivalent to the space of maps from S^{n-1} to $\mathrm{SO}(n)$. So we have

$$\begin{aligned} \mathrm{imm}(S^{n-1}, \mathbb{R}^n) &\simeq \mathrm{fimm}(S^{n-1}, \mathbb{R}^n) \\ &\simeq \mathrm{map}(S^{n-1}, \mathrm{SO}(n)) \\ &\simeq \mathrm{SO}(n) \times \Omega^{n-1}\mathrm{SO}(n). \end{aligned}$$

In particular $\pi_0 \mathrm{imm}(S^{n-1}, \mathbb{R}^n)$ is identified with $\pi_{n-1}\mathrm{SO}(n)$. If $n = 3$, this is a trivial group. Hence all immersions of S^2 in $\mathbb{R}^3$ are in the same *regular homotopy class* (a regular homotopy is a 1-parameter family of smooth immersions, or a path in the space of smooth immersions). It follows that the inclusion $S^2 \to \mathbb{R}^3$ is regularly homotopic to the composition of the same inclusion with any orientation-reversing linear isomorphism $\mathbb{R}^3 \to \mathbb{R}^3$, for example the one taking (x_1, x_2, x_3) to $(-x_1, -x_2, -x_3)$. In that sense, an eversion of the 2-sphere in $\mathbb{R}^3$ is possible, and that is what the movie [26] illustrates.

Exercise 1.9. (Use theorem 1.3.)
By example 1.8, the set of regular homotopy classes of immersions $S^{n-1} \to \mathbb{R}^n$ is in a preferred bijection with $\pi_{n-1}\mathrm{SO}(n) \cong \pi_n B\mathrm{SO}(n)$.

(i) Under this bijection, the regular homotopy class of the standard inclusion $f : S^{n-1} \hookrightarrow \mathbb{R}^n$ (viewed as an immersion) corresponds to the neutral element of $\pi_n B\mathrm{SO}(n)$.

(ii) The composition Af (where A is an orientation-reversing linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$) corresponds to the element $t_n \in \pi_n B\mathrm{SO}(n)$ which classifies the tangent bundle of S^n.

(iii) If n is even, then t_n has infinite order in $\pi_n B\mathrm{SO}(n)$. Consequently, an eversion of S^{n-1} in $\mathbb{R}^n$ (as discussed in example 1.8) is impossible.

(iv) If n is odd, then t_n has order at most 2 in $\pi_n B\mathrm{SO}(n)$.

It is shown in [22, Thm.2] that t_n is the neutral element of $\pi_n B\mathrm{SO}(n)$ if and only if $n = 1, 3, 7$. Related reading: [16], [1]. Conclusion: an eversion of S^{n-1} in $\mathbb{R}^n$ exists if and only if $n = 1, 3, 7$.

Exercise 1.10. (Use theorem 1.3.)
Let M be a closed *oriented* smooth manifold of dimension $n - 1$. Let $f : M \to \mathbb{R}^n$ be a smooth immmersion. The immersion f has a preferred unit normal vector field ξ, also known as the *Gauss map* associated with f. In other words, ξ is a map from M to S^{n-1} such that $\xi(z)$ is perpendicular to the linear subspace $df(z)(T_zM)$ of $\mathbb{R}^n$ and the linear map $T_zM \times \mathbb{R} \to \mathbb{R}^n$ defined by $(v, t) \mapsto df(z)(v) + t\xi(z)$ is an *orientation-preserving* linear isomorphism (for all $z \in M$).

(i) For $n = 2$ and $n = 6$, immersions $S^{n-1} \to \mathbb{R}^n$ are classified up to regular homotopy by the degree of their Gauss map, which can be any integer.

(ii) For $n = 4$ and $n = 8$, immersions $S^{n-1} \to \mathbb{R}^n$ are not classified up to regular homotopy by the degree of their Gauss map. The degree of the Gauss map can still be any integer.

(iii) For odd $n > 2$, the degree of the Gauss map of an immersion $S^{n-1} \to \mathbb{R}^n$ is always 1.

Exercise 1.11. (Use theorem 1.3.)
Let M be a smooth n-manifold whose tangent bundle TM admits a vector bundle trivialization. (Interesting examples: the projective spaces $\mathbb{R}P^3$ and $\mathbb{R}P^7$.) Then there exists a smooth immersion $M \to \mathbb{R}^{n+1}$. If in addition M has no closed component, then there exists a smooth immersion $M \to \mathbb{R}^n$. (*Closed* in this context means: compact and with empty boundary.)

Exercise 1.12. (Use theorem 1.3.)
For even n, the projective space $\mathbb{R}P^n$ can be smoothly immersed in $\mathbb{R}^{n+1}$ if and only if $n = 0, 2, 6$. [This exercise requires some knowledge of real K-theory, more specifically, the stable classification of real vector bundles on $\mathbb{R}P^n$. The Grothendieck group of the monoid of isomorphism classes of real vector bundles on $\mathbb{R}P^n$, modulo the subgroup determined by the trivial vector bundles, is a finite cyclic group of order $2^{\varphi(n)}$ generated by the class of the canonical line bundle. Here $\varphi(n)$ is defined as the number of integers s with $0 < s \leq n$ and $s \equiv 0, 1, 2, 4 \mod 8$. See [1, Thm.7.4]. Adams attributes this to Bott and Shapiro.]

Exercise 1.13. (Use theorem 1.3.)
An oriented surface M of genus g admits exactly 2^{2g} distinct regular homotopy classes of smooth immersions $M \to \mathbb{R}^3$.

Exercise 1.14. The projective plane $\mathbb{R}P^2$ admits exactly two regular homotopy classes of immersions in $\mathbb{R}^3$. Is it true that one can be obtained from the other by composing with an orientation-reversing linear isomorphism $\mathbb{R}^3 \to \mathbb{R}^3$? (This rumour arose in connection with the steel model of an immersed $\mathbb{R}P^2$ in $\mathbb{R}^3$ gracing the grounds of the Mathematisches Forschungsinstitut Oberwolfach. One may wonder whether this is an instance of Boy's surface or the mirror image of Boy's surface.)

The Smale-Hirsch theorem 1.3 has a relatively easy and well-known (to experts) reduction to a fibration statement which we now formulate. Let $A^p \subset D^p$ be the p-dimensional annulus obtained by removing an open ball of radius 1/2 about the origin from the standard disk D^p of radius 1.

Proposition 1.15. *For smooth N^n without boundary and integers $p \geq 0$, $q > 0$ such that $p+q=n$, the restriction map* $\mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(A^p \times D^q, N)$ *is a Serre fibration.*

It is straightforward to show that the restriction map from $\mathrm{imm}(D^p \times D^q, N)$ to $\mathrm{imm}(A^p \times D^q, N)$ is a *Serre microfibration*, i.e., has the homotopy lifting property (in the sense of Serre) for sufficiently small homotopies. See definition 3.20 and corollary 5.6 for the details. This is the starting point for the proof of proposition 1.15 (which will be given in section 5). Namely, in section 4 we develop methods which help us to show that some Serre microfibrations are in fact Serre fibrations.

Here is a brief explanation of how proposition 1.15 is used. First of all, the case $m < n$ in theorem 1.3 has a reduction to the case where $m = n$. The idea is to replace M by the total space of a suitable disk bundle on M. Suppose therefore that $m = n$. Let us also assume that M is compact (for the general case, see section 6). Then M admits a finite handle decomposition where each handle has index $< n$. It can be set up in such a way that handles of lower index are attached before those of higher index are attached. (Such a handle decomposition can be obtained from a Morse function $g\colon M \to (-\infty, c]$ having c as regular value with $g^{-1}(c) = \partial M$. Critical points of index n can be avoided or removed by cancellation, since we are assuming that $H_n(M) = 0$. The Morse function can be made self-indexing, so that $g(x) = j$ if x is a critical point of index j. See [24], [25, I§3] for details.) Then there is a filtration

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} = M$$

where $M_p \subset M$ is the handlebody made up from all handles of index $\leq p$. We now prove by induction on p that theorem 1.3 holds with M_p in place of M.

The induction beginning at ($p = 0$) reduces to the statement that the 1-jet prolongation map from $\mathrm{imm}(D^n, N)$ to $\mathrm{fimm}(D^n, N)$ is a weak equivalence. This has a rather short geometric proof based on the observation that the identity $D^n \to D^n$ is isotopic (as a smooth embedding) to an embedding with image contained in an arbitrarily small neighborhood of the center of D^n.

To make the induction step we set up a commutative square of restriction maps

$$\begin{tikzcd}
\mathrm{imm}(M_p, N) \arrow[r] \arrow[d] & \mathrm{imm}(M_{p-1}, N) \arrow[d] \\
\prod_{\lambda \in S_p} \mathrm{imm}(D^p \times D^q, N) \arrow[r] & \prod_{\lambda \in S_p} \mathrm{imm}(A^p \times D^q, N)
\end{tikzcd}$$

where S_p is an indexing set for the p-handles, and $A^p \times D^q$ is (diffeomorphic to) the intersection of any particular p-handle with M_{p-1}. Picture of M_p in the situation where $p = 1$ and $q = 2$ and S_p has just one element whereas $S_{p-1} = S_0$ has two elements:

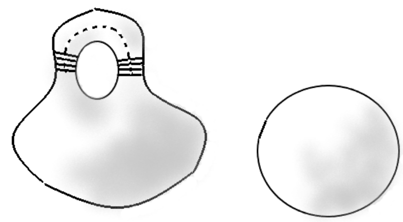


Fig 1. The darkly shaded region is $A^p \times D^q$. The dotted curve wants to be the core of the p-handle, $D^p \times \{0\} \subset D^p \times D^q$. Note that M_{p-1} is obtained by deleting the unshaded part of the p-handle. It has been demoted to the status of a smooth manifold with boundary and corners in the boundary.

It is clear from the definition of an immersion that the square is a pullback square. By proposition 1.15, the two horizontal arrows are Serre fibrations. Therefore we have a *homotopy pullback* square, a.k.a. *homotopy cartesian square*, [9, Def.1.3]. We compare with the square of restriction maps

$$\begin{tikzcd}
\mathrm{fimm}(M_p, N) \arrow[r] \arrow[d] & \mathrm{fimm}(M_{p-1}, N) \arrow[d] \\
\prod_{\lambda \in S_p} \mathrm{fimm}(D^p \times D^q, N) \arrow[r] & \prod_{\lambda \in S_p} \mathrm{fimm}(A^p \times D^q, N)
\end{tikzcd}$$

which is also a pullback square in which the two horizontal maps are Serre fibrations (almost by definition), and so again a homotopy pullback square. Jet prolongation gives us a (natural) map from the first square to the second. The maps

$$\begin{array}{rcl}
\mathrm{imm}(M_{p-1}, N) & \longrightarrow & \mathrm{fimm}(M_{p-1}, N)\ , \\
\prod_\lambda \mathrm{imm}(D^p \times D^q, N) & \longrightarrow & \prod_\lambda \mathrm{fimm}(D^p \times D^q, N)\ , \\
\prod_\lambda \mathrm{imm}(A^p \times D^q, N) & \longrightarrow & \prod_\lambda \mathrm{fimm}(A^p \times D^q, N)
\end{array}$$

are all weak homotopy equivalences. (For the first of these maps this is the inductive hypothesis. The second is a homotopy equivalence by an easy shrinking argument, lemmas 6.1 and 6.2 below. The third is again covered by our inductive hypothesis, which applies because $A^p \times D^q$ has a handle decomposition with handles of index less than p only. [To be more precise, $A^p \times D^q$ is a product of two manifolds with nonempty boundary, and so has corners in the boundary. For our purposes it makes no difference if we smoothen the corners. The result is something diffeomorphic to $S^{p-1} \times D^{q+1}$, and this has a handle decomposition with two handles, one of index 0 and the other of index $p-1$.] It follows that the jet prolongation map $\mathrm{imm}(M_p, N) \to \mathrm{fimm}(M_p, N)$ is a weak homotopy equivalence. That is the kind of conclusion which we can draw when we have a map between two homotopy pullback squares.

Remark 1.16. The reduction to the cases where $m = n$ in the above sketch proof is both traditional and convenient. But it is slightly misleading. Proposition 1.15 admits a mild generalization (exercise 5.9 below) where the condition $p + q = n$ is dropped entirely or replaced by $p + q \leq n$. (For us, $p + q$ is m.) In the light of this, the true goal of the reduction was not to achieve $m = n$. The true goal of the reduction was to get into a situation where M has a handle decomposition with handles of index less than $m = \dim(M)$ only.

