Graduate J. Math. **1** (2016), 18 – 27

Methods of Equivariant Topology in Two Nice Discrete Geometry Problems

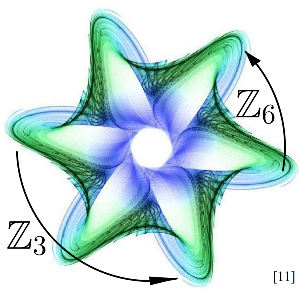
Marija Jelić

Abstract

We present the basic concepts of equivariant index theory, numerical and cohomological. We then show how these indices can be used to give solutions to some special cases of Knaster's problem, or to the prime case of Nandakumar & Ramana Rao problem.

MSC 2010. Primary 52A35, 05E18; Secondary 55N91, 52A38.

1 Introduction



Equivariant topology studies topological spaces with some kind of symmetry reflected in the action of a group, together with mappings among these spaces that "respect" such actions, called *equivariant mappings*. The development of equivariant topology was followed by its many applications in discrete geometry and combinatorics. Our aim is to illustrate how to transform a problem from discrete geometry into a problem in equivariant topology (about the nonexistence of certain maps) and then solve it. Section 2 contains the basic definitions and describes the general approach for reducing a given geometric problem to the nonexistence of an equivariant mapping. In order to prove that such a map doesn't exist, we need some measure of complexity of a space with group action. Here we introduce two such measures: the numerical index function in section 3, and the cohomological index in section 5. The main objective of our paper is to offer applications of these indices. Section 4 discusses some cases of the famous Knaster's problem. It will be shown that for every continuous map $f : S^{n-1} \to \mathbb{R}$, and for every prime p, $2 < p < n$, there exists a regular p-gon on a great circle of S^{n-1}, whose vertices have the same image under f. The last section is about the Nandakumar & Ramana Rao's conjecture from 2006, which asks whether for a convex polygon and $n \in \mathbb{N}$, there exists some partition of the polygon into n pieces with the same area and the same perimeter. For prime n we present a proof that uses the cohomological index. Generally, we will skip technical details and focus on geometrical ideas and creative constructions. We hope that this introduction will encourage the reader to further develop knowledge in equivariant topology, and make it part of his/her mathematical tool kit.

2 Topological Group Actions

2.1 Definitions and terminology

Our basic concept, the action of a group, is defined for topological groups, i.e. groups that are Hausdorff topological spaces with continuous group and inverse operations. We will work with finite (discrete) groups which are obviously topological groups.

Let X be a topological space and G a topological group. A *G-action* of G on X is a continuous map $\varphi : G \times X \to X$ such that the restrictions $\varphi_g = \varphi(g, -)$ for $g \in G$, are self-homeomorphisms of X satisfying:

1) $\varphi_e = \mathbf{1}_X$, for all $x \in X$,

2) $\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 g_2}$, for all $g_1, g_2 \in G$, $x \in X$.

The pair (X, φ) is called a *G-space*. The (left) action of g on x; $\varphi(g, x) = \varphi_g(x)$, is denoted by gx.

If X is a simplicial (or cell) complex, and if all mappings φ_g, for $g \in G$, are simplicial (resp. cell) mappings, then X is called a *simplicial* (or *cell*) *G-complex*.

A G-action on X is *free* (and X is a *free* G-space) if for every $g \in G$, $g \neq e$, the homeomorphism φ_g has no fixed points.

For every $x \in X$, the set $\{\varphi_g(x) \mid g \in G\}$ is called the *orbit* of x. When the action of a finite group is free, every orbit has the same cardinality as the group.

Definition 2.1. Let (X, φ) and (Y, ψ) be two G-spaces. A continuous mapping $f : X \to Y$ is *G-equivariant* if $f \circ \varphi_g = \psi_g \circ f$, for all $g \in G$, i.e. if $f(g \cdot x) = g \cdot f(x)$, for $\forall g \in G$, $\forall x \in X$. If G-equivariant mapping exists, we write $X \xrightarrow{G} Y$.

Equivariant topology is the study of G-spaces and G-equivariant maps between them.

2.2 Relevant Examples

We recall a few relevant but basic group actions.

• The first important group action is the antipodal action of $\mathbb{Z}_2 = \{0, 1\}$ on the sphere S^n, where 0 acts as the identity, while 1 acts as the antipodal mapping $x \mapsto -x$. This action is free.

The famous Borsuk-Ulam theorem says that *for every continuous mapping $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ such that $f(x) = f(-x)$*. It is well-known (see [21], 2.1.1) that this statement is equivalent to the following one: *There is no antipodal continuous mapping $g : S^n \to S^{n-1}$, i.e. the mapping which satisfies $g(-x) = -g(x)$, for all $x \in S^n$.* In the language of equivariant topology, this is stated as follows:

There is no $\mathbb{Z}_2$-equivariant mapping from S^n to S^{n-1}.

This example illustrates how it is possible to phrase important results in topology in terms of equivariant topology. More on this in §2.3.

• The actions of other cyclic groups $\mathbb{Z}_n$ are also very important. A $\mathbb{Z}_n$-action is completely determined by the homeomorphism φ_1, since $\varphi_k = \varphi_1 \circ \cdots \circ \varphi_1 = \varphi_1^k$, for all k. It is easily seen that for prime p the action of group $\mathbb{Z}_p$ is free if and only if the action of generator φ_1 is free ([21], 6.1.3).

• There is a free action of the group $\mathbb{Z}_n$ on the circle S^1, where the generator acts as the rotation by $\frac{2\pi}{n}$. Also, the group $SO(2)$ of all rotations around the origin acts freely on S^1, and generally, the special orthogonal group $SO(n)$ acts on S^{n-1}, but this action is not free for $n > 2$.

• If X and Y are two G-spaces, we have a natural *diagonal action* on the product $X \times Y$, given by: $g \cdot (x, y) := (g \cdot x, g \cdot y)$. Also, G acts on the geometrical join $X * Y$ in a similar way: $g \cdot (tx \oplus (1-t)y) := t(g \cdot x) \oplus (1-t)(g \cdot y)$, for all $t \in [0, 1]$, $x \in X$, $y \in Y$. (Remind that $X * Y$ can be seen as the union of all segments connecting a point of X to a point of Y, see [21],4.2.4.)

• For every topological space X, there is an action of symmetric group Σ_n on the product X^n given by permuting the coordinates. Precisely, for a permutation $\pi \in \Sigma_n$, $\varphi_\pi(x_1, \dots, x_n) := (x_{\pi(1)}, \dots, x_{\pi(n)})$. This Σ_n-action is not free, but it is free on the following very important subspace of X^n, called *the configuration space of n pairwise distinct points in X*;

$$F(X, n) := \{(x_1, x_2, \dots, x_n) \in X^n \mid, x_i \neq x_j \text{for all } i \neq j\}.$$

The cyclic group $\mathbb{Z}_n$ can be viewed as a subgroup of Σ_n generated by the permutation

$$\pi(1, 2, \dots, n) = (2, 3, \dots, n, 1)$$

(i.e generated by the cycle $(1, 2, \dots, n)$), and so $\mathbb{Z}_n$ acts on both X^n and $F(X, n)$.

2.3 Configuration Space - Test Map Method

At the foundation of our subject lies the question: how to transform a problem from discrete geometry or combinatorics into a problem in equivariant topology? One systematic approach which is both useful and powerful is the so called *configuration space - test map method* (see [28]). We briefly describe how this method works.

