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An introduction to Schur polynomials

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Abstract

This is a self-contained account of the theory of Schur polynomials, leading up to a proof of the Littlewood-Richardson rule. It has been attempted to give the simplest, most elegant known proofs of all results along the way. The way the RSK correspondence is introduced and then used to prove the Littlewood-Richardson rule in this article is new.

MSC 2010. Primary: 05E05.

Contents

1 Symmetric Polynomials

Consider polynomials in n variables $x_1, \dots, x_n$ having integer coefficients. Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, let x^α denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. A *symmetric polynomial* is a polynomial of the form

$$f(x_1, \dots, x_n) = \sum_\alpha c_\alpha x^\alpha, \text{ with } c_\alpha \in \mathbf{Z},$$

where, for any permutation $w \in S_n$, $c_{(\alpha_1, \dots, \alpha_n)} = c_{(\alpha_{w(1)}, \dots, \alpha_{w(n)})}$.

The integer partition λ obtained by sorting the coordinates of α is called the *shape* of α, denoted $\lambda(\alpha)$. The most obvious example of a symmetric polynomial in n variables is the *monomial symmetric polynomial*, defined for each integer partition λ:

$$m_\lambda = \sum_{\lambda(\alpha)=\lambda} x^\alpha.$$

Note that m_λ is homogeneous of degree $|\lambda|$ (the sum of the parts of λ).

Exercise 1.1. Take $n = 4$. Compute the monomial symmetric polynomials $m_{(3)}$, $m_{(2,1)}$, and $m_{(1^3)}$.

Theorem 1.2. *The polynomials $m_\lambda(x_1, \dots, x_n)$, as λ runs over all the integer partitions of d, form a basis for the space of homogeneous symmetric polynomials of degree d in n variables.*

2 Complete and Elementary Symmetric Polynomials

Recall that the coefficients of a polynomial are symmetric polynomials in its roots:

$$(t - x_1)(t - x_2)\cdots(t - x_n) = t^n - e_1(x_1, \dots, x_n)t^{n-1} + \cdots + (-1)^n e_n(x_1, \dots, x_n), \tag{2.1}$$

where the expression $e_i(x_1, \dots, x_n)$ in the coefficient of t^{n-i} is given by:

$$e_i(x_1, \dots, x_n) = \sum_{1\leq j_1<\cdots<j_i\leq n} x_{j_1}x_{j_2}\cdots x_{j_i}. \tag{2.2}$$

The polynomial e_i is called the ith *elementary symmetric polynomial*. By convention, $e_i(x_1, \dots, x_n) = 0$, for $i > n$.

The identity (2.1) can be written more elegantly as:

$$(1 + tx_1)\cdots(1 + tx_n) = \sum_{i=0}^{\infty} e_i(x_1, \cdots, x_n)t^i.$$

Dually, the *complete symmetric polynomials* are defined by the formal identity:

$$\frac{1}{(1 - x_1t)\cdots(1 - x_nt)} = \sum_{i=0}^{\infty} h_i(x_1, \cdots, x_n)t^i.$$

Example 2.1. In three variables:

$$\begin{aligned} e_2(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\ h_2(x_1, x_2, x_3) &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2. \end{aligned}$$

Exercise 2.2. Show that

$$h_i(x_1, \dots, x_n) = \sum_{1\leq j_1\leq\cdots\leq j_i\leq n} x_{j_1}\cdots x_{j_i},$$

and that $e_i(x_1, \dots, x_n) = \sum_{1\leq j_1<\cdots<j_i\leq n} x_{j_1}\cdots x_{j_i}$.