One of three questions asked has not yet been answered: what is an h-principle. Since we are (or pretend to be) homotopy theorists, we must look for a homotopy theoretic answer. Let F be a contravariant functor from a certain category $\mathscr{C}$ of manifolds of a fixed dimension m to the category of spaces. Suppose that all morphisms in $\mathscr{C}$ are smooth embeddings. The h-principle (for F) is the statement that F satisfies homotopy invariance and excision. *Homotopy invariance* means that for a morphism $M_0 \to M_1$ in $\mathscr{C}$ which happens to be a homotopy equivalence of spaces, the induced map $F(M_1) \to F(M_0)$ of spaces is a weak equivalence. *Excision* means that, if U and V are open subsets of a manifold M such that $U \hookrightarrow M$ and $V \hookrightarrow M$ are morphisms in $\mathscr{C}$, then the inclusions $U \cap V \to U, V$ and $U, V \to U \cup V$ are also morphisms in $\mathscr{C}$ and the commutative square

$$\begin{tikzcd} F(U \cup V) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(U \cap V) \end{tikzcd}$$

determined by these inclusions is a (weak) homotopy pullback square. *Excision* should also include the condition that $F(\emptyset)$ is a contractible or weakly contractible space, and that F takes disjoint unions to products, up to weak equivalence. So if M in $\mathscr{C}$ is the disjoint union of m-dimensional submanifolds U_α, then the inclusions $U_\alpha \to U$ (which, we hope, are morphisms in $\mathscr{C}$) determine a weak equivalence $F(U) \to \prod_\alpha F(U_\alpha)$. (The excision property is reminiscent of the properties of a sheaf. The h-principle is a homotopy sheaf principle.)

This interpretation of *h-principle* will not appeal to everybody. It is vague (especially by making few assumptions about the category $\mathscr{C}$), though probably not vague enough to cover all cases that should be covered. How does theorem 1.3 fit in? Fix N in theorem 1.3 and the dimension m of M, but allow M to vary among smooth manifolds of dimension m. For $\mathscr{C}$ take the category of all smooth m-manifolds, with smooth embeddings as morphisms. (Boundaries can be allowed.) Then we have two contravariant functors F_0 and F_1:

$$F_0(M) = \mathrm{imm}(M, N), \quad F_1(M) = \mathrm{fimm}(M, N).$$

It is very easy to show that F_1 satisfies homotopy invariance and excision. The theorem states that F_0 is naturally weakly equivalent to F_1 by means of the 1-jet prolongation map; therefore F_0 satisfies homotopy invariance and excision. [Conversely, if we *assume* that F_0 satisfies homotopy invariance and excision, and *observe* that F_1 satisfies homotopy invariance and excision, then it is merely an exercise to deduce that the 1-jet prolongation map $F_0 \to F_1$ is a natural weak equivalence. The key point to verify is that the 1-jet prolongation map $F_0(M) \to F_1(M)$ is a homotopy equivalence for $M \cong \mathbb{R}^m$. This is carried out in lemma 6.2.]

To conclude the introduction, here are a few sentences to clarify what will be done (and where) in the following sections, and what will not be done. Section 2 is all about corrugation, but it is written in strictly homotopy theoretic terms. We meet the corrugation method in lemma 2.6 as a criterion helping us to recognize certain maps as (Serre) fibrations. Section 3 is a compendium of mostly well-known results on fibrations and related concepts, with unusual emphasis on the concept of microfibration. These results are put to use in section 4, where lemma 2.6 is proved at last. In section 5 we prove proposition 1.15 by interpreting it as an instance of lemma 2.6. In section 6 we review the proof of theorem 1.3 from proposition 1.15. Section 7 is an unsystematic excursion into the wider universe of h-principles.

The literature on h-principles is vast and includes some encyclopedic treatises. The most encyclopedic is Gromov's book [10]. Not only does it describe many different h-principles, but it also describes many different methods used to establish h-principles. The book [6] is more self-contained, but it still makes a point of advertising the great diversity of h-principles. A fascinating new method for proving h-principles, probably not foreseen in [10], was introduced in [34],[35]. (To some extent, [18] makes the point that these new h-principles are are also accessible with older methods.) An untraditional compression method for proving theorem 1.3 was found by Rourke and Sanderson [29],[30]. Last not least there is a historical survey [33] covering the time 1959–1973.

In writing these lecture notes I could not do justice to so many contributions and developments. The goal was to write something relatively short, held together by a few ideas from homotopy theory.

2 Composition structures

How can proposition 1.15 be proved? We will adopt a categorical viewpoint by regarding $\mathrm{imm}(A^p \times D^q, N)$ as the space of objects of something like a category. (Deficiencies notwithstanding, this will have a *space* of objects and a *space* of morphisms.)

Definition 2.1. Let $f, g \in \mathrm{imm}(A^p \times D^q, N)$. A *morphism* from f to g is an immersion $h : S^{p-1} \times [0,3] \times D^q \to N$ such that

$$\begin{aligned} h(x,t,y) &= f(sx,y) \quad \text{for } t \in [0,1], \quad s = \frac{t+1}{2}, \\ h(x,t,y) &= g(sx,y) \quad \text{for } t \in [2,3], \quad s = \frac{t-1}{2}. \end{aligned}$$

It should be clear that there is some kind of composition of morphisms by concatenation. More details are given towards the end of section 5. Whatever precise formula is used to define composition, it is unlikely to be strictly associative, but it will at least be associative up to homotopy. The question of existence of identity morphisms is quite another matter. We will take that up later (section 5).

Next, we also want to use the restriction map

$$\mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(A^p \times D^q, N)$$

to create something like a functor Φ. More precisely, for each "object"

$$f \in \mathrm{imm}(A^p \times D^q, N),$$

the fiber of that restriction map over f (i.e., the space of immersions $D^p \times D^q \to N$ extending f) will be viewed as something like the value $\Phi(f)$ of a functor Φ. Indeed, it should be clear that a morphism $h : f \to g$ induces a map $\Phi(h) : \Phi(f) \to \Phi(g)$ by concatenation with h.

Following is a description of the situation in an abstract setting. In this, we assume optimistically that the "identity morphisms" which are so far missing can be found.

Definition 2.2. Let E, B be spaces and let $\sigma, \tau : E \to B$ be maps. Let

$$E\,{}_\sigma\!\times_\tau E := \{(x,y) \in E \times E \mid \sigma(x) = \tau(y)\}\,.$$

Both E and $E\,{}_\sigma\!\times_\tau E$ will be regarded as spaces over $B \times B$ using $x \mapsto (\tau(x), \sigma(x))$ for E, and $(x,y) \mapsto (\tau(x), \sigma(y))$ for $E\,{}_\sigma\!\times_\tau E$. A *composition structure* on (E, B, σ, τ) consists of

- a map $\kappa : E\,{}_\sigma\!\times_\tau E \longrightarrow E$ over $B \times B$,
- a map $\iota : B \to E$ such that $\sigma\iota = \mathrm{id}_B = \tau\iota$.

These are subject to the condition that the maps $E \to E$ given by $x \mapsto \kappa(\iota\tau(x), x)$ and $x \mapsto \kappa(x, \iota\sigma(x))$ are both homotopic to the identity, over $B \times B$.

Remark 2.3. The idea is that E is the space of morphisms of *something like* a topological category, and B is the space of objects. The maps σ and τ are *source* and *target*. The map ι is the map which to each object assigns its identity morphism. The map κ is *composition of morphisms*. It is not required to be associative.

Definition 2.4. Keep the notation of definition 2.2. Let $\pi : Z \to B$ be some map. Let

$$E\,{}_\sigma\!\times_\pi Z = \{(x,z) \in E \times Z \mid \sigma(x) = \pi(z)\}$$

and make this into a space over B by $(x,z) \mapsto \tau(x)$. — An *action* of E on Z (more precisely, a left action of $(E, B, \sigma, \tau, \kappa, \iota)$ on $\pi : Z \to B$) is a map $\alpha : E\,{}_\sigma\!\times_\pi Z \to Z$ over B, subject to the following conditions.

- the maps $E\,{}_\sigma\!\times_\tau E\,{}_\sigma\!\times_\pi Z \to Z$ given by $(x,y,z) \mapsto \alpha(\kappa(x,y),z)$ and $(x,y,z) \mapsto \alpha(x, \alpha(y,z))$ are homotopic over B, where the reference map from $E\,{}_\sigma\!\times_\tau E\,{}_\sigma\!\times_\pi Z$ to B is $(x,y,z) \mapsto \tau(x)$;
- the map $Z \to Z$ given by $z \mapsto \alpha(\iota(\pi(z)), z)$ for $z \in Z$ is homotopic over B to the identity id_Z.

Remark 2.5. The idea is that $b \mapsto Z_b = \pi^{-1}(b)$ for $b \in B$ defines something like a functor on the (almost-)category defined by $E, B, \sigma, \tau, \iota, \kappa$. In particular, for a "morphism" e in E with "source" $\sigma(e) = b$ and "target" $\tau(e) = c$, the "induced map" $e_* : Z_b \to Z_c$ is given by $z \mapsto \alpha(e,z)$.

The main lemma is as follows:

Lemma 2.6. *Keep the notation and assumptions of definition 2.4. Suppose that $(\tau, \sigma) : E \to B \times B$ and $\pi : Z \to B$ are Serre microfibrations. Then π is actually a Serre fibration.*

This will be proved in section 4.

Exercise 2.7. There is a (relatively unexciting) converse to lemma 2.6. For simplicity, suppose that $\pi : Z \to B$ is a Hurewicz fibration. Let E be the space of all paths in B, in other words the space of all continuous maps from $I = [0,1]$ to B with the compact-open topology. Define $\tau, \sigma : E \to B$ by $\sigma(\omega) = \omega(0)$ and $\tau(\omega) = \omega(1)$.

(i) Concatenation of paths (and some form of reparametrization) can be used to define a composition structure on (E, B, σ, τ).

(ii) Show that (E, B, σ, τ) with the composition structure in (i) admits an action on $\pi : Z \to B$.

Here is a preview of the proof of proposition 1.15. (The details can be found in section 5.) We just apply lemma 2.6, taking

- $E = \mathrm{imm}(S^{p-1} \times [0,3] \times D^q, N)$;
- $B = \mathrm{imm}(A^p \times D^q, N)$;
- $\sigma(h) = ((sx, y) \mapsto h(x,t,y))$ for $h \in E$, where $t \in [0,1]$ and $s = (t+1)/2$;

- $\tau(h) = ((sx, y) \mapsto h(x, t, y))$ for $h \in E$, where $t \in [2, 3]$ and $s = (t-1)/2$;
- $Z = \mathrm{imm}(D^p \times D^q, N)$;
- $\pi : Z \to B$ equal to the restriction map.

The composition rule κ and the action map α are both given by concatenation. They do not present a challenge. Constructing $\iota : B \to E$ is more of a challenge. But it can be met and it is at this point that corrugation (as in *corrugated cardboard*) comes in. The identity morphism $\iota(b)$ of an object $b \in B$ (an immersion from $A^p \times D^q$ to N) is obtained by corrugating that object.

Exercise 2.8. Let B be the space of smooth immersions from $[0, 1]$ to $\mathbb{R}^2$ and let E be the space of smooth immersions from $[0, 3]$ to $\mathbb{R}^2$. Define $\sigma, \tau : E \to B$ by $\sigma(g)(t) = g(t)$ and $\tau(g)(t) = g(t + 2)$ for $g \in E$ and $t \in [0, 1]$. Choose a diffeomorphism $\psi : [0, 3] \to [0, 5]$ once and for all such that $\psi(t) = t$ for $t \in [0, 1]$ and $\psi(t) = t + 2$ for $t \in [2, 3]$. Define $\kappa : E\, {}_\sigma{\times}_\tau\, E \longrightarrow E$ by

$$\kappa(f, g)(t) = \begin{cases} g(\psi(t)) & \text{if} \quad \psi(t) \in [0, 3] \\ f(\psi(t) - 2) & \text{if} \quad \psi(t) \in [2, 5]. \end{cases}$$

Construct a map $\iota : B \to E$ such that $(E, B, \sigma, \tau, \kappa, \iota)$ is a composition structure.

Remark 2.9. Revaz Kurdiani pointed out (in 2004) that definitions 2.2 and 2.4 can be formalized in the following manner. The category $\mathscr{C}$ of spaces over $B \times B$ is a *monoidal category* [21, §VII.1] with monoidal operation $\square$ given by

$$E \square E' := \{(x, y) \in E \times E' \mid \sigma(x) = \tau'(y)\}$$

for spaces E and E' over $B \times B$, with reference maps $(\tau, \sigma) : E \to B \times B$ and $(\tau', \sigma') : E' \to B \times B$. Here $E \square E'$ is again a space over $B \times B$ with reference map $(x, y) \mapsto (\tau(x), \sigma'(y))$. There is a two-sided *unit object* $1_{\mathscr{C}}$ for the monoidal operation, given by the diagonal $B \to B \times B$, viewed as a space over $B \times B$.

Next, the category $\mathscr{C}$ with the above monoidal operation *acts* [36, Ch.4 Defn.4.7] on the category $\mathscr{D}$ of spaces over B by

$$E \square Z := \{(x, z) \in E \times Z \mid \sigma(x) = \pi(z)\}$$

for spaces E over $B \times B$ and Z over B, with reference maps $(\tau, \sigma) : E \to B \times B$ and $\pi : Z \to B$. Here $E \square Z$ is meant to be a new object of $\mathscr{D}$ with reference map $(x, z) \mapsto \tau(x)$.