Suppose we wish to show that there exists some configuration of points in the plane, or a set of vertices in a graph, that satisfies a certain special property. That desired configuration is called *the solution* for the sought property/problem. To find it, one can proceed in steps:

• At first, construct a space X which is encoding all possible candidates for the solution. This space is called *the configuration space.*

• Construct a continuous mapping from X into some carefully chosen space Y. This map $f : X \to Y$ should tell us whether some configuration is the solution or not, in the following manner: $x \in X$ is a solution if and only if $f(x) \in Z$, where $Z \subset Y$ (some "discriminant" subspace). The space Y is called *the test space*, while f is called *the test map.*

• Failure to have solutions to the initial problem means that f maps X into $Y \backslash Z$.

• In the presence of G-symmetry, the test map can be enriched to a G-equivariant map $f : X \xrightarrow{G} Y$ or $f : X \xrightarrow{G} Y \backslash Z$, with G acting on all spaces in sight.

• If we can prove by methods of equivariant topology that there is no equivariant map $f : X \xrightarrow{G} Y \backslash Z$, then this would mean that f takes some value in Z and a solution exists.

There are several standard techniques for proving the nonexistence of equivariant mappings, including some index theories and the equivariant obstruction theory. Here, we will describe two kinds of equivariant indexes: the numerical ind_G and the cohomological Ind_G.

3 Numerical G-index

In order to define our first measure of G-complexity, the numerical index function, we need one special class of G-spaces.

Definition 3.1. Let G be a finite group, $|G| > 1$ and $n \in \mathbb{N}_0$. A G-space X is an *E_nG-space* if it satisfies the following conditions:

1° X is a free G-space,

2° X is a finite simplicial (or cell) G-complex such that $\dim X = n$,

3° X is $(n-1)$-connected.

A standard example of E_nG-space is the $(n+1)$-fold join $G^{*(n+1)}$, where G is considered as discrete 0-dimensional complex. For $n = 1$, G^{*2} is the complete bipartite graph $G * G$. Inductively we get that $G^{*(n+1)}$ is an n-dimensional complex. A free action of a group

G on itself is given by group multiplication $\varphi_g(x) := gx$, and it naturally induces a G-action on the join $G^{*(n+1)}$, which is also free. Also, since the join of a k-connected complex with an l-connected complex is $(k+l+1)$-connected ([21], 4.4.3), we inductively get that $G^{*(n+1)}$ is $(n-1)$-connected, so it is an E_nG-space. For example, an $E_n\mathbb{Z}_2$-space is the sphere S^n with antipodal action. This is in fact clear since $\mathbb{Z}_2$ is S^0 topologically, while the $(n+1)$-fold join $S^0 * \cdots * S^0$ is S^n.

Now we will state two important results without proofs. The proofs use standard topological techniques that are beyond our topic. They can be found in [21] (section 6) which provides an elegant exposition of topological methods.

If K is a finite simplicial complex, we will write $\|K\|$ for its realization.

Theorem 3.2. *Let K be a free finite simplicial (or cell) G-complex,* $\dim K \leqslant n$, *and let X be an $(n-1)$-connected G-space. Then $||K|| \xrightarrow{G} X$.*

In particular, $X \xrightarrow{G} Y$ holds for every two E_nG-spaces X and Y.

Theorem 3.3 (Borsuk-Ulam theorem for G-spaces)**.** *There is no G-equivariant mapping from an E_nG-space to an $E_{n-1}G$-space.*

Notice that for $G = \mathbb{Z}_2$ this is the standard Borsuk-Ulam theorem.

We can now define the index function.

Definition 3.4. Let X be a G-space. *The numerical index* of X is defined as

$$\mathrm{ind}_G(X) := \min\{n \in \mathbb{N}_0 \mid X \xrightarrow{G} E_nG\}.$$

We can choose any E_nG-space here because from Theorem 3.2 E_nG-spaces map G-equivariantly to each other. The G-index can be a natural number or ∞. For example, the disc with antipodal $\mathbb{Z}_2$-action has infinite $\mathbb{Z}_2$-index, since any $\mathbb{Z}_2$-equivariant mapping to $E_n\mathbb{Z}_2 = S^n$ should satisfy $f(0) = -f(0) \in S^n$, which is not possible. Similarly, all non-free G-spaces have infinite G-index.

Here are the most important properties of $\mathrm{ind}_G(X)$, which show that it is indeed a measure of G-complexity.

Theorem 3.5. *Let G be a finite nontrivial group.*

1) (Monotonicity) If $X \xrightarrow{G} Y$, then $\mathrm{ind}_G(X) \leqslant \mathrm{ind}_G(Y)$.

2) $\mathrm{ind}_G(E_nG) = n$.

3) $\mathrm{ind}_G(X * Y) \leqslant \mathrm{ind}_G(X) + \mathrm{ind}_G(Y) + 1$.

4) If X is $(n-1)$-connected, then $\mathrm{ind}_G(X) \geqslant n$.

5) If K is a free simplicial (cell) G-complex of dimension n, then $\mathrm{ind}_G(K) \leqslant n$.

Proof. The first property follows from the definition directly. The second is a direct consequence of Theorem 3.3 ($\mathrm{ind}_G(E_nG) \leqslant n$ because the identity map is G-equivariant). For 3), we take $E_nG = G^{*(n+1)}$, and let $\mathrm{ind}_G(X) = n$ and $\mathrm{ind}_G(Y) = m$. Then there exist $f_1 : X \xrightarrow{G} G^{*(n+1)}$ and $f_2 : Y \xrightarrow{G} G^{*(m+1)}$. Hence we have $f_1 * f_2 : X * Y \xrightarrow{G} G^{*(n+1)} * G^{*(m+1)} \cong G^{*(n+m+2)} = E_{n+m+1}G$, so $\mathrm{ind}_G(X*Y) \leqslant n+m+1$. Also, 4) and 5) follow directly from Theorem 3.2 and property 2): we have $E_nG \xrightarrow{G} X$ in 4), and $\|K\| \xrightarrow{G} E_nG$ in 5). $\square$

Finally, we can prove one of the most useful theorems of equivariant topology.

Theorem 3.6 (Dold's theorem)**.** *Let G be a finite nontrivial group. Let X be an n-connected G-space, and let Y be a free simplicial (or cell) G-complex such that* $\dim Y \leqslant n$. *Then there is no G-equivariant mapping from X to Y.*

Proof. The property 4) implies $\mathrm{ind}_G(X) \geqslant n+1$, while 5) implies $\mathrm{ind}_G(Y) \leqslant n$. Then from 1) there cannot exist a G-equivariant mapping from X to Y. $\square$

4 Application to Knaster's Problem

Consider some finite set of points $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ on the sphere S^{n-1}, and let us fix some dimension $m \in \mathbb{N}$. Knaster's problem asks the following. Given a continuous mapping $f : S^{n-1} \to \mathbb{R}^m$, does there exist a rotation $\rho \in SO(n)$, such that $f(\rho(A_1)) = f(\rho(A_2)) = \cdots = f(\rho(A_k))$? If the answer is "yes" for all such mappings f, set $\mathcal{A}$ is called *the solution of Knaster's problem for n and m.*

In [17], Knaster originally asked this question for $k = n-m+1$, and he asked whether every configuration of $n-m+1$ points is a solution or not. The answer to his question is negative, and counterexamples were found by Makeev in [19] (see also [2] and [7]). However, the problem of finding all solutions for given k, n and m is very interesting, and open in many cases.