More generally, for any integer partition $\lambda = (\lambda_1, \dots, \lambda_l)$, define:

$$\begin{aligned} h_\lambda &= h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l}, \\ e_\lambda &= e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_l}. \end{aligned}$$

Theorem 2.3. *Given partitions $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \cdots, \mu_m)$ of d, let $M_{\lambda\mu}$ denote the number of matrices (a_{ij}) with non-negative integer entries whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j. Then*

$$h_\lambda = \sum_\mu M_{\lambda\mu}m_\mu.$$

Dually, let $N_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with entries 0 or 1, whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j. Then

$$e_\lambda = \sum_\mu N_{\lambda\mu}m_\mu.$$

Proof. To prove the second identity involving elementary symmetric polynomials, note that a monomial in the expansion of

$$e_\lambda = \prod_{i=1}^{l} \sum_{j_1 < \cdots < j_{\lambda_i}} x_{j_1} \cdots x_{j_{\lambda_i}}$$

is a product of summands, one chosen from each of the l factors. Construct an $l \times m$ matrix (a_{ij}) corresponding to such a choice as follows: if the summand $x_{j_1} \cdots x_{j_{\lambda_i}}$ is chosen from the ith factor, then set the entries $a_{i,j_1}, \dots, a_{i,j_{\lambda_j}}$ to be 1 (the remaining entries of the ith row are 0). Clearly, the ith row of such a matrix sums to λ_i. The monomial corresponding to this choice is x^μ if, for each j, the number of i for which x_j appears in the monomial corresponding to the ith row is μ_j. This is just the sum of the jth column of the matrix (a_{ij}). It follows that the coefficient of x^μ, and hence the coefficient of m_μ in the expansion of e_λ in the basis of monomial symmetric polynomials of degree n, is $N_{\lambda\mu}$.

A similar proof can be given for the first identity involving complete symmetric polynomials. The only difference is that variables may be repeated in the monomials that appear in h_i. Counting the number of repetitions (instead of just recording 0 or 1) gives non-negative integer matrices. ☐

3 Alternating Polynomials

An *alternating polynomial* in $x_1, \dots, x_n$ is of the form:

$$f(x_1, \dots, x_n) = \sum_\alpha c_\alpha x_\alpha, \tag{3.1}$$

where, $c_{(\alpha_{w(1)}, \dots, \alpha_{w(n)})} = \epsilon(w) c_{(\alpha_1, \dots, \alpha_n)}$ for every multiindex α as in Section 1, and every permutation $w \in S_n$. Here $\epsilon : S_n \to \{\pm 1\}$ denotes the sign function. Equivalently, an alternating polynomial is one whose sign is reversed upon the interchange of any two variables.

Exercise 3.1. If α is a multiindex with $\alpha_i = \alpha_j$ for some $i \neq j$, then $c_\alpha = 0$.

In particular, every monomial in an alternating polynomial must be composed of distinct powers. Moreover, the polynomial is completely determined by the coefficients with strictly decreasing multiindices, namely, multiindices of the form c_α, where $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1 > \cdots > \alpha_n$.

Exercise 3.2. Let $\delta = (n-1, n-2, \dots, 1, 0)$. Given an integer partition with at most n parts, we will pad it with 0's so that it can be regarded as a weakly decreasing multiindex of length n. Then $\lambda \mapsto \lambda + \delta$ is a bijection from the set of integer partitions with at most n parts onto the set of strictly decreasing multiindices.

Example 3.3. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a weakly decreasing multiindex. The *alternant* corresponding to λ, which is defined as:

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})$$

is alternating, with unique strictly decreasing monomial $x^{\lambda+\delta}$.

Exercise 3.4. The alternating polynomial of the form (3.1) is equal to $\sum_\lambda c_{\lambda+\delta} a_{\lambda+\delta}$, the sum being over all weakly decreasing multiindices λ.

4 Interpretation of Alternants with Labeled Abaci

A labeled abacus with n beads is a word $w = (w_k; k \geq 0)$ with letters $w_i \in \{0, \dots, n\}$ such that the subword of non-zero letters is a permutation of $1, 2, \dots, n$. The sign $\epsilon(w)$ of the abacus is the sign of this permutation, the support is the set $\operatorname{supp}(w) = \{k \mid w_k > 0\}$, and the weight is defined as:

$$\operatorname{wt}(w) = \prod_{k \in \operatorname{supp}(w)} x_{w_k}^k .$$

The shape of the abacus, $\operatorname{shape}(w)$ is the unique partition λ such that the components of $\lambda + \delta$ form the support of w.