In the categories $\mathscr{C}$ and $\mathscr{D}$, there are notions of homotopy between morphisms. (I hear the category enthusiasts shouting at me that $\mathscr{C}$ and $\mathscr{D}$ are categories enriched over simplicial sets.) What we see in definitions 2.2 and 2.4 can therefore be reformulated as follows.

- We have an object E in $\mathscr{C}$, a morphism $\kappa : E \square E \to E$ and a morphism ι from $1_{\mathscr{C}}$ to E.
- The morphism obtained by composing
$$E \;\cong 1_{\mathscr{C}} \square E \xrightarrow{\iota \square \mathrm{id}_E} E \square E \xrightarrow{\kappa} E$$
is homotopic to id_E , and the morphism obtained by composing
$$E \;\cong\; E \square 1_{\mathscr{C}} \xrightarrow{\mathrm{id}_E \square \iota} E \square E \xrightarrow{\kappa} E$$
is homotopic to id_E .
- We have an object Z in $\mathscr{D}$ and a morphism $\alpha : E \square Z \to Z$ in $\mathscr{D}$.
- The morphisms from $E \square E \square Z$ to Z given by composing, respectively,
$$E \square (E \square Z) \xrightarrow{\mathrm{id}_E \square \alpha} E \square Z \xrightarrow{\alpha} Z$$
$$(E \square E) \square Z \xrightarrow{\kappa \square \mathrm{id}_Z} E \square Z \xrightarrow{\alpha} Z$$
are homotopic.
- The morphism $Z \to Z$ given by composing
$$Z \cong 1_{\mathscr{C}} \xrightarrow{\iota \square \mathrm{id}_Z} E \square Z \xrightarrow{\alpha} Z$$
is homotopic to id_Z.

3 Fibrations and related notions

In this section and the next, the main results from fibration theory which we will need are collected.

Definition 3.1. A map $p : E \to B$ has the *homotopy lifting property*, HLP (also known as the *covering homotopy property*), if the following holds. Given any space X and (continuous) maps

$$f : X \times [0, 1] \to B, \qquad \bar{f}_0 : X \to E$$

such that $p\bar{f}_0(x) = f(x, 0)$ for all $x \in X$, there exists a map $\bar{f} : X \times [0, 1] \to E$ such that $p\bar{f} = f$ and $\bar{f}(x, 0) = \bar{f}_0(x)$ for all $x \in X$.

If p has the HLP, it is called a fibration.

(HLP)
$$\begin{tikzcd} X \arrow[r, "\bar{f}_0"] \arrow[d, "x \mapsto (x,0)"'] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "f"'] \arrow[ur, dotted, "\bar{f}"] & B \end{tikzcd}$$

More vocabulary. It is also common to say *Hurewicz fibration* in the above circumstances. If p satisfies the above whenever X is a *CW*–space, then p is a *Serre fibration.*

There is a useful extension of the HLP which is called HELP: *homotopy extension lifting property.* See for example [38, I.7.16]. We formulate a special case of this for Serre fibrations. Let X be any CW-space, $A \subset X$ a CW-subspace.

Proposition 3.2. *Let $p\colon E \to B$ be a Serre fibration. Let X be a CW-space with a CW-subspace A and let $Z = X \times \{0\} \cup A \times [0,1]$, a subspace of $X \times [0,1]$. Given maps*

$$f\colon X \times [0,1] \to B, \qquad \bar{f}_Z\colon Z \to E$$

such that $p\bar{f}_Z(x,t) = f(x,t)$ for all $(x,t) \in Z$, there exists a map $\bar{f}\colon X \times [0,1] \to E$ such that $p\bar{f} = f$ and $\bar{f}(x,t) = \bar{f}_Z(x,t)$ for all $x \in Z$.

$$\text{(HELP)} \qquad \begin{tikzcd} Z = \begin{array}{c} X \times \{0\} \\ \cup\, A \times [0,1] \end{array} \arrow[r, "\bar{f}_Z"] \arrow[d, "\text{inclusion}"'] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "f"] \arrow[ur, dotted, "\bar{f}"] & B \end{tikzcd}$$

Remark 3.3. The following is a very useful fact: a map $p : E \to B$ is a Serre fibration if it satisfies the HLP above in all cases where X is a (compact) disk D^n, any $n \geq 0$. Most of the proof of that can be found in or near [32, 7.2.5]. More precisely, it is shown there that if $p\colon E \to B$ has the HLP in all cases where the test space X is a disk D^n, then it has the HELP in all cases where the test pair (X, A) is of the form (D^n, S^{n-1}). From there it is easy to deduce by induction over skeleta that $p\colon E \to B$ has the HELP whenever the test pair (X, A) consists of a CW-space X and a CW-subspace A. The induction is in fact carried out in [32] in those cases where (X, A) is a pair of simplicial complexes.

The concept of Serre fibration has a lot to do with the notion of a *weak homotopy equivalence*, as we shall see. A map $f\colon Y \to Z$ is a weak homotopy equivalence if for every CW-space X, the induced map $f_*\colon [X, Y] \to [X, Z]$ is a bijection, where $[-,-]$ denotes homotopy classes of maps. According to JHC Whitehead's theorem, there is another characterization: a map $f\colon Y \to Z$ is a weak homotopy equivalence if and only if, for every choice of base point y in Y and every $n \geq 0$, the map of homotopy groups or homotopy sets $\pi_n(Y, y) \to \pi_n(Z, f(y))$ induced by f is an isomorphism (bijection of pointed sets when $n = 0$).

If Y and Z are themselves CW-spaces, or homotopy equivalent to CW-spaces, then $f\colon Y \to Z$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.

Remark 3.4. If $p\colon E \to B$ is a fiber bundle and B is paracompact Hausdorff, then p is a fibration. This is well known, but the proof is not easy. Spanier [32, 2.7.13] proves more (and refers to Hurewicz and others as the originators). Let $p\colon E \to B$ a map where B is paracompact Hausdorff. Suppose that every $b \in B$ admits a neighborhood U in B such that the map $p^{-1}(U) \to U$ obtained by restricting p is a fibration. *Then p is a fibration.* This is applicable to fiber bundles because, if $p\colon E \to B$ is a fiber bundle, then every $b \in B$ admits a neighborhood U in B such that $p^{-1}(U)$ is homeomorphic, *over* U, to a product $U \times F$. In this situation $p^{-1}(U) \to U$ is clearly a fibration. We will not need this but we shall need some variants and weaker forms.

Exercise 3.5. Let $p\colon E \to B$ be a Serre fibration, where B is path connected. Let $x, y \in B$. Show that the spaces $p^{-1}(x)$ and $p^{-1}(y)$ are homotopy equivalent.

Example 3.6. The map $p\colon S^1 \to S^1$ given by $z \mapsto z^2$ (in complex number notation) is a fiber bundle, therefore a fibration. The evaluation map $\mathrm{O}(n) \to S^n$ given by $A \mapsto A\mathbf{e}_1$ is a fiber bundle (where $\mathbf{e}_1$ is the first standard basis vector). The projection from the triangle $\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, x + y \leq 1\}$ to the interval $[0, 1]$ given by $(x, y) \mapsto x$ is a fibration, but not a fiber bundle.

Example 3.7. Let $f\colon X \to Y$ be any (continuous) map of spaces. Following Serre, we can write f as a composition $X \to X^\sharp \to Y$ where the first arrow is a homotopy equivalence and the second arrow, $X^\sharp \to Y$, is a fibration. The definition of $X^\sharp$ is

$$X^\sharp = \{(x, w) \mid x \in X,\ w\colon [0,1] \to Y,\ w(0) = f(x)\ \}.$$

In words: an element of $X^\sharp$ is a pair (x, w) consisting of an $x \in X$ and a path w in Y starting at $f(x)$. The space $X^\sharp$ is to be topologized as a subspace of $X \times Y^{[0,1]}$ where $Y^{[0,1]}$ denotes the space of continuous maps from $[0, 1]$ to Y, with the compact-open topology. The map $X^\sharp \to Y$ in the promised factorization of f is defined by

$$(x, w) \mapsto w(1)\ .$$

It was shown by Serre and is also shown for example in Spanier [32, 2.8.9] that $X^\sharp \to Y$ is a fibration. The Serre construction is a very basic tool in homotopy theory and does, obviously, provide many examples of fibrations.

Example 3.8. Let $p\colon E \to B$ be a fibration and let $g\colon A \to B$ be any map. Let g^*E be the subspace of $A \times E$ consisting of $(x, y) \in A \times E$ with $g(x) = p(y)$. There is a forgetful projection $g^*E \to A$. It is again a fibration. (Exercise.) We say that $g^*E \to A$ is the fibration obtained from $p\colon E \to B$ by pullback along $g\colon A \to B$.

Remark. The definition of the pullback as a space is obviously quite symmetric, despite the asymmetrical designation g^*E chosen above. Quite generally, suppose that $u\colon X \to Z$ and $v\colon Y \to Z$ are maps of spaces. Their *pullback* is the subspace of $X \times Y$ consisting of the pairs (x, y) which satisfy $u(x) = v(y)$.

Exercise 3.9. [5, ch.XX Thm.4.5] Let $p\colon M \to X$ be a fibration, where M is a closed nonempty manifold and X is any path–connected space having more than one point. Show that p is not nullhomotopic. [Hint: Suppose for a contradiction that p is nullhomotopic. Make a homotopy lifting problem out of a nullhomotopy for p.]

Exercise 3.10. An observation related to Proposition 1.15: Show that the restriction map $\mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(A^p \times D^q, N)$ is not a fibration when $N = \mathbb{R}$, $p = 1$ and $q = 0$.

Exercise 3.11. Similar to the previous exercise, but harder: Show that the restriction map $\mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(A^p \times D^q, N)$ is not a fibration when $N = \mathbb{R}^2$, $p = 2$ and $q = 0$. [Hint: It is enough to exhibit two immersions $f, g\colon A^2 \to \mathbb{R}^2$ which are in the same path component of $\mathrm{imm}(A^2, \mathbb{R}^2)$, in such a way that f extends to an immersion $D^2 \to S^2$ whereas g does not. Find conditions which ensure that an immersion $A^2 \to \mathbb{R}^2$ does not extend to an immersion $D^2 \to \mathbb{R}^2$.]

Definition 3.12. A map $p\colon E \to B$ is also called a *space over* B. Given two spaces over B, say $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$, a map over B from p_1 to p_2 is a map $f\colon E_1 \to E_2$ such that $p_2 f = p_1$. To be honest, the standard expression is: "... a map from E_1 to E_2 over B ...". Given two maps $f, g\colon E_1 \to E_2$, both over B, and a homotopy $h\colon E_1 \times [0,1] \to E_2$ from f to g, we say that h is a homotopy *over* B if $p_2 h(x,t) = p_1(x)$ for all $x \in E_1$ and $t \in [0,1]$. In this situation we also say that h is a *vertical homotopy*.

Definition 3.13. Let two spaces over B be given, $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$. They are *fiberwise homotopy equivalent* or *homotopy equivalent over* B if there exist maps $u\colon E_1 \to E_2$, $v\colon E_2 \to E_1$ and homotopies α from vu to id_{E_1}, and β from uv to id_{E_2}, such that u, v and α, β are all over B.

Example 3.14. Let $B = \mathbb{R}$, $E_1 = \mathbb{R}$, $p_1 = \mathrm{id}_{\mathbb{R}}$, $E_2 = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ and let $p_2\colon E_2 \to \mathbb{R}$ be given by $(x,y) \mapsto x$. Then p_1 and p_2 are fiberwise homotopy equivalent. Namely, define $u\colon E_1 \to E_2$ by $u(x) = (x,0)$ and $v\colon E_2 \to E_1$ by $v(x,y) = x$ and $\alpha(x,t) = x$ for $x \in E_1$, and $\beta((x,y),t) = (x,(1-t)y)$ for $(x,y) \in E_2$. However: p_1 is a fibration and p_2 is not a fibration. (The path $[0,1] \to B$ given by $t \mapsto t$ cannot be lifted to a path in E_2 with prescribed initial position $(0,1)$, for example.)

Corollary 3.15. *The HLP is not a fibrewise homotopy invariant.*

Definition 3.16. [4, §5]. Let $p\colon E \to B$ be a map. We say that p has the *weak homotopy lifting property*, WHLP, if for every space X and maps

$$f\colon X \times [0,1] \to B, \qquad \bar{f}_0\colon X \to E$$

such that $p\bar{f}_0(x) = f(x,0)$ for all $x \in X$, there exists a map $\bar{f}\colon X \times [0,1] \to E$ such that $p\bar{f} = f$ and the map $x \mapsto \bar{f}(x,0)$ from X to E is *vertically* homotopic to $\bar{f}_0$. In that situation, the map p is called a *weak fibration*.