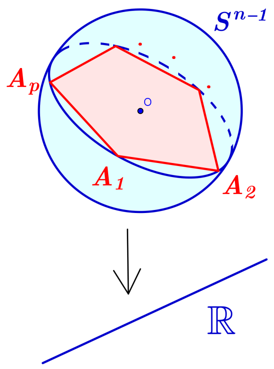
It is easily observed that if $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ contains $k = n-m+2$ linearly independent points; meaning the corresponding vectors OA_i are independent, then it is not a solution. Indeed, in that case points $A_1, \ldots, A_{k-1}$ determine $(n-m)$-dimensional affine subspace $\Pi \subset \mathbb{R}^n$, $A_k \notin \Pi$. The orthogonal projection to the orthogonal complement $f : S^{n-1} \to \Pi^{\perp} \cong \mathbb{R}^m$ is a function for which desired rotation ρ doesn't exist.

The Borsuk-Ulam theorem is a special case of the affirmative Knaster problem. It corresponds to the case $m = n-1$ and $\mathcal{A} = \{e, -e\}$ is a set of two antipodal points. This case was generalized by Hopf [13] for any two points $A, B \in S^{n-1}$.

Even in the first interesting case of spaces S^2 and $\mathbb{R}$, all Knaster's solutions are not known. Kakutani [15] proved that the vertices A_1, A_2, A_3 of an orthonormal frame are solution, and in [10] this was generalized to the case of every 3-element subset of S^2. On the other hand, it is known that every five points on S^2 are not solution, as well as every four non-planar points (for dimensional reasons). The remaining cases of four planar points is still open, with some partial results (for example the set $\{A, -A, B, -B\}$ is solution for all $A, B \in S^2$; see [18]).

For $m = 1$, analogously to the Kakutani case, $\mathcal{A} = \{e_1, e_2, \ldots, e_n\} \subset S^{n-1}$ being the standard orthonormal basis is a solution ([27]); $\mathcal{A} \subset S^{n-1}$ being the set of vertices of any regular $(n-1)$-simplex ([6]) as well, etc.

Generally, there are many nice situations where $\mathcal{A}$ can be a solution to Knaster's problem and to which methods of equivariant topology can be applied. We discuss two examples from Makeev's papers [19] and [20]. In so doing, we give a detailed exposition of the configuration space - test map method.



Theorem 4.1. *Let p be an odd prime, $n \in \mathbb{N}$, $p < n$. Let $A_1, A_2, \ldots, A_p$ be the vertices of a regular polygon with p sides, on a great circle of sphere S^{n-1}. Then for every continuous mapping $f : S^{n-1} \to \mathbb{R}$ there exists a rotation $\rho \in SO(n)$ such that $f(\rho(A_1)) = f(\rho(A_2)) = \ldots = f(\rho(A_p))$.*

An interesting reformulation is that *for every continuous map $f : S^{n-1} \to \mathbb{R}$, and an odd prime $p < n$, there exists a regular p-gon on a great circle of S^{n-1} such that all its vertices have the same image under f.*

Proof. Each regular polygon with p sides and vertices $X_1, \ldots, X_p$ lying on a great circle of the sphere S^{n-1} defines in some cyclic ordering a p-tuple $(X_1, \ldots, X_p)$ of points of S^{n-1}. Each such tuple is uniquely determined by the pair (X_1, X_2). Conversely, any pair of points (X_1, X_2) that make an angle of $\frac{2\pi}{p}$ determine points $X_3, \ldots, X_p$ uniquely so that $(X_1, X_2, \ldots, X_p)$ is an ordered vertex set of a regular polygon. Then we can identify the configuration space of all p-tuples $(X_1, \ldots, X_p)$ of points of S^{n-1}, which form an ordered vertex set of some regular polygon with p vertices on a great circle, with $V_2(\mathbb{R}^n)$, the Stiefel variety of orthonormal 2-frames in $\mathbb{R}^n$.

Suppose to the contrary that a given configuration $(A_1, A_2, \ldots, A_p)$, thought of as an element of $V_2(\mathbb{R}^n)$, is not a solution to Knaster's problem. Then there is a continuous map $f : S^{n-1} \to \mathbb{R}$ for which the desired rotation doesn't exist. We can define a test map $F : V_2(\mathbb{R}^n) \to \mathbb{R}^p$,

$$F(X_1, X_2, \ldots, X_p) := (f(X_1), f(X_2), \ldots, f(X_p)).$$

By assumption, $F(V_2(\mathbb{R}^n))$ has empty intersection with the diagonal subspace Δ in $\mathbb{R}^p$, for otherwise there would be a regular polygon with p sides and vertices having all the same image under f. But every such polygon can be obtained as some rotation of our polygon with vertex set $\{A_1, A_2, \ldots, A_p\}$, which gives a contradiction to the assumption. So, we can consider the same mapping F with smaller codomain, i.e. $F : V_2(\mathbb{R}^n) \to \mathbb{R}^p \backslash \Delta$.

Now consider some appropriate group actions. The group $\mathbb{Z}_p$ acts freely on both spaces. It cyclically permutes vectors of $V_2(\mathbb{R}^n)$; i.e after our identification of the configuration space with $V_2(\mathbb{R}^n)$, the generator of $\mathbb{Z}_p$ acts according to

$$g_{\mathbb{Z}_p}(X_1, \ldots, X_p) := (X_2, X_3, \ldots, X_p, X_1)$$

This action is obviously free. The action on $\mathbb{R}^p \backslash \Delta$ is also cyclic permutation of coordinates. Note that this action is also free, and that it wouldn't be free if p was not a prime number. Also, it is obvious that F is a $\mathbb{Z}_p$-equivariant mapping. We would like to prove that such an equivariant mapping cannot exist. In order to apply Dold's theorem, we shall equivariantly modify the codomain to a CW-complex. At first, take the orthogonal projection $\pi : \mathbb{R}^p \backslash \Delta \to \Delta^{\perp} \backslash \{0\}$, and then radial projection r from $\Delta^{\perp} \backslash \{0\}$ onto the unit sphere in $\Delta^{\perp}$, which is S^{p-2}. Spaces $\Delta^{\perp} \backslash \{0\}$ and S^{p-2} have inherited $\mathbb{Z}_p$-actions from $\mathbb{R}^p \backslash \Delta$, which are both well-defined and free. Also, π and r are $\mathbb{Z}_p$-equivariant deformations. So, we have the following $\mathbb{Z}_p$-equivariant composition:

$$\phi = r \circ \pi \circ F : V_2(\mathbb{R}^n) \xrightarrow{\mathbb{Z}_p} S^{p-2}.$$

The Stiefel manifold $V_k(\mathbb{R}^n)$ is an $(n-k-1)$-connected space ([12], p. 382), so our domain is $(n-3)$-connected, while the dimension of the codomain is $p - 2 < n - 2$. Since the action on this codomain is free, by Dold's theorem (Theorem 3.6) we obtain a contradiction, and this proves our claim. $\square$