Example 4.1. Consider the labeled abacus $w = 510032046000\cdots$. Its underlying permutation is 513246, which has sign -1, so $\epsilon(w) = -1$. Also, $\mathrm{supp}(w) = \{0, 1, 4, 5, 7, 8\}$, $\mathrm{shape}(w) = (3, 3, 2, 2)$ (indeed, $(3, 3, 2, 2, 0, 0) + (5, 4, 3, 2, 1, 0) = (8, 7, 5, 4, 1, 0)$) and $\mathrm{wt}(w) = x_5^0x_1^1x_3^4x_2^5x_4^7x_6^8$. We visualize the abacus w as a configuration of beads on a single runner, with possible positions of beads numbered $1, 2, 3, \ldots$. If $w_k = i$ where $i > 0$, then a bead labeled i is placed in position k on the runner. If $w_k = 0$, then the position k is unoccupied. In the running example the visualization is:

0	1	2	3	4	5	6	7	8	9
⑤	①	•	•	③	②	•	④	⑥	•

The first row shows the positions $k = 0, 1, \ldots$ on the runner and the second row shows the beads.

Theorem 4.2. *For every partition λ the alternant in n variables,*

$$a_{\lambda+\delta} = (-1)^{\lfloor n/2 \rfloor} \sum_{w} \epsilon(w)\mathrm{wt}(w),$$

the sum being over all labeled abaci with n beads and shape λ.

Proof. The theorem follows from the expansion of the determinant. □

5 Cauchy's Bialternant Form of a Schur Polynomial

The lowest degree polynomial of the form $a_{\lambda+\delta}$ arises when $\lambda = 0$; a_δ is the Vandermonde determinant:

$$a_\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Exercise 5.1. Show that, for every weakly decreasing multiindex λ, $a_{\lambda+\delta}$ is divisible by a_δ in the ring of polynomials in $x_1, \ldots, x_n$.

Exercise 5.2. Show that $f \mapsto fa_\delta$ is an isomorphism of the space of homogeneous symmetric polynomials in $x_1, \ldots, x_n$ of degree d onto the space of homogeneous alternating polynomials of degree $d + \binom{n}{2}$.

This motivates the historically oldest definition of Schur polynomials—*Cauchy's bialternant formula*:

$$s_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_\delta, \tag{5.1}$$

for any partition λ with at most n parts. If λ has more than n parts, set $s_\lambda(x_1, \ldots, x_n) = 0$. This is clearly a symmetric polynomial of degree $|\lambda|$.

Theorem 5.3. *As λ runs over all integer partitions of d with at most n parts, the Schur polynomials $s_\lambda(x_1, \ldots, x_n)$ form a basis of the space of all homogeneous symmetric polynomials in $x_1, \ldots, x_n$ of degree d.*

Proof. This follows from Exercises 3.4 and 5.2. □

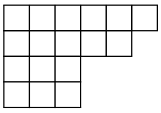
Exercise 5.4. [Stability of Schur polynomials] Show that substituting $x_n = 0$ in the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ with n variables gives the corresponding Schur polynomial $s_\lambda(x_1, \ldots, x_{n-1})$ with $n - 1$ variables.

6 Pieri's rule

The set of integer partitions is endowed with the *containment order*. We say that a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ *contains* a partition $\mu = (\mu_1, \ldots, \mu_m)$ if $l \geq m$, and $\lambda_i \geq \mu_i$ for every $i = 1, \cdots, m$. We write $\lambda \supset \mu$ or $\mu \subset \lambda$. Recall that the Young diagram of the partition λ is the set of points

$$\{(i, j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

Visually, each node (i, j) of the Young diagram is replaced by a box, and the box corresponding to (i, j) is placed in the ith row and jth column (matrix notation). Thus, the Young diagram of $\lambda = (6, 5, 3, 3)$ is:



The containment of partitions is nothing but the containment relation on their Young diagrams. Henceforth, for a partition λ, the symbol λ will also be used to refer to its Young diagram.