Proposition 3.17. *Suppose that $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ are homotopy equivalent over B. If p_1 has the WHLP, then so does p_2.*

Proof. For each $s \in [0,1]$ let $\iota_s\colon X \to X \times [0,1]$ be given by $\iota_s(x) = (x,s)$. Let u, v, α, β be maps and homotopies as in definition 3.13. Let $f\colon X \times [0,1] \to B$ and $\bar{f}_0\colon X \to E_2$ be given such that $p_2\bar{f}_0(x) = f(x,0)$ for all $x \in X$. Together, f and $\bar{f}_0$ make up a homotopy lifting problem for p_2. Then f together with $v\bar{f}_0$ constitute a homotopy lifting problem for p_1. Since p_1 has the WHLP, there exists $\bar{f}\colon X \times [0,1] \to E_1$ such that $p_1\bar{f} = f$ and $\bar{f} \circ \iota_0$ is vertically homotopic to $v\bar{f}_0$. Then $u\bar{f}\colon X \times [0,1] \to E_2$ is a homotopy such that $p_2 \circ (u\bar{f}) = f$ and $u\bar{f} \circ \iota_0$ is vertically homotopic to $u \circ (v\bar{f}_0) = (uv) \circ \bar{f}_0$, which is vertically homotopic to $\bar{f}_0$. This homotopy solves the homotopy lifting problem for p_2 that we started with, in the weak sense of the WHLP. ☐

It follows from proposition 3.17 that the map p_2 in example 3.14 has the WHLP, because p_1 in the same example has the HLP (and consequently, the WHLP). This is one of the easiest examples of a map having the WHLP but not the HLP.

Exercise 3.18. Suppose that $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ are both weak fibrations. Let $u\colon E_1 \to E_2$ be a map over B which is an ordinary homotopy equivalence. Then u is a fiberwise homotopy equivalence (i.e., homotopy equivalence over B).

Proposition 3.19. *Let $p\colon E \to B$ be a weak fibration. Let $b \in B$ and let $F = p^{-1}(b) \subset E$. Then for any choice of base point $c \in F$, and any $n \geq 0$, we have $\pi_n(E,F,c) \cong \pi_n(B,b)$ (isomorphism induced by p).*

This was originally proved by Serre for Serre fibrations. There is a proof (for Serre fibrations) in [32, 7.2.9]. Be warned that Spanier uses the expression *weak fibration* for a Serre fibration.

Definition 3.20. A map $p\colon E \to B$ is a *microfibration* (has the micro-HLP) if, for every $f\colon X \times [0,1] \to B$ and $\bar{f}_0\colon X \to E$ with $p\bar{f}_0 = f\iota_0$, there exist a neighborhood U of $X \times \{0\}$ in $X \times [0,1]$ and $\bar{f}\colon U \to E$ such that $p\bar{f} = f|_U$ and $\bar{f}\iota_0 = \bar{f}_0$.

There is also a notion of *Serre microfibration*. Namely, $p\colon E \to B$ is a Serre microfibration if, for every $f\colon X \times [0,1] \to B$ and $\bar{f}_0\colon X \to E$ with $p\bar{f}_0 = f\iota_0$ where X is a CW-space, there exist a neighborhood U of $X \times \{0\}$ and $\bar{f}\colon U \to E$ such that $p\bar{f} = f|_U$ and $\bar{f}\iota_0 = \bar{f}_0$.

Gromov advertised and used Serre microfibrations in [10, 1.4.2]. For some recent theory and applications related to the notion, see [37], [28], [7, §2.2].

Exercise 3.21. Show that if $p\colon E \to B$ is a fibration and $V \subset E$ is open, then $p|_V$ is a microfibration.

Example 3.22. Let $p\colon \mathbb{R}^2 \to \mathbb{R}$ be the standard projection given by $(x,y) \mapsto x$. This is obviously a (trivial) fibration. Let $V = \mathbb{R}^2 \smallsetminus \{(0,0)\}$. This is open in $\mathbb{R}^2$, so $p|V$ is a microfibration. But $p|V$ is not a fibration, since the fiber of $p|V$ over $0 \in \mathbb{R}$ is not homotopy equivalent to the fiber of $p|V$ over $1 \in \mathbb{R}$.

Exercise 3.23. Let M, N be smooth manifolds and let $f\colon M \to N$ be a smooth submersion (i.e., for every $x \in M$ the differential $df(x)\colon T_xM \to T_{f(x)}M$ is surjective). Show that f is a Serre microfibration.

Lemma 3.24. A map $p\colon E \to B$ which is a weak fibration and a Serre microfibration is a Serre fibration.

Proof. First we discuss the case where the test space is a point. Let $w\colon [0,1] \to B$ and $\bar{w}_0 \in E$ be given, with $p(\bar{w}_0) = w(0) \in B$. Using the WHLP we obtain

$$v\colon [0,1] \to E$$

with $p \circ v = w$, and a path $u\colon [0,1] \to p^{-1}(w(0)) \subset E$ such that $u(0) = v(0)$ and $u(1) = \bar{w}_0$. Choose $\delta \in (0,1/2]$ and let

$$L_\delta = \{(x,y) \in [0,1]^2 \mid xy = 0, \quad x \leq \delta\}\,.$$

Together u and v define a map

$$q\colon L_\delta \to E$$

where $q(x,y) = u(y)$ if $x = 0$ while $q(x,y) = v(x)$ if $y = 0$. Then $p \circ q$ is the map $(x,y) \mapsto w(x)$. Let

$$h\colon [0,\delta] \times L_\delta \longrightarrow B$$

be given by $h(s,x,y) = w(x+s)$. Together, h and q make a homotopy lifting problem which we can micro-solve using the micro-HLP. We obtain

$$H\colon [0,\varepsilon] \times L_\delta \longrightarrow E$$

(for some ε such that $0 < \varepsilon \leq \delta$) so that $p \circ H = h$ where applicable, and H restricted to $\{0\} \times L_\delta$ agrees with q. Now we make a map

$$\Phi\colon [0,\varepsilon] \times [0,1] \longrightarrow [0,\varepsilon] \times L_\delta$$

using the formula $(s,t) \mapsto (s,0,t-s)$ when $t \geq s$, and $(s,t) \mapsto (t, s-t, 0)$ when $t \leq s$. Then

$$(p \circ H \circ \Phi)(x,y) = (h \circ \Phi)(x,y) = w(x) \in B$$

where applicable, and also $(H \circ \Phi)(x,y) = q(x,y) \in E$ if in addition $xy = 0$. Therefore the continuous map $W\colon [0,1] \to E$ defined by

$$W(s) = \begin{cases} H(\Phi(s, 1 - s/\varepsilon)) & \text{if} \quad s \in [0,\varepsilon] \\ q(s,0) = v(s) & \text{if} \quad s \in [\varepsilon, 1] \end{cases}$$

satisfies $p \circ W = w$ and $W(0) = q(0,1) = u(1) = \bar{w}_0$. It is a solution of our path lifting problem consisting of w and $\bar{w}_0$.

The general case where the test space is a disk D^k is very similar. Start with $w\colon [0,1] \times D^k \to B$ and $\bar{w}_0\colon D^k \to E$, where $p(\bar{w}_0(z)) = w(0,z)$ for all $z \in D^k$. Using the WHLP we obtain $v\colon [0,1] \times D^k \to E$ with $p \circ v = w$, and a homotopy $u\colon [0,1] \times D^k \to E$ such that $u(0,z) = v(0,z)$ and $u(1,z) = \bar{w}_0(z)$ for all $z \in D^k$, and $p(u(t,z)) = \bar{w}_0(z)$ for all $z \in D^k$. Together u and v define a map

$$q\colon L_\delta \times D^k \to E\,.$$

Define $h\colon [0,\delta] \times L_\delta \times D^k \longrightarrow B$ by $h(s,x,y,z) = w(x+s,z)$. Together, h and q make a homotopy lifting problem which we can micro-solve using the micro-HLP. We obtain $H\colon [0,\varepsilon] \times L_\delta \times D^k \longrightarrow E$. Then the continuous map $W\colon [0,1] \times D^k \to E$ defined by

$$W(s,z) = \begin{cases} H((\Phi(s, 1 - s/\varepsilon), z) & \text{if} \quad s \in [0,\varepsilon] \\ q(s,0,z) = v(s,z) & \text{if} \quad s \in [\varepsilon, 1] \end{cases}$$

satisfies $p \circ W = w$ and $W(0,z) = q(0,1,z) = u(1,z) = \bar{w}_0(z)$. ☐

Corollary 3.25. *Let $p\colon E \to B$ be a map where B is paracompact. Suppose that*

- *p is locally fiber homotopy trivial, that is, every $b \in B$ admits an open neighborhood U_b in B such that the restriction $p^{-1}(U_b) \to U_b$ of p is homotopy equivalent over U_b to a trivial fiber bundle;*
- *p is a Serre microfibration.*

Then p is actually a Serre fibration.

Proof. Let $b \in B$ and choose an open neighborhood U_b so that $p^{-1}(U_b) \to U_b$ is homotopy equivalent over U_b to a trivial fiber bundle. Then $p^{-1}(U_b) \to U_b$ is a weak fibration (has the WHLP) because the WHLP is a fiberwise homotopy invariant. Since p is a Serre microfibration, $p^{-1}(U_b) \to U_b$ is also a Serre microfibration. Therefore $p^{-1}(U_b) \to U_b$ is a Serre fibration by lemma 3.24. Now a map which is locally a Serre fibration is globally a Serre fibration (exercise 3.26). ☐

Exercise 3.26. Let $p\colon E \to B$ be a map and suppose that every $b \in B$ admits an open neighborhood U_b in B such that the restriction $p^{-1}(U_b) \to U_b$ of p is a Serre fibration. Show that p is a Serre fibration. [*Hint.* Write $I := [0,1]$. Begin with a lifting problem consisting of $f\colon I^n \to E$ and $h\colon I^n \times I \to B$ such that $f(x) = h(x,0)$ for all $x \in I^n$. For a sufficiently large positive integer q, the image under h of each of the cubes

$$\prod_{s=1}^{n+1} [\, a_s q^{-1}, b_s q^{-1}]$$

(where $a_1, a_2, \dots, a_{n+1} \in \{0,1,2,\dots,q-1\}$ and $b_s = a_s + 1$) is contained in a subset U of B such that the map $p^{-1}(U) \to U$ obtained by restricting p is a Serre fibration. Now use proposition 3.2 repeatedly.]

4 Composition structures and fibrations

For the following lemma, we adopt the notation and assumptions of definition 2.2. Suppose in addition that B is a disk D^i, and that $(\tau,\sigma)\colon E \to B \times B$ is a Serre microfibration.

Lemma 4.1. *Then for every $b \in B$ there exists a neighborhood U_b in B and maps*

- $m_{\text{out}}\colon U_b \to E$,
- $m_{\text{in}}\colon U_b \to E$,
- $h_{\text{gen}}\colon U_b \times [0,1] \to E$,
- $h_{\text{spec}}\colon U_b \times [0,1] \to E$

such that for any $c \in U_b$,

- *$m_{\text{out}}(c)$ maps to (c,b) under $(\tau,\sigma)\colon E \to B \times B$,*
- *$m_{\text{in}}(c)$ maps to (b,c) under $(\tau,\sigma)\colon E \to B \times B$,*
- *the path $t \mapsto h_{\text{gen}}(c,t)$ begins at $\kappa(m_{\text{out}}(c), m_{\text{in}}(c))$, ends at $\iota(c)$ and runs in the fiber of $(\tau,\sigma)\colon E \to B \times B$ over (c,c);*
- *the path $t \mapsto h_{\text{spec}}(c,t)$ begins at $\kappa((m_{\text{in}}(c), m_{\text{out}}(c))$, ends at $\iota(b)$ and runs in the fiber of $(\tau,\sigma)\colon E \to B \times B$ over (b,b).*

(Reasons for the notation: m_{out} selects *out*going *m*orphisms, from the fixed b to other objects; m_{in} selects *in*coming *m*orphisms, from other objects to the fixed object b; next, h_{gen} selects paths in the endomorphism space of *general* objects c and h_{spec} selects paths in the endomorphism space of the *specified* object b.)

Proof. Assume first that b is in the interior of the disk $B = D^i$. Then without loss of generality it is the center $\underline{0}$ of the disk, and the disk is the cone on a sphere S^{i-1}. We write

$$B \cong [0,1] \times S^{i-1}/\sim$$

where $\sim$ identifies all points of the form $(0,c)$ with $\underline{0}$. Thus the identity $B \to B$ can be regarded as a homotopy from a constant map $S^{i-1} \to B$ with value $\underline{0}$ to the inclusion $S^{i-1} \to B$. This homotopy gives us *two* homotopies,

$$(g_s\colon S^{i-1} \to B\times B)_{s\in[0,1]} \quad\text{and}\quad (h_s\colon S^{i-1} \to B\times B)_{s\in[0,1]}\,,$$

where $g_s(c) = (sc, \underline{0})$ and and $h_s(c) = (\underline{0}, sc)$. For these homotopies, we have initial lifts $\bar{g}_0\colon S^{i-1} \to E$ and $\bar{h}_0\colon S^{i-1} \to E$ respectively, which are constant with value $\iota(\underline{0})$. (*Lift* means that if we compose on the left with the map $(\tau,\sigma)\colon E \to B \times B$, we obtain g_0 and h_0, respectively.) Using the micro–HLP for $(\tau,\sigma)\colon E \to B \times B$, we therefore get the map m_{out} as a micro–lift of the homotopy (g_s), and m_{in} as a micro–lift of the homotopy (h_s). The maps m_{out}, m_{in} are defined on a neighborhood of $\underline{0}$ in B. We obtain this as a neighborhood of $\{0\} \times S^{i-1}$ in $[0,1] \times S^{i-1}$, divided out by $\{0\} \times S^{i-1}$.)