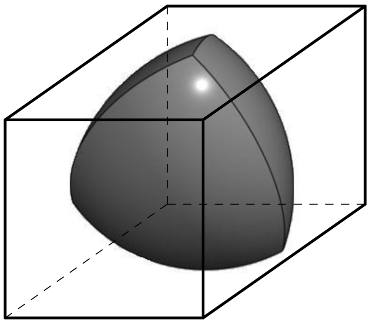
Theorem 4.2. *Let p be an odd prime and $n \in \mathbb{N}$, such that $2p < n + 1$. Let $A_1, A_2, \ldots, A_p$ be the vertices of a regular $(p-1)$-simplex, whose center is not at the origin. Then for every continuous mapping $f : S^{n-1} \to \mathbb{R}$ there exists a rotation $\rho \in SO(n)$ such that $f(\rho(A_1)) = f(\rho(A_2)) = \ldots = f(\rho(A_p))$.*

Proof. The configuration space of all p-tuples $(X_1, \ldots, X_p)$ which form a regular $(p-1)$-simplex congruent to the simplex $(A_1, \ldots, A_p)$, is the Stiefel manifold $V_p(\mathbb{R}^n)$. (This is obvious if $A_1, \ldots, A_p$ are the vertices of an orthonormal frame. Otherwise, every such simplex $(X_1, \ldots, X_p)$ has its unique corresponding simplex $(X_1', \ldots, X_p')$, formed by vertices of an orthonormal frame, and whose center is collinear with the origin and the center of $(X_1, \ldots, X_p)$.)

Now suppose to the contrary that there is a continuous mapping f for which there is no desired rotation, and follow the construction of the previous proof. Since the vertex set $\{X_1, \ldots, X_p\}$ of every considered $(p-1)$-simplex is $\rho(\{A_1, \ldots, A_p\})$, for some $\rho \in SO(n)$, then $X_1, \ldots, X_p$ never have the same image under f. Therefore the following mapping is well defined:

$$F : V_p(\mathbb{R}^n) \to \mathbb{R}^p \backslash \Delta,$$
$$F(X_1, \ldots, X_p) := (f(X_1), \ldots, f(X_p)).$$

The group $\mathbb{Z}_p$ acts again freely on the domain and codomain by cyclically permuting the vectors and coordinates, respectively. Also, F is an $\mathbb{Z}_p$-equivariant mapping. As in the previous proof, the composition of the orthogonal projection and the radial projection is a $\mathbb{Z}_p$-equivariant retraction $q : \mathbb{R}^p \backslash \Delta \to S^{p-2}$. The composition $q \circ F : V_p(\mathbb{R}^n) \to S^{p-2}$ is a $\mathbb{Z}_p$-equivariant mapping. Since $V_p(\mathbb{R}^n)$ is $(n-p-1)$-connected, and $n-p-1 > p-2 = \dim S^{p-2}$, we obtain a contradiction thanks to Dold's theorem. □



Let us illustrate some results on the Knaster's problem with an attractive application. There is a natural question whether every convex body can be inscribed into a cube. By *convex body* in $\mathbb{R}^d$ we consider every compact, convex set in $\mathbb{R}^d$, with nonempty interior. We say that a body is inscribed into a cube if all faces of the cube belong to supporting planes of the body.

Theorem 4.3 (Kakutani, [15])**.** *Every convex body K in $\mathbb{R}^3$ can be inscribed into a cube.*

Proof. Consider the function $f : S^2 \to \mathbb{R}$, where for each vector $v \in S^2$, $f(v)$ is defined as the *width* of K in the direction of vector v, i.e. the distance between the supporting planes, orthogonal to the vector v. Since K is convex, f is continuous function. Now take an arbitrary orthonormal frame (e_1, e_2, e_3) on S^2. We know that $\{e_1, e_2, e_3\}$ is the solution of Knaster's problem for S^2 and $\mathbb{R}$, so there exists a rotation $\rho \in SO(3)$, such that $f(\rho(e_1)) = f(\rho(e_2)) = f(\rho(e_3))$. Put $\widetilde{e_i} = \rho(e_i)$, for $i = 1, 2, 3$. Then $\{\widetilde{e_1}, \widetilde{e_2}, \widetilde{e_3}\}$ is a new orthonormal frame whose vectors are parallel to directions in which our body has the same width. This means that along those directions we can put a cube which is circumscribed around the body. □

By using the same idea of width function $f : S^{n-1} \to \mathbb{R}$, now with supporting hyperplanes, and with the mentioned result that the vertices of an orthonormal basis are Knaster's solution for S^{n-1} and $\mathbb{R}$, we have the general result: *Every convex body in $\mathbb{R}^n$ can be inscribed into an n-dimensional cube.*

5 Cohomological Fadell-Husseini Index

5.1 Definition and basic properties

The (ideal-valued) cohomological index was introduced by E. Fadell and S. Husseini [9] in relation to critical point theory. In its general form, it is defined for compact lie groups G, paracompact G-pairs (X, A) and generalized multiplicative cohomology theories. Below we restrict ourselves to finite groups G and to singular cohomology. To define the cohomological index in this context, we need the concept of *universal principal G-bundle $EG \to BG$* (see [26], I.8). The space EG is a "direct limit" of spaces E_nG used in the definition of ind_G. It is a *contractible* free cell G-complex which can be obtained by taking the infinite join: $EG := G * G * G * \cdots$, where G-action is the standard action on a join. The orbit space $BG := EG/G$ is called *the classifying space of the group G*. For example, $E\mathbb{Z}_2 = S^\infty$, and $B\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^\infty$; the infinite real projective space. A detailed theory of G-bundles in given in [26].

Now, let X be a G-space and $\mathbb{K}$ a commutative ring with unit. A constant map $c : X \to *$ is obviously G-equivariant. Taking the product with EG gives the second projection $p : X \times EG \to EG$, which is also G-equivariant (on domain we consider the diagonal action). Therefore, the induced map $(X \times EG)/G \to EG/G = BG$ of orbit spaces is well-defined. A standard notation is $X_G := (X \times EG)/G$. Since cohomology is contravariant, this last mapping induces a homomorphism

$$p^* : H^*(BG; \mathbb{K}) \to H^*(X_G; \mathbb{K}).$$

Definition 5.1. *The Fadell-Husseini cohomological index* of a G-space X, with respect to coefficients $\mathbb{K}$, is defined as the kernel ideal:

$$\mathrm{Ind}_G(X; \mathbb{K}) := \ker(p^* : H^*(BG; \mathbb{K}) \to H^*(X_G; \mathbb{K})).$$

We will usually write just $\mathrm{Ind}_G(X)$, and keep in mind the ring of coefficients $\mathbb{K}$.

When X is a free G-space, instead of X_G we can work with X/G. Indeed, both spaces are homotopy equivalent and this can be seen as follows. The first projection $\pi : X \times EG \to X$ is G-equivariant so it induces a map of orbit spaces $\pi' : X_G \to X/G$. When the action on X is free, this is a bundle projection with fiber EG, which is a contractible space. This implies that π' is a weak homotopy equivalence and thus, for spaces of the homotopy type of a CW-complex, this map is a homotopy equivalence. Consequently, for free G-action we can write

$$\mathrm{Ind}_G(X; \mathbb{K}) = \ker(p^* : H^*(BG; \mathbb{K}) \to H^*(X/G; \mathbb{K})).$$

The main property of the cohomological index, which gives a necessary condition for the existence of G-map, is given in the following proposition.

Proposition 5.2 (Monotonicity)**.** *If $X \xrightarrow{G} Y$, then* $\mathrm{Ind}_G(X) \supset \mathrm{Ind}_G(Y)$.