A *skew-shape* is a difference of Young diagrams $\lambda \setminus \mu$, where $\lambda \supset \mu$. Write λ/μ for this skew-shape. A skew-shape is called a *horizontal strip* (respectively, a *vertical strip*) if it has at most one box in each vertical column (respectively, horizontal row).

Theorem 6.1. *For every partition λ, and every positive integer k,*

$$s_\lambda h_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip of size k. Dually,

$$s_\lambda e_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \supset \lambda$ such that μ/λ is a vertical strip of size k.

Proof. Let $\mathrm{Abc}(\lambda)$ denote the set of all n-bead labeled abaci (see Section 4) of shape λ. Let $M(n,k)$ denote the set of all vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ with non-negative integer coordinates and sum k. Set $\mathrm{wt}(\alpha) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Using the abacus interpretation of alternants (Theorem 4.2), the first identity is equivalent to showing:

$$\sum_{w \in \mathrm{Abc}(\lambda)} \epsilon(w) \sum_{\alpha \in M(n,k)} \mathrm{wt}(\alpha) = \sum_\mu \sum_{w \in \mathrm{Abc}(\mu)} \epsilon(w)\mathrm{wt}(w),$$

the sum on the right being over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip. We will define an involution I on the $\mathrm{Abc}(\lambda) \times M(n,k)$ whose fixed points correspond to elements of

$$\coprod_{\mu/\lambda \text{ is a horiz. strip of size } k} \mathrm{Abc}(\mu) \times M(n,k)$$

under a bijection that preserves weights and signs, and such that if $I(w,\alpha) = (w', \alpha')$ then $\mathrm{wt}(w)\mathrm{wt}(\alpha) = \mathrm{wt}(w')\mathrm{wt}(\alpha')$ and $\epsilon(w') = -\epsilon(w)$. Then all terms on the left hand side, except for those which do not correspond to fixed points, will cancel, and the surviving terms will give the right hand side.

To construct I, scan the abacus from left to right. Upon encountering a bead numbered j, move the bead α_j steps to the right, one step at a time. If this process completes without this bead colliding with another bead, (w,α) is a fixed point of I. The new abacus w^* has $\epsilon(w^*) = \epsilon(w)$ (the underlying permutation remains unchanged), and $\mathrm{shape}(w^*)/\mathrm{shape}(w)$ is a horizontal strip of size k.

However, suppose a collision does occur, say the first collision is when bead j hits bead k that is located $p \leq \alpha_j$ places to the right of its initial position. Define $I(w,\alpha) = (w', \alpha')$, where w' is w with the beads i and j interchanged, $\alpha'_j = \alpha_j - p$, $\alpha'_k = \alpha_k + p$ and all other coordinates of α and α' are equal. Clearly w' has the opposite sign from w, and $\mathrm{wt}(w)\mathrm{wt}(\alpha) = \mathrm{wt}(w')\mathrm{wt}(\alpha')$. It is not hard to see that $I(w', \alpha') = (w, \alpha)$.

Example 6.2. Let $n = 6$, $\lambda = (3,3,2,2,0,0)$, $k = 3$, and

$$(w, \alpha) = (51003204600\cdots, (2,1,0,0,0,0)).$$

Scanning the abacus from left to right, the first bead to be moved is numbered 1. It can be moved 2 places to the right without any collisions. After that the bead numbered 2 can be moved 1 place to the right, again without collisions. So (w,α) is a fixed point for I. The new abacus $50013024600\cdots$ has shape $(3,3,3,2,2,0)$ obtained by adding a horizontal 3-strip to $(3,3,2,2,0,0)$.

On the other hand, if $\alpha = (1,1,1,0,0,0)$, then the first collision is of the bead numbered 3 with the bead numbered 2. Interchanging the beads numbered 2 and 3, and modifying the weights as prescribed gives $I(w,\alpha) = (51002304600\cdots, (1,2,0,0,0,0))$.