The construction of the homotopies h_{gen} and h_{spec} is similar, except that we need the relative HLP (alias HELP). Let

$$K = B \times [0,1], \qquad L = S^{i-1} \times [0,1]\,,$$

so that we can identify K with a quotient of $[0,1] \times L$. In particular the two maps from K to $B \times B$ given by $(c,t) \mapsto (c,c)$ and $(c,t) \mapsto (\underline{0},\underline{0})$ can then be regarded as two homotopies

$$(G_s\colon L \longrightarrow B \times B)_{s\in[0,1]}\,, \quad (H_s\colon L \longrightarrow B \times B)_{s\in[0,1]}$$

given by $G_s(c,t) = (sc, sc)$ and $H_s(c,t) = (\underline{0},\underline{0})$, for all $c \in S^{i-1}$. We have an initial lift for both, given by the map $L \to E$ taking $(c,t) \in L = S^{i-1} \times [0,1]$ to $\omega(t)$, where ω is a path in E such that $\omega(0) = \kappa(\iota(\underline{0}), \iota(\underline{0}))$ and $\omega(1) = \iota(\underline{0})$. (The path ω is meant to run in the fiber of the map $(\tau,\sigma)\colon E \to B \times B$ over the point $(\underline{0},\underline{0})$; its existence is guaranteed by definition 2.2.) We also have micro-lifts for the restricted homotopies $(G_s|S^{i-1} \times \{0,1\})$ and $(H_s|S^{i-1} \times \{0,1\})$. These lifts can be defined by the formulae

$$(c,t) \mapsto \begin{cases} \kappa(m_{\text{out}}(sc), m_{\text{in}}(sc)) & t = 0,\ \text{case of } (G_s) \\ \iota(sc) & t = 1,\ \text{case of } (G_s) \\ \kappa(m_{\text{in}}(sc), m_{\text{out}}(sc)) & t = 0,\ \text{case of } (H_s) \\ \iota(\underline{0}) & t = 1,\ \text{case of } (H_s) \end{cases}$$

where we are assuming $s \in [0, \varepsilon_1]$, for an $\varepsilon_1 > 0$ which we have from the earlier construction of m_{out} and m_{in}. (Note that $\iota(\underline{0}) = m_{\text{in}}(\underline{0}) = m_{\text{out}}(\underline{0})$ by construction.) From the micro–HELP of proposition 3.2, we obtain lifted (micro)–homotopies, micro–lifting (G_s) and (H_s). We call them h_{gen} and h_{spec}, respectively. In more detail, we get two maps defined on $[0,\varepsilon_2] \times S^{i-1} \times [0,1]$ for some $\varepsilon_2 > 0$ which is $\le \varepsilon_1$. Their restriction to $\{0\} \times S^{i-1} \times [0,1]$ is given by $(0,c,t) \mapsto \omega(t)$, independent of $c \in S^{i-1}$. Since $B \cong [0,1] \times S^{i-1}/\sim$, we can view these maps as being defined on $U \times [0,1] \subset B \times [0,1]$, where U is the closed disk of radius ε_2 about $\underline{0}$. In other words, our maps can be viewed as two homotopies h_{gen} and h_{spec} between certain maps from U to E.

The case where b is on the boundary of $B = D^i$ is very similar. We may think of B as the cone on a hemisphere (closed upper half of S^{i-1}) and of $b = \underline{0}$ as the apex alias center of the cone. The details are left to the reader. □

Proof of lemma 2.6. We begin with some easy reductions. Firstly, we can reduce to the case where B is a disk. Namely, suppose that $f\colon X \times [0,1] \to B$ is a homotopy (with B still arbitrary) which we want to lift across $\pi\colon Z \to B$, with an initial lift $\bar{f}_0\colon X \to Z$. Since we are going for the Serre fibration property, we may assume that X is a disk D^i. But then $B' := X \times [0,1]$ is also (homeomorphic to) a disk D^{i+1}. We can now use $f\colon B' \to B$ to pull the entire homotopy lifting problem and the composition structure and action data back to B'. Thus we replace B by B' and Z by

$$Z' = f^*Z = \{(c, z) \in B' \times Z \mid f(c) = \pi(z)\}$$

and E by

$$E' = \{(e, c, d) \in E \times B' \times B' \mid \tau(e) = f(c),\ \sigma(e) = f(d)\}\,.$$

The homotopy f itself can be replaced by the identity $X \times [0, 1] \to B'$ (but try to forget that it is an identity map) and the initial lift $\bar{f}_0$ can be replaced by the map

$$X \longrightarrow Z' \subset B' \times Z$$

whose second coordinate is $\bar{f}_0$ and whose first coordinate is the inclusion of $X \cong X\times\{0\}$ to $X\times[0, 1]$. If this new homotopy lifting problem with base space B' has a solution, then that solution determines a solution for the old homotopy lifting problem: just compose with projection $Z' \to Z$.

Secondly, to show that $\pi\colon Z \to B$ is a Serre fibration when B is a disk, we only need to show that it is locally fiber homotopy trivial, because of corollary 3.25. But this is almost obvious from lemma 4.1. Given $b \in B$ we choose U_b as in that lemma, along with the maps and homotopies m_{out}, m_{in}, h_{gen} and h_{spec}. Now fix some $c \in U_b$. Then we have a preferred choice of maps

$$\pi^{-1}(b) \longrightarrow \pi^{-1}(c)\,,\ \pi^{-1}(c) \longrightarrow \pi^{-1}(b)$$

given by acting on the left with $m_{\text{out}}(c)$ and $m_{\text{in}}(c)$, respectively (using the "action" α described in definition 2.4). These two maps are homotopy inverses of each other. Indeed, the composite maps $\pi^{-1}(c) \longrightarrow \pi^{-1}(c)$ and $\pi^{-1}(b) \longrightarrow \pi^{-1}(b)$ are homotopic to the respective identity maps by means of the homotopies given by the actions of $h_{\text{gen}}(t, c)$ on $\pi^{-1}(c)$, and $h_{\text{spec}}(t, c)$ on $\pi^{-1}(b)$, respectively, for $t \in [0, 1]$. Allowing c to vary, we have a homotopy equivalence

$$\pi^{-1}(b) \times U_b \longrightarrow \pi^{-1}(U_b)$$

which is, by construction, a homotopy equivalence over U_b. This completes the proof. □

Exercise 4.2. Let $(E, B, \sigma, \tau, \kappa, \iota)$ be a composition structure as in definition 2.2. Suppose that $(\tau, \sigma)\colon E \to B \times B$ is a Serre microfibration. Show that $\tau\colon E \to B$ and $\sigma\colon E \to B$ are Serre fibrations.

5 Spaces of smooth maps and spaces of immersions

Our main goal in this section is to prove proposition 1.15 by deducing it from lemma 2.6. We have to start with a lengthy discussion of spaces of smooth maps and spaces of immersions.

Let M and N be smooth manifolds. Assume $\partial N = \emptyset$, but let's not assume that $\partial M = \emptyset$. We denote by $C^\infty(M, N)$ the set of all smooth maps from M to N. This comes with a preferred topology, the compact–open C^∞ topology. This is described in [15, §2.1].

Example 5.1. If M is compact, then $\text{imm}(M, N) \subset C^\infty(M, N)$ is an open subset. If M is noncompact, this is usually not the case.

Exercise 5.2. Show that $\text{imm}(\mathbb{R}^m, \mathbb{R}^m)$ is not open in $C^\infty(\mathbb{R}^m, \mathbb{R}^m)$ for $m > 0$.

Next we have a well-known lemma about constructing smooth functions with prescribed higher derivatives. This is essentially due to E. Borel.

Lemma 5.3. *Let L be a smooth compact manifold. For $i = 0, 1, 2, \ldots$ let $f_i\colon L \to \mathbb{R}$ be smooth functions. Then there exists a smooth $F\colon L \times \mathbb{R} \to \mathbb{R}$ such that the i–th partial derivative of F in the $\mathbb{R}$ direction, evaluated along $L \times \{0\} \cong L$, equals f_i.*

Proof. (The proof is reproduced from [8, IV, Lem.2.5] because it proves more than the lemma states, and we need that extra information.) There is no loss of generality in assuming that L is a codimension zero compact smooth submanifold of a euclidean space $\mathbb{R}^n$. (Otherwise, embed L in a euclidean space, and replace it by the total space of a normal disk bundle in the euclidean space. This will have corners if ∂L is nonempty; remove them by rounding corners without touching the zero section. Replace f_i by $f_i p$ where p is the disk bundle projection.) The advantage which we have from that is that we can use standard notation for partial derivatives.

To begin with fix a smooth function $\rho\colon \mathbb{R} \to \mathbb{R}$ such that $\rho(t) = 1$ for $|t| \le \frac{1}{2}$ and $\rho(t) = 0$ for $|t| \ge 1$. Set

$$(*) \qquad F(x, t) = \sum_{i=0}^{\infty} \frac{t^i}{i!}\rho(\mu_i t) f_i(x)$$

where the (large) real numbers $\mu_i \ge 1$ are yet to be determined. We want to choose them in such a way that the series

$$(**) \qquad \sum_{i=0}^{\infty} D^\alpha\!\left(\frac{t^i}{i!}\rho(\mu_i t) f_i(x)\right)$$

is uniformly convergent for every multi–index

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}).$$

If that can be achieved, then F is well defined and $(*)$ can be differentiated term by term, and F solves our problem.

To determine the numbers μ_i, write the i–th term in $(*)$ in the form

$$(\mu_i)^{-i} f_i(x) \cdot (i!)^{-1}(\mu_i t)^i \rho(\mu_i t) = (\mu_i)^{-i} f_i(x) \cdot \psi_i(\mu_i t)$$

where $\psi_i(t) = (i!)^{-1} t^i \rho(t)$. Next let

$$M_i = \max\left\{ D^\alpha\left(f_i(x)\psi_i(t)\right) \,\Big|\, \quad (x, t) \in L \times \mathbb{R},\ |\alpha| < i \right\}.$$

(Because ψ_i vanishes outside $[-1, 1]$, the maximum can be taken over (x, t) in the compact set $L \times [-1, 1]$ and over the finitely many α which satisfy $|\alpha| < i$.) Since $\mu_i > 1$, it follows for $|\alpha| < i$ that

$$|i\text{–th element in } (**)| \le (\mu_i)^{|\alpha|}(\mu_i)^{-i}M_i \le M_i\mu_i^{-1}\,.$$

Now choose $\mu_i = \max\{1, 2^iM_i\}$. Then for fixed α and for any $i > |\alpha|$, the i–th term of $(**)$ is bounded by 2^{-i}. □

But let's not stop there. The construction of F in terms of the f_i is quite explicit. The only "random" choice which we made was the choice of the function ρ, which really should be made once and for all at the beginning. Then the construction amounts to a map of the form

$$\textstyle\prod_{i=0}^{\infty} C^\infty(L,\mathbb{R}) \quad\longrightarrow\quad C^\infty(L\times\mathbb{R},\mathbb{R})\ \ .$$

This map is *continuous* (with the product topology in the LHS). To verify this, let's first observe that the formula for each number $\mu_i = \mu_i(f_0, f_1, f_2, \dots)$ is continuous as a function of the variables $f_0, f_1, f_2, \dots$. In fact it depends only on f_i and that dependence can be expressed in terms of the values of f_i and the partial derivatives of f_i , of order $< i$. (The derivatives of ρ are also involved in that expression, but only those of order $< i$. Each of them has a maximum since ρ vanishes outside a compact interval.) Next we need to know that $(**)$ depends continuously on $(f_0, f_1, f_2, \dots)$ for fixed α. Write $(**)$ as a sum

$$\sum_{i=0}^{k} D^\alpha\bigg(\frac{t^i}{i!}\rho(\mu_i t)f_i(x)\bigg) + \sum_{i=k+1}^{\infty} D^\alpha\bigg(\frac{t^i}{i!}\rho(\mu_i t)f_i(x)\bigg)$$

where k is larger than $|\alpha|$. The continuous dependence of each μ_i on $(f_0, f_1, f_2, \dots)$ implies that the first of the two summands depends continuously on $(f_0, f_1, f_2, \dots)$. For the other summand, we have the bound $2^{-k} + 2^{-k-1} + 2^{-k-2} + \cdots = 2^{1-k}$, which we can make as small as we like by choosing k large.

We are therefore in a position to formulate the following astonishing corollary to (the proof of) E. Borel's lemma:

Corollary 5.4. *The map from $C^\infty(L\times\mathbb{R},\mathbb{R})$ to $\prod_{i\ge0} C^\infty(L,\mathbb{R})$ given by*

$$F \mapsto \left(\frac{\partial^i F}{\partial t^i}\,\Big|_{t=0}\right)_{i=0,1,2,\dots}$$

has a continuous right inverse.