Proof. Suppose that there exists a map $f : X \xrightarrow{G} Y$. The proof follows from a sequence of commutative diagrams, see below. The first diagram with constant maps commutes. When we product with EG, we get a commutative diagram of G-spaces and G-mappings. This induces the third commutative diagram of orbit spaces after passing to the quotients. Finally, we apply cohomology and revert arrows (coefficients are in $\mathbb{K}$). We obtain that $f^* \circ p_Y^* = p_X^*$, and thus $\ker p_Y^* \subset \ker p_X^*$.

$$\begin{tikzcd} X \arrow[rr, "G"] \arrow[dr] & & Y \arrow[dl] \\ & * & \end{tikzcd} \qquad \begin{tikzcd} X \times EG \arrow[rr, "G"] \arrow[dr] & & Y \times EG \arrow[dl] \\ & EG & \end{tikzcd}$$

$$\begin{tikzcd} X_G \arrow[r] \arrow[dr] & Y_G \arrow[d] \\ & BG \end{tikzcd} \qquad \begin{tikzcd} H^*(X_G) & H^*(Y_G) \arrow[l, "f^*"'] \\ H^*(BG) \arrow[u, "p_X^*"] \arrow[ur, "p_Y^*"'] & \end{tikzcd}$$

□

Besides monotonicity, the following three properties are usually called *the axioms of cohomological index*. We state them for completeness but we omit the proofs.

• (*Additivity*) If G-spaces X_1 and X_2 are open in $X_1 \cup X_2$, or they are CW-subspaces of a CW-complex $X_1 \cup X_2$, then $\mathrm{Ind}_G(X_1) \cdot \mathrm{Ind}_G(X_2) \subset \mathrm{Ind}_G(X_1 \cup X_2)$.

• (*Continuity*) If $A \subset X$ is a closed G-invariant subspace of X, then for some open G-invariant U, $U \supset A$, holds: $\mathrm{Ind}_G(\overline{U}) = \mathrm{Ind}_G(A)$.

• (*The Index theorem*) Let $f : X \to Y$ be a G-map, B is a closed G-invariant subspace of Y, and $A = f^{-1}(B) \subset X$. Then: $\mathrm{Ind}_G(A) \cdot \mathrm{Ind}_G(Y \backslash B) \subset \mathrm{Ind}_G(X)$.

There are also formulas for computing the cohomological indices of products and joins. For a detailed exposition of this topic see [9] and [29].

5.2 Examples of cohomological index

The cohomological index gives better classification of G-spaces than the numerical index. Cohomology rings of most classifying spaces of interest have been computed, and their cohomological indices determined. This turns out to be very useful in applications. The following important theorem is proved in [26], III(2.5).

Theorem 5.3. *For a prime number $p \neq 2$,*

$$H^*(B\mathbb{Z}_p; \mathbb{Z}_p) \cong \mathbb{Z}_p[t] \otimes_{\mathbb{Z}_p} \Lambda[s],$$

where $\deg t = 2$, $\deg s = 1$, and Λ is the exterior algebra over $\mathbb{Z}_p$ ($s^2 = 0$). Hence H^{2i} is generated by t^i and H^{2i+1} is generated by $t^i s$.

For $p = 2$, $H^*(B\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[t]$, where $\deg t = 1$.

Related to this theorem, there are several well-known indices.

• The index of antipodal action on sphere S^n with respect to coefficients $\mathbb{Z}_2$ is an ideal generated by t^{n+1}, i.e. $\mathrm{Ind}_{\mathbb{Z}_2}(S^n; \mathbb{Z}_2) = \langle t^{n+1} \rangle \subset \mathbb{Z}_2[t]$.

Also, for prime $p \neq 2$, the sphere $S^{2n-1} \subset \mathbb{C}^n$ is a $\mathbb{Z}_p$-space if $\mathbb{Z}_p$ is interpreted as a subgroup of $S^1 \subset \mathbb{C}$, and S^1 acts on S^{2n-1} by complex multiplication. In this case $\mathrm{Ind}_{\mathbb{Z}_p}(S^{2n-1}; \mathbb{Z}_p) = \langle t^n \rangle \subset \mathbb{Z}_p[t] \otimes_{\mathbb{Z}_p} \Lambda[s]$.

• The index which we need in the next chapter is the $\mathbb{Z}_p$-index of the configuration space of p points in $\mathbb{R}^d$ (with the aforementioned cyclic action). It is computed in [4], via application of spectral sequences and a strong machinery. Here is the theorem.

Theorem 5.4 ([4] 6.2)**.** *Let p be a prime and $d > 1$. Then for $\mathbb{K} = \mathbb{Z}_p$ holds*

$$\mathrm{Ind}_{\mathbb{Z}_p}(F(\mathbb{R}^d, p)) = H^{\geqslant (d-1)(p-1)+1}(B\mathbb{Z}_p; \mathbb{Z}_p)$$
$$= \begin{cases} \langle t^{(d-1)(p-1)+1} \rangle, & p = 2, \\ \langle s t^{\frac{(d-1)(p-1)}{2}}, t^{\frac{(d-1)(p-1)}{2}+1} \rangle, & p > 2. \end{cases}$$

6 The Nandakumar & Ramana Rao Conjecture

In 2006, Nandakumar and Ramana Rao [22] asked a very simple, but interesting question: given a positive integer n, can any convex polygon be partitioned into n convex pieces such that all pieces have the same area and the same perimeter? Such partition is called *the convex fair partition*. The pieces in the partition could have different shapes, of course. They proposed the following conjecture.

Conjecture 6.1 (Nandakumar & Ramana Rao)*.* For a given planar convex polygon K and any natural number $n > 1$ there exists a partition of the plane into n convex pieces $C_1, \ldots, C_n$ such that

$$\mathrm{area}(C_1 \cap K) = \cdots = \mathrm{area}(C_n \cap K) \quad \textit{and}$$

$$\mathrm{perimeter}(C_1 \cap K) = \cdots = \mathrm{perimeter}(C_n \cap K).$$

For example, for any rectangle the conjecture is true. Also, if n is a perfect square, there is a fair partition of every triangle T into n congruent small triangles. But what about an arbitrary polygon? For $n = 2$ Nandakumar and Ramana Rao [24] gave the following elementary proof. For every point A on the boundary ∂K, there exists a unique point $f(A) \in \partial K$, such that line $Af(A)$ divides K into two pieces of equal area. We move point A continuously clockwise along ∂K, until it reaches the point $f(A)$. The function $P_r(X) - P_l(X)$, where $P_r(X)(P_l(X))$ is the perimeter of the piece to the right(left) of line $Xf(X)$ (direction is important) has changed sign during this movement, so from intermediate value theorem there is a point $C \in \partial K$, between A and $f(A)$, such that line $Cf(C)$ divides K into two pieces of equal area and equal perimeter.

Nandakumar and Ramana Rao [23] also gave elementary arguments for the case $n = 2^k$. Afterwards, many experts in equivariant topology worked on this problem. The case $n = 3$ was settled in [3]. Also, an analogous question was asked for higher dimensions. In [14], the 3-dimensional case is presented as *the spicy-chicken problem*: is it possible to cut a chicken fillet, with surface marinated in a sauce, such that each among n people gets the same amount of chicken and the same amount of sauce? In fact, we want a partition of 3-dimensional convex body into n pieces of equal volume and equal surface area. Naturally, one can think of adding an extra condition in 3-dimensions, beside the volume and the surface area. To generalize Conjecture 6.1 formally, we need several notions. Let $\mathrm{Conv}(\mathbb{R}^d)$ denotes the metric space of all d-dimensional convex bodies with the Hausdorff metric. Instead of the area or volume, generally we can speak about nice measures.