Let M and N be smooth as before. Let $M_0 \subset M$ be a compact codimension zero submanifold. (Boundary legislation is as follows: We require that there exist a smooth map $f\colon M \to \mathbb{R}$ such that f and $f|\partial M$ are transverse to 0, and such that $M_0 = \{x \in M \mid f(x) \le 0\}$. This is equivalent to saying that M_0 looks "locally" like $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{m-2}$ in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^{m-2}$, where $\mathbb{R}_+ = \{z \in \mathbb{R} \mid z \ge 0\}$. In particular M_0 is a manifold with corners if $M_0 \cap \partial M$ is nonempty.)

Lemma 5.5. *The restriction map $\rho\colon C^\infty(M,N) \to C^\infty(M_0,N)$ is a Serre fibration. If $N = \mathbb{R}^i$ for some $i \ge 0$, it admits a (continuous) section.*

Proof. First sentence (the restriction map is a Serre fibration): this is left as an exercise with the following organizational hints.

Step 1. Reduce to the case where $M = L \times \mathbb{R}$ and $M_0 = L \times [0,\infty)$ for a compact smooth manifold with boundary L. This reduction will not be used heavily until later in step 4.

Step 2. Using corollary 5.4, show that

$$\rho\colon C^\infty(L\times\mathbb{R},\mathbb{R}^i) \to C^\infty(L\times[0,\infty),\mathbb{R}^i)$$

admits a section.

Step 3. Show that $\rho\colon C^\infty(L\times\mathbb{R},\mathbb{R}^i) \to C^\infty(L\times[0,\infty),\mathbb{R}^i)$ is a fibration, using the section from step 2 and the vector space structure on $C^\infty(L\times\mathbb{R},\mathbb{R}^i)$. (Indeed, any continuous linear map between topological vector spaces which admits a section is a fibration.)

Step 4. Using step 3 show that $\rho\colon C^\infty(L\times\mathbb{R},V) \to C^\infty(L\times[0,\infty),V)$ is a fibration whenever V is an open subset of $\mathbb{R}^i$. (Smooth maps $L\times\mathbb{R} \to \mathbb{R}^i$ which take $L\times[0,\infty)$ to V can be precomposed with smooth maps $L\times\mathbb{R} \to L\times\mathbb{R}$ which agree with the identity on $L\times[0,\infty)$, in such a way that the composition lands in V.)

Step 5. Show that $\rho\colon C^\infty(L\times\mathbb{R},N) \to C^\infty(L\times[0,\infty),N)$ is a fibration for arbitrary N. Without loss of generality, N is a smooth submanifold of $\mathbb{R}^i$ which is also a closed subset of $\mathbb{R}^i$, and it has a tubular neighborhood V in $\mathbb{R}^i$.

For the second sentence (existence of a section): reduce to the case $i = 1$. That follows easily from corollary 5.4. □

Corollary 5.6. *If M is compact, then the restriction map* $\mathrm{imm}(M,N) \to \mathrm{imm}(M_0,N)$ *is a Serre microfibration.*

Proof. See example 5.1 and exercise 3.21. □

Proof of proposition 1.15. As explained in section 2, this will be deduced from lemma 2.6. The appropriate interpretations of E, B, Z etc. in lemma 2.6 are given in section 2; see especially definitions 2.1 and definition 2.2. To obtain $\kappa\colon E\,{}_\sigma\!\times_\tau E \longrightarrow E$ choose an identification of the colimit (pushout) of

$$[0,3] \xleftarrow{\ t+2 \leftarrow\!\shortmid\ t\ } [0,1] \xrightarrow{\ t\mapsto t\ } [0,3]$$

with $[0,3]$ which extends the identity on the left–hand copy of $[0,1]$ and on the right–hand copy of $[2,3]$. Similarly, to obtain the action map $\alpha\colon E\,{}_\sigma\!\times_\pi Z \to Z$ choose an appropriate identification of the colimit of

$$D^p \xleftarrow{\ (t+1)x/2 \leftarrow\!\shortmid\ (x,t)\ } S^{p-1}\times[0,1] \xrightarrow{\ (x,t)\mapsto(x,t)\ } S^{p-1}\times[0,3]$$

with D^p. All that is straightforward.

It remains to construct $\iota\colon B \to E$. Let $U = (\mathbb{R}^p \smallsetminus 0) \times \mathbb{R}^q$, an open neighbourhood of $A^p \times D^q$ in $\mathbb{R}^p \times \mathbb{R}^q$. The restriction map

$$r : \mathrm{imm}(U, N) \to B = \mathrm{imm}(A^p \times D^q, N)$$

admits a section, $s : B \to \mathrm{imm}(U, N)$ such that $rs = \mathrm{id}_B$. (See exercise 5.7 for instructions.) For $f \in B$ we want to define $\iota(f)$ by

$$\iota(f) = s(f) \circ v$$

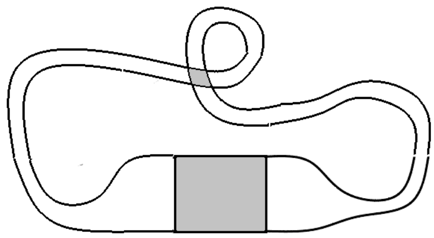
where

$$v\colon S^{p-1} \times [0, 3] \times D^q \longrightarrow U$$

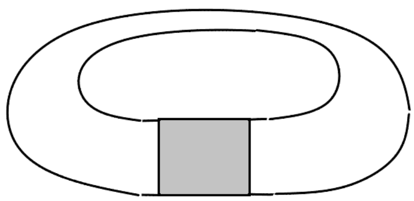
is an immersion yet to be defined, independent of f. We take out a common factor S^{p-1} on both sides, and so we proceed to construct an immersion

$$w\colon [0, 3] \times D^q \to \mathbb{R}_+ \times \mathbb{R}^q \; .$$

Because the target τ and source σ of $\iota(f) = s(f) \circ v$ are prescribed, v is prescribed on $S^{p-1} \times [0, 1] \times D^q$ and on $S^{p-1} \times [2, 3] \times D^q$, and so w is prescribed on $[0, 1] \times D^q$ and on $[2, 3] \times D^q$. Therefore we must have $w(t, y) = (s, y)$ with $s = (t + 1)/2$ for $t \in [0, 1]$, and $w(t, y) = w(s, y)$ with $s = (t - 1)/2$ for $t \in [2, 3]$; in particular $w(t, y) = w(t-2, y)$ for $t \in [2, 3]$. A solution is shown in the following picture, where $q = 1$:



(The shaded square is the image of $[0, 1] \times D^1$ and also the image of $[2, 3] \times D^1$ under w. The white background is an artist's impression of $\mathbb{R}_+ \times \mathbb{R}^1$.) The next picture describes a non-solution:



(Reader: if this is not clear, you may find a return to exercise 2.8 illuminating.) The cases $q > 1$ are similar. To show that ι satisfies (for example) the last condition in definition 2.4, we make the following auxiliary choices.

1. Choice of a section $\zeta : \mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(\mathbb{R}^p \times \mathbb{R}^q, N)$ of the restriction $\mathrm{imm}(\mathbb{R}^p \times \mathbb{R}^q, N) \to \mathrm{imm}(D^p \times D^q, N)$. Constructing ζ is an exercise very similar to the construction of the section s, exercise 5.7.

2. For every $g \in \mathrm{imm}(D^p \times D^q, N)$, with $f = g_{|A^p \times D^q}$, a path γ_g in $\mathrm{imm}(U, N)$ from $\zeta(g)_{|U}$ to $s(f)$ projecting to a constant path in $\mathrm{imm}(A^p \times D^q, N)$, and depending continuously on g. See instructions in exercise 5.8.

Now given $g \in Z = \mathrm{imm}(D^p \times D^q, N)$ with image $\pi(g) = f \in \mathrm{imm}(A^p \times D^q, N)$ we need to construct a path in Z from $\alpha(\iota(f), g)$ to g which projects to a constant path in B. In other words, we need to construct a regular homotopy from the concatenation of $\iota(f)$ and g to g itself which is stationary on $A^p \times D^q$. Let R be the region of $D^p \times D^q$, diffeomorphic to $A^p \times D^q$ but larger than $A^p \times D^q$, where the concatenation $\alpha(\iota(f), g)$ of $\iota(f)$ and g agrees by construction with $\iota(f)$. The restriction of $\alpha(\iota(f), g) \in \mathrm{imm}(D^p \times D^q, N)$ to R is therefore given by

$$s(f) \circ e$$

where e is a fixed codimension zero immersion $R \to U$, not dependent on g or f. But $s(f)$ is $\gamma_g(0)$ and so we have a path in $\mathrm{imm}(R, N)$ given by

$$t \mapsto \gamma_g(t) \circ e$$

where $t \in [0, 1]$. We lift this to a path ω_g in $Z = \mathrm{imm}(D^p \times D^q, N)$ in a trivial manner, by not changing anything on the complement of R. The path ω_g starts therefore with $\alpha(\iota(f), g)$ and ends with an immersion which has the form

$$\zeta(g) \circ u$$

where $u\colon D^p \times D^q \to \mathbb{R}^p \times \mathbb{R}^q$ is an immersion which no longer depends on g, and which agrees with the standard inclusion on $A^p \times D^q$. Finally we choose a path ω' from $u \in \mathrm{imm}(D^p \times D^q, \mathbb{R}^p \times \mathbb{R}^q)$ to the standard inclusion, projecting to the constant path in $\mathrm{imm}(A^p \times D^q, N)$. (Here we need remember that u is ultimately defined in terms of v, so we are using a good property of v which, with hindsight, explains why v was defined the way it was defined.) Let γ_g be the concatenation of paths, $\omega' * \omega_g$. This is the solution to our problem.

The verification that ι also satisfies the remaining conditions is similar. □

Exercise 5.7. Existence of a section s for the restriction map

$$r\colon \mathrm{imm}(U, N) \to \mathrm{imm}(A^p \times D^q, N) \; ;$$

instructions.

(1) Show that the restriction $C^\infty(U, N) \to C^\infty(A^p \times D^q, N)$ admits a (continuous) section, alias right inverse. This is similar to lemma 5.5. Restrict this section to obtain a map $s_1\colon \mathrm{imm}(A^p \times D^q, N) \to C^\infty(U, N)$.

(2) Construct an isotopy of smooth embeddings $v_t\colon U \to U$, where $t \in [0, \infty)$, such that $v_0 = \mathrm{id}$ and such that $v_t(U)$ for $t \geq 1$ is contained in an ε_t-neighborhood of $A^p \times D^q$, where $\varepsilon_t = t^{-1}$. Also ensure $v_t \equiv \mathrm{id}$ on $A^p \times D^q$ for all t.

(3) Let $W \subset \mathrm{imm}(A^p \times D^q, N) \times U$ be the open subset consisting of all (f, z) such that the differential of

the smooth map $s_1(f) : U \to N$ at $z \in U$ is injective, alias invertible. Note that W contains $\mathrm{imm}(A^p \times D^q, N) \times A^p \times D^q$. Construct a (continuous) function τ from $\mathrm{imm}(A^p \times D^q, N)$ to $[1, \infty)$ so that the set of pairs $(f, z) \in \mathrm{imm}(A^p \times D^q, N) \times U$ where the distance from z to $A^p \times D^q$ is $\leq 1/\tau(f)$ is contained in W. Define the section s by $s(f) = s_1(f) \circ v_{\tau(f)}$.

Exercise 5.8. Existence of a homotopy γ; instructions. The problem can be generalized as follows. Let X be any paracompact space and let $\alpha, \beta : X \to \mathrm{imm}(U, N)$ be maps such that $r\alpha = r\beta$, where r is the restriction map from $\mathrm{imm}(U, N)$ to $\mathrm{imm}(A^p \times D^q, N)$. We want to show that there exists a homotopy from α to β over $\mathrm{imm}(A^p \times D^q, N)$.

(1) Show that there is an open neighborhood W of $X \times (A^p \times D^q)$ in $X \times U$ so that the equations $\alpha(x)(z) = \beta(x)(z')$ for $(x, z) \in W$ can be solved simultaneously and continuously for the unknown z'. This defines a map $z \mapsto z'$ from W to $X \times U$, over X.

(2) Let (v_t) be the isotopy from step (2) of exercise 5.7. Construct a function $\psi : X \to [1, \infty)$ such that the image of

$$X \times U \longrightarrow X \times U \ ; \ (x, z) \mapsto (x, v_{\psi(x)}(z))$$

is contained in W. Here the paracompactness of X should be used.

(3) Evidently α is homotopic over $\mathrm{imm}(A^p \times D^q, N)$ to the map taking $x \in X$ to $\alpha(x) \circ v_{\psi(x)}$. By step (1), for each $x \in X$, the immersion $\alpha(x) \circ v_{\psi(x)}$ can be written in the form $\beta(x) \circ \sigma(x)$ for some smooth immersion $\sigma(x) : U \to U$, depending continuously on $x \in X$.

(4) Therefore it only remains to show that the space of immersions $U \to U$ which restrict to the identity on $A^p \times D^q$ is contractible. For that, use the following procedure. For $g \in \mathrm{imm}(U, U)$, choose $t(g) \in [1, \infty)$ so large that the family

$$s \mapsto ((1 - s)g + s \cdot \mathrm{id}_U) \circ v_{t(g)}$$

where $0 \leq s \leq 1$ defines a path in $\mathrm{imm}(U, U)$. Use the vector space structure in $\mathbb{R}^p \times \mathbb{R}^q \supset U$ to make sense of $(1 - s)g + s \cdot \mathrm{id}_U$. Now we have obvious paths from g to $g \circ v_{t(g)}$, and from $g \circ v_{t(g)}$ to $v_{t(g)}$, and from there back to id_U.

Exercise 5.9. Verify that the proof of proposition 1.15 which we have given does not really use the condition $p+q = n$. Consequently that condition can be dropped. (Or it can be replaced by $p + q \leq n$, since the cases where $p + q > n$ are uninteresting.)

6 Reduction to codimension zero and completion of proof

Lemma 6.1. *The space* $\mathrm{imm}(\mathbb{R}^m, \mathbb{R}^n)$ *is homotopy equivalent to the space of injective linear maps from* $\mathbb{R}^m$ *to* $\mathbb{R}^n$.

Proof. Clearly $\mathrm{imm}(\mathbb{R}^m, \mathbb{R}^n) \simeq \mathrm{imm}_0(\mathbb{R}^m, \mathbb{R}^n)$ where $\mathrm{imm}_0(\mathbb{R}^m, \mathbb{R}^n)$ is the subspace of $\mathrm{imm}(\mathbb{R}^m, \mathbb{R}^n)$ consisting of those immersions which take 0 to 0. Let X be the space of injective linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. The inclusion of X in $\mathrm{imm}_0(\mathbb{R}^m, \mathbb{R}^n)$ has a left inverse $\mathrm{imm}_0(\mathbb{R}^m, \mathbb{R}^n) \to X$ given by taking the differential at 0. It remains to show that the composition

$$\mathrm{imm}_0(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow X \longrightarrow \mathrm{imm}_0(\mathbb{R}^m, \mathbb{R}^n)$$

is homotopic to the identity. An explicit homotopy (h_t) is as follows. For an immersion $f \in \mathrm{imm}_0(\mathbb{R}^m, \mathbb{R}^n)$ and $x \in \mathbb{R}^m$ we define $h_t(f)(x)$ by $t^{-1}f(tx)$ provided $0 < t \leq 1$, and $h_0(f)(x) = df(0)(x)$. Continuity of (h_t) at $t = 0$ is a consequence of the defining properties of $df(0)$. ☐

The next lemma is a variant of lemma 6.1 and for $m = n$ it is also the induction beginning in the handle induction strategy described in the introduction.

Lemma 6.2. *For any smooth* N *without boundary, the inclusion*

$$\mathrm{imm}(D^m, N) \to \mathrm{fimm}(D^m, N)$$

is a homotopy equivalence.

Proof. Let X_N be the space of pairs (x, v) where $x \in N$ and $v : \mathbb{R}^m \to T_xN$ is a linear injection. We can make a commutative diagram as follows:

$$\begin{tikzcd}
\mathrm{imm}(D^m, N) \arrow[rr] & & \mathrm{fimm}(D^m, N) \\
\mathrm{imm}(\mathbb{R}^m, N) \arrow[u] \arrow[rr] \arrow[dr] & & \mathrm{fimm}(\mathbb{R}^m, N) \arrow[u] \arrow[dl] \\
& X_N &
\end{tikzcd}$$

The vertical arrows are given by restriction (of immersions and formal immersions) from $\mathbb{R}^m$ to D^m. The map from $\mathrm{fimm}(\mathbb{R}^m, N)$ to X_N is given by evaluation at the origin $0 \in \mathbb{R}^m$. It is easy to see that the left-hand vertical arrow is a homotopy equivalence. Indeed there is a map in the other direction given by pre-composition with a fixed smooth embedding $\mathbb{R}^m \to D^m$; for convenience, choose this in such a way that it agrees with the identity on a small neighborhood of $\in \mathbb{R}^n$. The same argument takes care of the right-hand vertical arrow, so that this is a homotopy equivalence, too. Next, it is true by inspection that the arrow from $\mathrm{fimm}(\mathbb{R}^m, N)$ to X_N is a homotopy equivalence. Therefore it suffices to show that the arrow from $\mathrm{imm}(\mathbb{R}^m, N)$ to X_N, defined by $f \mapsto df(0)$, is a homotopy equivalence.

Choose an exponential map $TN \to N$, more precisely, a smooth map $e : TN \to N$ such that for all $x \in N$ the restriction of e to the tangent space T_xN is a smooth embedding whose value at 0 is $x \in N$, and whose differential at $0 \in T_xN$ is the standard isomorphism $T_0(T_xN) \to T_xN$ (a.k.a. the identity).

Let $\mathrm{imm}^1(\mathbb{R}^m, N) \subset \mathrm{imm}(\mathbb{R}^m, N)$ be the subspace consisting of those immersions g for which $\mathrm{im}(g) \subset e(T_{g(0)}N)$. The inclusion of $\mathrm{imm}^1(\mathbb{R}^m, N)$ in $\mathrm{imm}(\mathbb{R}^m, N)$ is a homotopy equivalence. (Prove this by pre-composing immersions $\mathbb{R}^m \to N$ with embeddings $\mathbb{R}^m \to \mathbb{R}^m$ whose image is a small neighborhood of the origin.) It remains to show that the composition

$$\mathrm{imm}^1(\mathbb{R}^m, N) \hookrightarrow \mathrm{imm}(\mathbb{R}^m, N) \to X_N$$

is a homotopy equivalence. A candidate for a homotopy inverse is the map from X_N to $\mathrm{imm}^1(\mathbb{R}^m, N)$ given by $(x, v) \mapsto e_x \circ v$, where e_x denotes the restriction of e to T_xN. This is indeed a homotopy inverse; the proof is similar to the proof of the previous lemma. $\square$

Now we fix M and N as in previous sections, of dimension m and n, respectively. Let $V \to M$ be a vector bundle, of fiber dimension $n - m$, with a Riemannian metric. Let $\bar{V}$ be the associated disk bundle. (The notation suggests that $V \subset \bar{V}$, which is correct up to diffeomorphism over M. On the other hand and more obviously, the disk bundle $\bar{V}$ is contained in V.) Let $\mathrm{imm}_V(M, N)$ be the space of pairs (g, ι) where g is a smooth immersion $M \to N$ and ι is an isomorphism of V with the normal bundle of g. (The normal bundle of the immersion g is $g^*TN/\mathrm{im}(dg)$, also known as the cokernel of the derivative $dg\colon TM \to g^*TN$.) A smooth immersion

$$f\colon \bar{V} \to N$$

determines an immersion $g\colon M \to N$ by restriction to the zero section, and an isomorphism ι of the vector bundle $V \to M$ with the normal bundle of g. The isomorphism ι is simply df, or more precisely, what we get when we "divide"

$$T\bar{V}_{|M} \xrightarrow{df} f^*TN_{|M} = g^*TN$$

by appropriate vector subbundles (namely, the tangent bundle TM in the source, and $\mathrm{im}(dg)$ in the target). In this way we obtain a map from $\mathrm{imm}(\bar{V}, N)$ to $\mathrm{imm}_V(M, N)$.

Lemma 6.3. *The above map* $\mathrm{imm}(\bar{V}, N) \longrightarrow \mathrm{imm}_V(M, N)$ *is a homotopy equivalence.*

Proof. (Sketch.) It is enough to show that a similarly defined map from $\mathrm{imm}(V, N)$ to $\mathrm{imm}_V(M, N)$ is a homotopy equivalence. We make a diagram chase in the commutative diagram

$$\begin{tikzcd} & \mathrm{imm}(V,N) \arrow[r] \arrow[d, "\simeq"] & \mathrm{imm}_V(M,N) \\ Y' \arrow[r, "\simeq"] & Y \arrow[r] & \mathrm{imm}^1_V(M,N) \arrow[u, "\simeq"'] \end{tikzcd}$$

constructed as follows.

- Y is the space of all smooth maps $f\colon V \to N$ such that the differential $df(x)$ is invertible for every x in the zero section of V.
- Choose an exponential map $e\colon TN \to N$ as in the proof of lemma 6.2. Let $Y' \subset Y$ consist of those $f \in Y$ which satisfy $f(V_x) \subset e(T_yN)$ for all $x \in M$, where $y = f(0_x)$.
- Let $\mathrm{imm}^1_V(M, N)$ be the space of pairs (g, j) where $g\colon M \to N$ is a smooth immersion and $j\colon V \to g^*(TN)$ is a vector bundle monomorphism (over M) such that $j(V_x) \cap dg(T_xM) = 0$ (in the tangent space T_xN) for all $x \in M$. There is a forgetful map from $\mathrm{imm}^1_V(M, N)$ to $\mathrm{imm}_V(M, N)$.

The inclusions $\mathrm{imm}(V, N) \to Y$ and $Y' \to Y$ are homotopy equivalences. (This is left to the reader.) The map from $\mathrm{imm}(V, N)$ to $\mathrm{imm}_V(M, N)$ extends to a map $Y \to \mathrm{imm}_V(M, N)$ (same formula), as indicated in the diagram. That extended map lifts to a map $Y \to \mathrm{imm}^1_V(M, N)$, as indicated in the diagram. The forgetful map from $\mathrm{imm}^1_V(M, N)$ to $\mathrm{imm}_V(M, N)$ is a homotopy equivalence by inspection. It remains to show that the composition

$$Y' \hookrightarrow Y \to \mathrm{imm}^1_V(M, N)$$

(lower row of the diagram) is a homotopy equivalence. A candidate for a homotopy inverse is the map u from $\mathrm{imm}^1_V(M, N)$ to Y' which takes $(g, j) \in \mathrm{imm}^1_V(M, N)$ to the smooth map $V \to N$ defined by

$$V_x \ni v \mapsto e(g(x), j(v)),$$

where $(g(x), j(v)) \in TN$. An argument by convexity, using straight line segments in T_yN for every $y \in N$, shows that u is a homotopy left inverse for the above map from Y' to $\mathrm{imm}^1_V(M, N)$. More obviously, it is also a right inverse. $\square$

Corollary 6.4. *The following is a homotopy pullback square:*

$$\begin{tikzcd} \mathrm{imm}(\bar{V}, N) \arrow[r, "\text{1-jet prolong.}"] \arrow[d, "\text{restr.}"'] & \mathrm{fimm}(\bar{V}, N) \arrow[d, "\text{restr.}"] \\ \mathrm{imm}(M, N) \arrow[r, "\text{1-jet prolong.}"] & \mathrm{fimm}(M, N)\,. \end{tikzcd}$$

Proof. By the lemma, we may replace $\mathrm{imm}(\bar{V}, N)$ by $\mathrm{imm}_V(M, N)$. By inspection, we may also replace $\mathrm{fimm}(\bar{V}, N)$ by the space $\mathrm{fimm}_V(M, N)$ of triples $(f, \delta f, \iota)$ where $(f, \delta f) \in \mathrm{fimm}(M, N)$ and ι is a vector bundle isomorphism from V to $\mathrm{coker}(\delta f) = f^*TN/\mathrm{im}(\delta f)$. Then our diagram turns into

$$\begin{tikzcd} \mathrm{imm}_V(M, N) \arrow[r, "\text{1-jet prolong.}"] \arrow[d, "\text{restr.}"'] & \mathrm{fimm}_V(M, N) \arrow[d, "\text{restr.}"] \\ \mathrm{imm}(M, N) \arrow[r, "\text{1-jet prolong.}"] & \mathrm{fimm}(M, N)\,. \end{tikzcd}$$

It is a strict pullback square and it is easy to verify that the vertical arrows are fibrations. Hence it is a homotopy pullback square. $\square$

Remark 6.5. Corollary 6.4 and lemma 6.3 go under the name *micro-extension trick*. Cf. [6, §8.1].

Exercise 6.6. Give an example of a smooth compact m-manifold M and a smooth vector bundle $V \to M$ of fiber dimension $k > 0$, for some m and k, such that the map $\mathrm{imm}(V, \mathbb{R}^{m+k}) \to \mathrm{imm}(M, \mathbb{R}^{m+k})$, restriction to the zero section, is not a weak homotopy equivalence. (Make it as simple as possible.)