A *nice* measure μ in $\mathbb{R}^d$ is an absolutely continuous probability measure (given by a nonnegative Lebesgue integrable density function with convex support) such that the measure of every hyperplane is zero.

A *convex partition of* $\mathbb{R}^d$ is a partition of $\mathbb{R}^d$ where all the parts are closed convex sets with pairwise disjoint interiors.

Conjecture 6.2 (Generalized N&RR). *Given a convex body $K \subset \mathbb{R}^d$ ($d \geqslant 2$), a nice measure μ on $\mathbb{R}^d$, any natural number $n > 1$ and any $d-1$ continuous functions $\varphi_1, \ldots, \varphi_{d-1} : \mathrm{Conv}(\mathbb{R}^d) \to \mathbb{R}$, there exists a partition of $\mathbb{R}^d$ into n convex pieces $C_1, \ldots, C_n$ such that*

$$\mu(C_1 \cap K) = \cdots = \mu(C_n \cap K) \text{ and}$$

$$\varphi_i(C_1 \cap K) = \cdots = \varphi_i(C_n \cap K), \text{ for all } i \in \{1, \ldots, d-1\}.$$

Notice that in $\mathbb{R}^2$ this generalizes the first conjecture for convex figures instead of polygons. The next steps in solving both the original and the generalized N&RR conjecture were done by Karasev [16], Hubard & Aronov [14] and Blagojević & Ziegler in [5]. All of them observed that both conjectures are true if there is no Σ_n-equivariant mapping $F(\mathbb{R}^d, n) \to S(W_n^{\oplus(d-1)})$. Here W_n is the Σ_n-representation

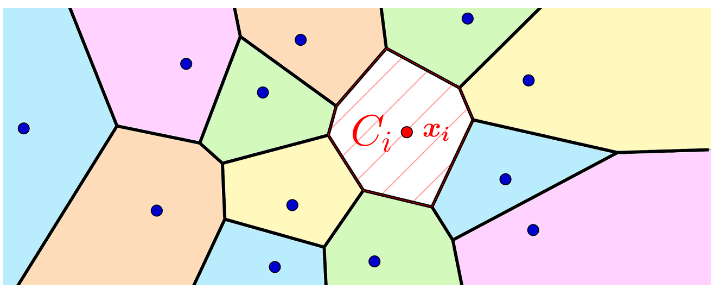
$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\},$$

where the Σ_n-action is given by permuting coordinates. Till now, it has been verified that both conjectures hold for $n = p^k$, where p is a prime number.

Our aim is to present the proofs for both conjectures in the case $n = p$, for prime p. There are two steps. First, we establish the claim that the conjectures are true if there is no Σ_n-equivariant map $F(\mathbb{R}^d, n) \to S(W_n^{\oplus(d-1)})$. The second step is to prove this claim for prime n. Before pursuing this, we analyze next one particular convex partition of $\mathbb{R}^d$ that we need, and which arises from every given finite set of points in $\mathbb{R}^d$.

6.1 Voronoi diagrams

Let S be an ordered n-tuple of distinct points in $\mathbb{R}^d$, $S = (x_1, x_2, \ldots, x_n) \in F(\mathbb{R}^d, n) \subset (\mathbb{R}^d)^n$. Points in S are called *sites*. For every site x_i, we define the set $C_i = \{x \in \mathbb{R}^d \mid \|x - x_i\| \leqslant \|x - x_j\| \text{ for all } 1 \leqslant j \leqslant n\}$, which is called *Voronoi region*. It contains all points x for which distance to x_i is minimal among all distances to the sites.



Each C_i is the intersection of half-spaces $H_{i,j}^+ = \{x \in \mathbb{R}^d \mid \|x - x_i\| \leqslant \|x - x_j\|\}$, so C_i must be a convex polyhedron (for $d = 2$ a convex polygon). Each two Voronoi regions intersect at most along their faces, therefore we obtain a convex partition of $\mathbb{R}^d$ called *Voronoi diagram*: $V(S) = (C_1, C_2, \ldots, C_n)$.

Now, let us take sites $S = (x_1, \ldots, x_n) \in F(\mathbb{R}^d, n)$ and a weight vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. We define power functions $h_i : \mathbb{R}^d \to \mathbb{R}$, $h_i(x) = \|x - x_i\|^2 - w_i$, $i = 1, \ldots, n$. Then we have *power regions* which minimize the power distance:

$$C_i = \{x \in \mathbb{R}^d \mid h_i(x) \leqslant h_j(x) \text{ for all } 1 \leqslant j \leqslant n\}, \quad i = 1, \ldots, n.$$

The set $V(S, w) = (C_1, C_2, \ldots, C_n)$ is called *generalized Voronoi diagram* or *power diagram*. Obviously, for $w = (0, 0, \ldots, 0)$, it is the standard Voronoi diagram with sites S. Again, every region C_i is a convex polyhedron, with at most $n-1$ facets because it is the intersection of halfspaces $H_{i,j}^+ = \{x \in \mathbb{R}^d \mid h_i(x) \leqslant h_j(x)\}$, bounded by hyperplanes: $H_{i,j} = \{x \in \mathbb{R}^d \mid h_i(x) = h_j(x)\}$, i.e. equivalently

$$H_{i,j} = \{x \in \mathbb{R}^d \mid 2\langle x, x_i - x_j\rangle = \|x_i\|^2 - \|x_j\|^2 + w_j - w_i\}.$$

So, $V(S, w)$ is a convex partition of $\mathbb{R}^d$. In contrast to standard case, some power regions may be empty. But visually, it is not easy to distinguish between standard and generalized Voronoi diagrams. A discussion of this construction in the plane is given in [8].

Notice that every dividing hyperplane $H_{i,j}$ is orthogonal to $x_i - x_j$, and that its position depends only on $w_j - w_i$ (for fixed S). So, if we add the same amount to every weight, diagram doesn't change. In other words, for all α, $V(S, w) = V(S, w + \alpha e)$, where $e = (1, 1, \ldots, 1)$. Hence for every power diagram we can choose w such that $w_1 + \cdots + w_n = 0$, i.e. $w \in W_n$. Even more, if $V(S, w) = V(S, w')$, then w' has to be this kind of translate of vector w ([25],[14]).

For us, the most important property of a power diagram is that it can equipartition nice measures. It has been proven that given a nice measure μ and the set of sites S, there is a weight vector w such that a power diagram $V(S, w)$ satisfies $\mu(C_i) = \frac{\mu(\mathbb{R}^d)}{n}$ for all $i = 1, \ldots, n$ ([1]). This w is unique up to translations by the diagonal e, so it is unique in W_n. Furthermore, when points in S move continuously (and remain different) then the weight vector w also moves continuously through W_n ([1], [14]). So, the function which to each $S = (x_1, x_2, \ldots, x_n) \in F(\mathbb{R}^d, n)$ associates the power diagram $V(S, w) = (C_1, \ldots, C_n)$ that equipartitions the measure, is continuous.

6.2 The construction of the mapping $F(\mathbb{R}^d, n) \xrightarrow{\Sigma_n} S(W_n^{\oplus(d-1)})$.

This is a construction that illustrates again the configuration space - test map method. We construct the test map in details when $d = 2$; the case $d > 2$ being entirely analogous. Proceeding by contradiction, we suppose that there is a convex polygon K that doesn't satisfy Conjecture 6.1 for some $n > 1$. We start from $S = (x_1, x_2, \ldots, x_n) \in F(\mathbb{R}^2, n)$. The area of K can be considered as a nice measure in $\mathbb{R}^2$, so from the previous section we have power diagram $V(S, w) = (C_1, \ldots, C_n)$ in the plane that equipartitions the area. If we put $C_i' := C_i \cap K$, we have the convex equal area partition of K:

$$(C'_1, C'_2, \dots, C'_n) \in CEAP(K, n),$$

where $CEAP(K,n) \subset \mathrm{Conv}(\mathbb{R}^2) \times \cdots \times \mathrm{Conv}(\mathbb{R}^2)$ is the space of convex equal area partitions of K, with suitably defined metric. Then we map this to n-tuple of perimeters ($p(-)$ denotes the perimeter):

$$\mapsto (p(C'_1), p(C'_2), \dots, p(C'_n)) \in \mathbb{R}^n,$$

and normalize it by subtracting $\frac{1}{n}\sum_{k=1}^n p(C'_k)$ in order to get an element of W_n:

$$\mapsto \left(p(C'_1) - \frac{1}{n}\sum_{k=1}^n p(C'_k), \dots, p(C'_n) - \frac{1}{n}\sum_{k=1}^n p(C'_k) \right).$$

By assumption, the perimeters of pieces are never all equal, so the image of this function is in $W_n \backslash \{0\}$. Therefore we can take the radial projection to sphere in W_n: $S(W_n) \cong S^{n-2}$. There is a natural Σ_n-action on all these spaces, given by permuting the coordinates, and we see that all considered mappings are Σ_n-equivariant. So, we have constructed an equivariant mapping:

$$\mathbf{F}(\mathbb{R}^{\mathbf{2}}, \mathbf{n}) \xrightarrow{\boldsymbol{\Sigma}_{\mathbf{n}}} \mathbf{S}(\mathbf{W}_{\mathbf{n}}).$$

It can be verified that this map is continuous and the reader is referred for details to [14].

In the case of Conjecture 6.2 we proceed analogously. Each $S = (x_1, x_2, \dots, x_n) \in F(\mathbb{R}^d, n)$ maps continuously to $(C'_1, C'_2, \dots, C'_n) \in CEAP(K, n)$, and then to

$$\begin{aligned}&[(\varphi_1(C'_1), \dots, \varphi_1(C'_n)),\\ &\qquad \dots\dots, (\varphi_{d-1}(C'_1), \dots, \varphi_{d-1}(C'_n))] \in (\mathbb{R}^n)^{d-1}.\end{aligned}$$

After that, in each n-tuple, we subtract the average value $\frac{1}{n}\sum_{k=1}^n \varphi_i(C'_k)$, and then we get an element of $W_n^{\oplus(d-1)}$. If we suppose to the contrary that there is no equipartiton on which all φ_i are equal (for all i), it is obvious that the obtained element of $W_n^{\oplus(d-1)}$ is not zero, so we can take the radial projection to the sphere $S(W_n^{\oplus(d-1)})$. And again, the group Σ_n acts naturally on all these spaces (permuting in each n-tuple). The corresponding mappings are Σ_n-equivariant, so we have: $\mathbf{F}(\mathbb{R}^{\mathbf{d}}, \mathbf{n}) \xrightarrow{\boldsymbol{\Sigma}_{\mathbf{n}}} \mathbf{S}(\mathbf{W}_{\mathbf{n}}^{\oplus(\mathbf{d-1})})$.

Notice that $S(W_n^{\oplus(d-1)}) \approx S^{(n-1)(d-1)-1}$, but we keep the notation $S(W_n^{\oplus(d-1)})$ since it contains the information about the Σ_n-action, which we will need.

6.3 Proof of conjectures for a prime number of pieces

The main theorem is based on cohomological index. It was proven in [4].

Theorem 6.1. *Let p be a prime number and $d \geqslant 2$. Then there is no $\mathbb{Z}_p$-equivariant mapping*

$$F(\mathbb{R}^d, p) \to S(W_p^{\oplus(d-1)}).$$

Consequently, there is no Σ_p-equivariant mapping

$$F(\mathbb{R}^d, p) \to S(W_p^{\oplus(d-1)}).$$

Proof. Suppose to the contrary that there exists $f : F(\mathbb{R}^d, p) \xrightarrow{\mathbb{Z}_p} S(W_p^{\oplus(d-1)})$. From the monotonicity property of cohomological index (Proposition 5.2) we have:

$$\mathrm{Ind}_{\mathbb{Z}_p}(F(\mathbb{R}^d, p); \mathbb{Z}_p) \supset \mathrm{Ind}_{\mathbb{Z}_p}(S(W_p^{\oplus(d-1)}); \mathbb{Z}_p).$$

We know that the index of configuration space $F(\mathbb{R}^d, p)$ with $\mathbb{Z}_p$ coefficients is:

$$\begin{aligned}\mathrm{Ind}_{\mathbb{Z}_p}(F(\mathbb{R}^d, p)) &= H^{\geqslant(d-1)(p-1)+1}(B\mathbb{Z}_p; \mathbb{Z}_p)\\ &= \begin{cases} \langle t^{(d-1)(p-1)+1} \rangle, & p = 2,\\ \langle st^{\frac{(d-1)(p-1)}{2}}, t^{\frac{(d-1)(p-1)}{2}+1} \rangle, & p > 2.\end{cases}\end{aligned}$$

On the other hand, observe that the $\mathbb{Z}_p$-action on $S(W_p^{\oplus(d-1)})$ is free. Indeed, since p is prime, it is enough to check that the generator of $\mathbb{Z}_p$ acts freely. The $\mathbb{Z}_p$-action cyclically permutes coordinates in each W_p. If the $\mathbb{Z}_p$-action of the generator fixes some vector $(w_1, \dots, w_p) \in W_p$, then $w_1 = \ldots = w_p$, so all of them are zeros. The same holds for all $d-1$ copies W_p, and we get the origin as the only element on which the action is not free, but it is not in $S(W_p^{\oplus(d-1)})$. So, the action if free indeed. Then according to section 5.1 we know that $E\mathbb{Z}_p \times_{\mathbb{Z}_p} S(W_p^{\oplus(d-1)}) \simeq S(W_p^{\oplus(d-1)})/\mathbb{Z}_p$. Therefore for all $l > \dim S(W_p^{\oplus(d-1)}) = (d-1)(p-1)-1$ we have

$$\begin{aligned}&H^l(E\mathbb{Z}_p \times_{\mathbb{Z}_p} S(W_p^{\oplus(d-1)})\ ; \mathbb{Z}_p)\\ &\quad = H^l(S(W_p^{\oplus(d-1)})/\mathbb{Z}_p\ ; \mathbb{Z}_p)\\ &\quad = 0.\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Ind}_{\mathbb{Z}_p}(S(W_p^{\oplus(d-1)}); \mathbb{Z}_p) =\\ &\quad \ker\left(H^*(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^*(E\mathbb{Z}_p \times_{\mathbb{Z}_p} S(W_p^{\oplus(d-1)}); \mathbb{Z}_p\right)\\ &\quad \supset H^{\geqslant(d-1)(p-1)}(B\mathbb{Z}_p; \mathbb{Z}_p).\end{aligned}$$