Reduction of theorem 1.3 to the case where $m = n$. Suppose theorem 1.3 known in the case $m = n$. For the general case $m \le n$, choose some vector bundle V on M of fiber dimension $n - m$. Then, by corollary 6.4, the homotopy fiber of the jet prolongation map

$$\mathrm{imm}(M, N) \longrightarrow \mathrm{fimm}(M, N)$$

over any point in the image of the forgetful map

$$\mathrm{fimm}_V(M, N) \to \mathrm{fimm}(M, N)$$

is weakly homotopy equivalent to a point. Since V was fairly arbitrary, this means that *all* homotopy fibers of $\mathrm{imm}(M, N) \longrightarrow \mathrm{fimm}(M, N)$ are weakly homotopy equivalent to a point. $\square$

Most of the details of the proof of theorem 1.3 outlined in the introduction have now been supplied. The case of a noncompact M (more precisely, an M such that no connected component of M is a closed manifold) can be reduced to the compact case as follows. We may assume that ∂M is empty (otherwise delete the boundary). We may assume that $m = n$ as before. Then we choose compact smooth codimension zero submanifolds M_i of M for $i = 0, 1, 2, \dots$ such that M_i is contained in the interior of M_{i+1} and such that $\bigcup_i M_i = M$. This leads to a commutative ladder

$$\begin{array}{ccccccc}
\mathrm{imm}(M_0, N) & \longleftarrow & \mathrm{imm}(M_1, N) & \longleftarrow & \mathrm{imm}(M_2, N) & \longleftarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\mathrm{fimm}(M_0, N) & \longleftarrow & \mathrm{fimm}(M_1, N) & \longleftarrow & \mathrm{fimm}(M_2, N) & \longleftarrow
\end{array}$$

where the horizontal arrows are restriction maps. The horizontal arrows are Serre fibrations. The vertical arrows are weak equivalences. It follows in a formal manner that the induced map from the (inverse) limit of the upper row to the (inverse) limit of the lower row is a weak equivalence. That amounts to saying that the standard map from $\mathrm{imm}(M, N)$ to $\mathrm{fimm}(M, N)$ is a weak equivalence. [One of the most convincing formal arguments, which is probably not one of the most elementary arguments, goes like this. Because the rows of the diagram are made up of fibrations, the standard comparison maps

$$\lim_i \mathrm{imm}(M_i, N) \to \operatorname*{holim}_i \mathrm{imm}(M_i, N),$$
$$\lim_i \mathrm{fimm}(M_i, N) \to \operatorname*{holim}_i \mathrm{fimm}(M_i, N)$$

are weak equivalences. Now it only remains to show that the map

$$\operatorname*{holim}_i \mathrm{imm}(M_i, N) \longrightarrow \operatorname*{holim}_i \mathrm{imm}(M_i, N)$$

induced by the vertical arrows in that diagram is a weak equivalence. But this follows from the well-known homotopy invariance properties of holim, since the 1-jet prolongation $\mathrm{imm}(M_i, N) \to \mathrm{fimm}(M_i, N)$ is a weak equivalence for all i.]

Exercise 6.7. Suppose that M and N are smooth manifolds, where $\partial N = \emptyset$ and $M \setminus \partial M$ has no compact component. Show (in outline) that the jet prolongation map $\mathrm{imm}(M, N) \to \mathrm{fimm}(M, N)$ is a weak equivalence *without* using reduction to the case(s) where $\dim(M) = \dim(N)$. Instead use exercise 5.9 and remark 1.16.

Exercise 6.8. Let M and N be smooth manifolds with boundary, of dimensions m and n, respectively. Let $\mathrm{imm}_{\mathsf{pair}}(M, N)$ be the space of *neat* smooth pairwise immersions (with the compact-open C^∞ topology) from M to N. Such a neat pairwise immersion is a smooth map of pairs $f : (M, \partial M) \to (N, \partial N)$ such that $f^{-1}(\partial N) = \partial M$ and $df(x) : T_xM \to T_{f(x)}N$ is injective, for every $x \in M$, and $df(x)(T_x\partial M) + T_{f(x)}\partial N = T_{f(x)}N$ for every $x \in \partial M$. In the same spirit, let $\mathrm{fimm}_{\mathsf{pair}}(M, N)$ be the space of pairwise formal immersions. Such a pairwise formal immersion consists of a map of pairs $g : (M, \partial M) \to (N, \partial N)$ and a pair of compatible vector bundle monomorphisms (over M and over ∂M, respectively) from TM to g^*TN, and from $T(\partial M)$ to $g^*(T(\partial N))$.

(i) Show that the 1-jet prolongation $\mathrm{imm}_{\mathsf{pair}}(M, N) \to \mathrm{fimm}_{\mathsf{pair}}(M, N)$ is a weak equivalence in the cases where $m < n$. [It is not absolutely necessary to imitate the proof of theorem 1.3. Instead, give alternative definitions of $\mathrm{imm}_{\mathsf{pair}}(M, N)$ and $\mathrm{fimm}_{\mathsf{pair}}(M, N)$ as homotopy pullbacks of more basic things.]

(ii) Using (i), show that a smooth immersion $S^1 \to \mathbb{R}^2$ extends to a neat pairwise immersion $(D^2, S^1) \to (\mathbb{R}^2 \times [0, \infty), \mathbb{R}^2 \times \{0\})$ if and only if the degree of the associated Gauss map $S^1 \to S^1$ is odd.

7 Submersion theory and Gromov's theorem

A *submersion* is a smooth map $f : M \to N$ with the property that, for each $x \in M$, the differential $T_xM \to T_{f(x)}M$ is surjective. Here, as before, N should be without boundary, M can have a nonempty boundary, but we pay no special attention to the tangent spaces $T_x\partial M$ for $x \in \partial M$. Hence the restriction of a submersion $M \to N$ to ∂M need not be a submersion.

Assuming that M is compact, let $\mathrm{subm}(M, N)$ be the space of smooth submersions from M to N, an open subspace of $C^\infty(M, N)$. Also let $\mathrm{fsubm}(M, N)$ be the space of formal submersions from M to N. An element in $\mathrm{fsubm}(M, N)$ is a pair $(f, \delta f)$ where $f : M \to N$ is a continuous map and $\delta f : TM \to f^*TN$ is a vector bundle surjection. There is a jet prolongation map

$$\mathrm{subm}(M,N) \to \mathrm{fsubm}(M,N)$$

given by $f \mapsto (f, df)$.

Theorem 7.1. *[27] Let M and N be smooth manifolds, with $\partial N = \emptyset$. Assume that $M \smallsetminus \partial M$ has no compact component. Then the jet prolongation map*

$$\mathrm{subm}(M,N) \to \mathrm{fsubm}(M,N)$$

is a weak homotopy equivalence.

Remark. This is uninteresting if $\dim(M) < \dim(N)$, because then both spaces $\mathrm{subm}(M,N)$ and $\mathrm{fsubm}(M,N)$ are empty. If $\dim(L) = \dim(N)$, we recover the special case $m = n$ of theorem 1.3.

The proof of theorem 7.1 is very similar to the proof of theorem 1.3. Nevertheless it is worth highlighting a few points.

First of all, if M is a compact smooth manifold, then $\mathrm{subm}(M,N)$ is an open subspace of $C^\infty(M,N)$. Hence the analogue of corollary 5.6 for submersions is valid. We need this, of course, in order to have access to lemma 2.6. Then we can prove the analogue of proposition 1.15 for submersions. The statement is that the restriction map

$$\mathrm{subm}(D^p \times D^q, N) \longrightarrow \mathrm{subm}(A^p \times D^q, N)$$

is a Serre fibration. Here $q > 0$, and we can assume $p + q \ge n$. For the proof, let $Z = \mathrm{subm}(D^p \times D^q, N)$ and $B = \mathrm{subm}(A^p \times D^q, N)$, with $\pi\colon Z \to B$ equal to the restriction map, and

$$E \quad = \quad \mathrm{subm}(S^{p-1} \times [0,3] \times D^q, N)\,.$$

There are maps $\sigma, \tau\colon E \to B$ and $\pi\colon Z \to B$ given by formulas similar to those we had in section 2 (preview of proof of proposition 1.15). There is a composition rule in the shape of a map $\kappa\colon E_\sigma \times_\tau E \to E$ over $B \times B$, and another composition rule in the shape of a map $\alpha\colon E_\sigma \times_\pi Z \to Z$ over B. The difficult thing is, as before, to find $\iota\colon B \to E$ such that the conditions in definitions 2.2 and 2.4 are satisfied. For $f \in B$, define $\iota(f)$ by a formula of type

$$\iota(f) = s(f) \circ v\,.$$

This is similar to the formula $\iota(f) = s(f) \circ v$ which we had in section 5, proof of proposition 1.15; in particular v has exactly the same meaning. The only difference is that $s(f)$ is now an extension of f to a *submersion* $U \to N$, where U is short for $(\mathbb{R}^p \smallsetminus \{0\}) \times \mathbb{R}^q$ as before. — When we constructed v in the course of proving proposition 1.15, we wanted it to be an immersion; here we want it to be a submersion. But of course it is both an immersion and a submersion. □

There is an even more general result which can be proved with exactly the same arguments. Fix a manifold N and integers $m, k > 0$ and a subset $\mathfrak{W}$ of the jet space $J^k(\mathbb{R}^m, N)$. Suppose that

(i) $\mathfrak{W}$ is open in $J^k(\mathbb{R}^m, N)$

(ii) $\mathfrak{W}$ is invariant under "local diffeomorphisms" of $\mathbb{R}^m$. [This means that if you have a diffeomorphism $\varphi\colon U \to V$ between open subsets of $\mathbb{R}^m$, and some jet $s \in J^k(V,N) \cap \mathfrak{W}$, then $s \circ \varphi \in J^k(U,N) \cap \mathfrak{W}$.]

Now suppose that M is any smooth m–manifold, possibly with boundary. Let $\mathfrak{W}_M \subset J^k(M,N)$ consist of all the jets which, in local coordinate charts about their source in M, belong to $\mathfrak{W} \subset J^k(\mathbb{R}^m, \mathbb{R}^n)$. [Because of the diffeomorphism invariance condition, it does not matter how you choose the coordinate charts. But see the remark just below.] Let $C^\infty_{\mathfrak{W}}(M,N)$ consist of all the smooth maps from M to N whose k–jets at any point $x \in M$ belong to $\mathfrak{W}_M$. Let $\Gamma(\mathfrak{W}_M)$ be the space of continuous sections (with the compact–open C^0 topology) of the bundle projection

$$\mathfrak{W}_M \longrightarrow M\,.$$

Remark. Officially, an element of $J^k(\mathbb{R}^m, \mathbb{R}^n)$ is represented by a triple (x, U, f) where $x \in \mathbb{R}^m$ and f is a smooth map from a neighborhood U of x to $\mathbb{R}^n$. Unofficially, however, an element of $J^k(\mathbb{R}^m, \mathbb{R}^n)$ is a collection of numbers, one for each possible "mixed partial derivative" $\partial^\alpha/\partial x_\alpha$ where $|\alpha| \le k$. With the second definition, it is easy to represent elements of $J^k(\mathbb{R}^m, \mathbb{R}^n)$ by slightly less than the above — for example, a triple (x, U, f) where x belongs to a hyperplane $\mathbb{R}^{m-1} \times \{0\} \subset \mathbb{R}^m$ and U is a neighborhood of x in $\mathbb{R}^{m-1} \times [0, \infty[$, and $f\colon U \to \mathbb{R}^n$ is smooth. Hence a smooth map $M \to N$ has well defined k–jets at any point of M, even at points in ∂M.

Theorem 7.2. *[11],[12]; [10, §2.2.2]. Suppose that $M \smallsetminus \partial M$ has no compact component. Then the jet prolongation map $C^\infty_{\mathfrak{W}}(M,N) \longrightarrow \Gamma(\mathfrak{W}_M)$ is a weak homotopy equivalence.*

Proof. Once you have unravelled the meaning, you will see that it can be proved exactly like theorem 7.1, and also like the variant of theorem 1.3 described in exercise 6.7. The following is important: Given

$$f \in C^\infty_{\mathfrak{W}}(M,N)$$

and given any codimension zero immersion/submersion $v\colon L \to M$, the composition $f \circ v$ belongs to $C^\infty_{\mathfrak{W}}(L,N)$. The reason is that codimension zero immersions / submersions are locally diffeomorphic (inverse function theorem) and our assumptions on $\mathfrak{W}$ include some diffeomorphism invariance. □

Remark 7.3. (Some important h-principles which are not covered here.) There is an h-principle for PL immersions of PL manifolds [13] and there is one for topological (locally flat) immersions [19], [20], [17, V App.A].

In the setting of theorem 1.3, questions of the following type can be asked. Given a formal immersion $(f, \delta f) \in \mathrm{fimm}(M,N)$ and a neighborhood U of f in

$C^0(M,N)$, is it possible to find $g \in \mathrm{imm}(M,N)$ and a path from (g,dg) to $(f,\delta f)$ in $\mathrm{fimm}(M,N)$ such that the underlying path in $C^0(M,N)$ from g to f is a path in U ? The answer is yes in the cases where $\dim(M) < \dim(N)$. Both [6] and [10] make a point of this, and use expressions like *the C^0-dense h-principle* (for smooth immersions). In the not-very-ambitious form just given, the statement can be deduced directly from proposition 1.15. [Hint. Choose a metric on N inducing the topology and find a handle decomposition of M such that $f(H)$ has small diameter in N for each handle H. Such a handle decomposition can be obtained from a smooth triangulation of M with small simplices. Without loss of generality, f is smooth. The formal immersion $(f,\delta f)$ determines a normal vector bundle $V \to M$ and an associated disk bundle $\bar{V} \to M$. A handle decomposition of M determines a handle decomposition of $\bar{V}$ with the same number of handles.]

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