But from the monotonicity property we conclude that

$$H^{\geqslant(d-1)(p-1)+1}(B\mathbb{Z}_p; \mathbb{Z}_p) \supset H^{\geqslant(d-1)(p-1)}(B\mathbb{Z}_p; \mathbb{Z}_p),$$

which is a contradiction since for $p > 2$ we have: $t^{\frac{(d-1)(p-1)}{2}} \in \mathrm{Ind}_{\mathbb{Z}_p}(S(W_p^{\oplus(d-1)}))$ and

$$t^{\frac{(d-1)(p-1)}{2}} \notin \mathrm{Ind}_{\mathbb{Z}_p}(F(\mathbb{R}^d, p));$$

while for $p = 2$ we have: $t^{(d-1)(p-1)} \in \mathrm{Ind}_{\mathbb{Z}_p}(S(W_p^{\oplus(d-1)}))$ and $t^{(d-1)(p-1)} \notin \mathrm{Ind}_{\mathbb{Z}_p}(F(\mathbb{R}^d, p)))$. This contradiction proves the theorem. □

The previous discussion together with Theorem 6.1 establishes the N&RR conjecture in full generality for all primes p. In summary we can state the following beautiful consequences.

Theorem 6.2. *For a given planar convex polygon K and any prime number p there exists a partition of the plane into p convex pieces $C_1, \dots, C_p$ such that*

$$\text{area}(C_1 \cap K) = \cdots = \text{area}(C_p \cap K) \quad \textit{and}$$

$$\text{perimeter}(C_1 \cap K) = \cdots = \text{perimeter}(C_p \cap K).$$

Theorem 6.3. *For a given convex body K in $\mathbb{R}^d$ ($d \geqslant 2$), a nice measure μ on $\mathbb{R}^d$, any prime p and any $d-1$ continuous functions $\varphi_1, \dots, \varphi_{d-1} : \text{Conv}(\mathbb{R}^d) \to \mathbb{R}$, there exists a partition of $\mathbb{R}^d$ into p convex pieces $C_1, \dots, C_p$ such that*

$$\mu(C_1 \cap K) = \cdots = \mu(C_p \cap K) \quad \textit{and}$$

$$\varphi_i(C_1 \cap K) = \cdots = \varphi_i(C_p \cap K), \forall\, i \in \{1, \dots, d-1\}.$$

7 An Open Problem

Finally we conclude with some simple questions that every high school student can understand but that are still unanswered. Can every triangle be partitioned into 6 pieces of equal area and equal perimeter? For 7 pieces we know the answer: yes. For 125 or 7^{2015} pieces, too. But for 6? Or 10 pieces? No one knows. Not yet.

Acknowledgement: The author would like to express gratitude to the members of the CGTA-seminar at the University of Belgrade for their lectures on the connections between topology and discrete geometry,; lectures that were the inspiration for studying topology.

References

[1] B. Aronov, F. Aurenhammer, F. Hoffmann, *Minkowski-type theorems and least-squares clustering*, Algoritmica **20(1)** (1998), 61-76.

[2] I. K. Babenko, S. A. Bogatyi, *On mapping a sphere into Euclidean space* (in Russian), Mat. Zametki **46(3)** (1989), 3–8.

[3] I. Bárány, P. V. M. Blagojević, A. Szűcs, *Equipartitioning by a convex 3-fan*, Advances in Math. **223** (2010), 579–593.

[4] P. V. M. Blagojević, W. Lück, G. M. Ziegler, *Equivariant Topology of Configuration Spaces*, Israel Journal of Mathematics **200** (2014), 49–77.

[5] P. V. M. Blagojević, G. M. Ziegler, *Convex equipartitions via equivariant obstruction theory*, February 2012, http://arxiv.org/abs/1202.5504.

[6] S. A. Bogatyi, G. N. Khimshiashvili, *A generalization of the Borsuk-Ulam theorem and a problem of Knaster* (in Russian), Soobsh. Akad. Nauk. Gruzinskoj SSR **123** (1986), 477–480.

[7] W. Chen, *Counterexamples to Knaster's conjecture*, Topology **37(2)**(1998), 401–405.

[8] H. Edelsbrunner, *A Short Course in Computational Geometry and Topology*, Springer (2014).

[9] E. Fadell, S. Husseini, *An ideal-valued cohomological index theory with applications to Borsuk-Ulam and Bourgin-Yang theorems*, Ergodic Th. and Dynam. Sys. **8*** (1988), 73–85.

[10] E. E. Floyd, *Real valued mappings on spheres*, Proc. Amer. Math. Soc. **6** (1955), 957–959.

[11] The image of fractal created by Fractal Science Kit: www.fractalsciencekit.com.

[12] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2001.

[13] H. Hopf, *Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze*, Portugal. Math. **4** (1944), 129–139.

[14] A. Hubard, B. Aronov, *Convex equipartitions of volume and surface area*, Preprint, October 2010; version 3, September 2011, http://arxiv.org/abs/1010.4611.

[15] S. Kakutani, *A proof that there exists a circumscribing cube around any bounded closed set in* $\mathbb{R}^3$, Ann. Math. **43** (1942), 739–741.

[16] R. N. Karasev, *Equipartition of several measures*, Preprint, November 2010; version 6, August 2011, http://arxiv.org/abs/1011.4762.

[17] B. Knaster, *Problem 4*, Colloq. Math. **30(1)** (1947), 30–31.

[18] G. R. Livesay, *On a theorem of F. J. Dyson*, Ann. of Math. **59** (1954), 227–229.

[19] V. V. Makeev, *On some properties of continuous mappings of spheres and problems of combinatorial geometry* (in Russian), Geometrical questions of the theory of functions and set theory, Kalinin (1986), 75–85.

[20] V. V. Makeev, *The Knaster problem and almost spherical sections* (in Russian), Mat. Sbornik, Kalinin **180(3)** (1989), 425–431.

[21] J. Matoušek, *Using the Borsuk-Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer-Verlag, Heidelberg, 2003.

[22] R. Nandakumar, *"Fair" partitions*, Blog entry, http://nandacumar.blogspot.de/2006/09/cutting-shapes.html, September 28, 2006.

[23] R. Nandakumar, N. Ramana Rao, *Fair partitions of polygons: An elementary introduction*, Proc. Indian Academy of Sciences (Mathematical Sciences) **122** (2012), 459–467.

[24] R. Nandakumar, N. Ramana Rao, *"Fair" Partitions of Polygons - an Introduction*, arxiv: 0812.2241 (2008).

[25] P. Soberón, *Balanced convex partitions of measures in* $\mathbb{R}^d$, Mathematatika **58** (2012), 71–76.

[26] T. tom Dieck, *Transformation groups*, Walter de Gruyter, Berlin, New York, 1987.

[27] H. Yamabe, Z. Yujobo, *On the continuous function defined on a sphere*, Osaka Math. **2** (1950), 19–22.

[28] R.T. Živaljević *User's guide to equivariant methods in combinatorics*, Publications de l'Institut mathematuque, **59(73)** (1996), 114–130.

[29] R.T. Živaljević *User's guide to equivariant methods in combinatorics II*, Publications de l'Institut mathematuque, **64(78)** (1998), 107–132.

Marija Jelić
Department of Topology
Faculty of Mathematics - University of Belgrade,
Studentski trg 16, 11000 Belgrade
Serbia
E-mail address: marijaj@matf.bg.ac.rs