Let $N(n,k)$ denote the set of vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\alpha_i \in \{0,1\}$ for each i, and $\alpha_1 + \cdots + \alpha_n = k$. In terms of Abaci, the second Pieri rule becomes:

$$\sum_{w \in \mathrm{Abc}(\lambda)} \epsilon(w) \sum_{\alpha \in N(n,k)} \mathrm{wt}(\alpha) = \sum_\mu \sum_{w \in \mathrm{Abc}(\mu)} \epsilon(w)\mathrm{wt}(w),$$

where μ runs over all partition such that μ/λ is a vertical strip of size k.

We construct an involution I on $\mathrm{Abc}(\lambda) \times N(n,k)$ as follows: scan the abacus from *right to left*. Upon encountering a bead numbered j, if $\alpha_j = 1$, try to move the bead one step to the right. If this process completes without collisions, then (w,α) is a fixed point of I. Otherwise, if the first collision occurs with bead numbered j colliding with bead numbered k, then define w' to be w with beads j and k interchanged. Also, since the kth bead was adjacent to the jth bead, it could not have been moved in its turn. So $\alpha_k = 0$. Let w' be the abacus obtained by interchanging beads numbered k and j in w, let α' be obtained by interchanging α_k and α_j in α, and set $I(w,\alpha) = (w',\alpha')$. ☐

Example 6.3. The pair $(51003204600\cdots, (1,1,1,0,0,0))$ is a fixed point for I, and the shifted abacus is $(50100324600\cdots)$ of shape $(3,3,3,3,1,0)$. On the other hand

$$I(51003204600\cdots, (0,0,1,0,1,1)) = (51002304600\cdots, (0,1,0,0,1,1)).$$

The following is a special case of Pieri's rule:

Corollary 6.4. *For every positive integer k,*

$$s_{(k)} = h_k, \text{ and } s_{(1^k)} = e_k.$$

Exercise 6.5. Use Pieri's rule to show that:

$$h_k e_l = s_{(k,1^l)} + s_{(k+1,1^{l-1})}.$$

Conclude that $s_{(j+1,1^k)} = \sum_{l=0}^{k} (-1)^l h_{j+l+1} e_{k-l}$.

7 Schur to Complete and Elementary via Tableaux

Pieri's rule allows us to compute the complete and elementary symmetric polynomials h_λ and e_λ in terms of Schur polynomials.

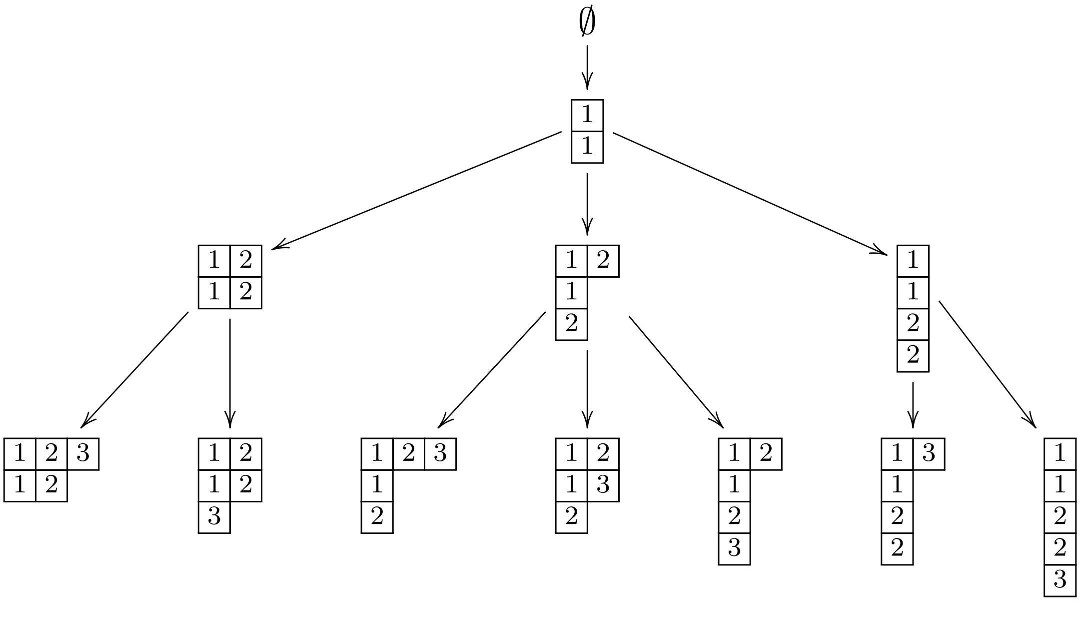
Example 7.1. Repeated application of Pieri's rule gives an expansion of $e_{(2,2,1)} = e_2e_2e_1$ as:

$$\begin{tikzcd}
 & & & e_2e_2e_1 \arrow[d] & & & \\
 & & & s_{(1^2)}e_2e_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & s_{(2^2)}e_1 \arrow[dl] \arrow[d] & & s_{(2,1^2)}e_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(1^4)}e_1 \arrow[d] \arrow[dr] & \\
s_{(3,2)} & s_{(2^2,1)} & s_{(3,1^2)} & s_{(2^2,1)} & s_{(2,1^3)} & s_{(2,1^3)} & s_{(1^5)}
\end{tikzcd}$$

giving:

$$e_{(2^2,1)} = s_{(3,2)} + 2s_{(2^2,1)} + s_{(3,1^2)} + 2s_{(2,1^3)} + s_{(1^5)}.$$

The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:



The boxes in the vertical strip added at the ith stage are filled with i.

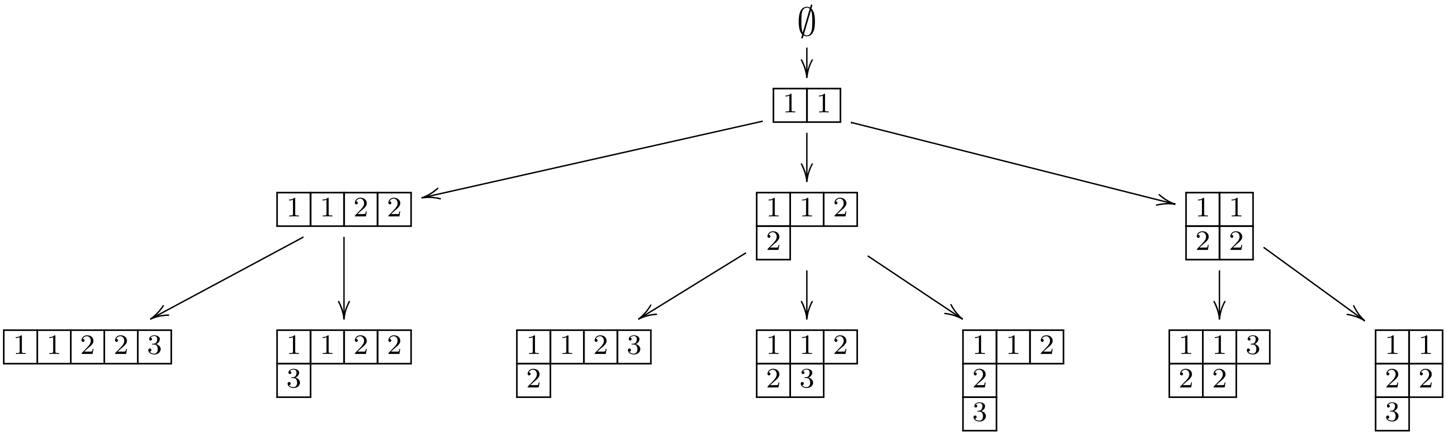
Example 7.2. Repeated application of Pieri's rule gives an expansion of $h_{(2,2,1)} = h_2h_2h_1$ as:

$$\begin{tikzcd}
 & & & h_2h_2h_1 \arrow[d] & & & \\
 & & & s_{(2)}h_2h_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & s_{(4)}h_1 \arrow[dl] \arrow[d] & & s_{(3,1)}h_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(3,2)}h_1 \arrow[d] \arrow[dr] & \\
s_{(5)} & s_{(4,1)} & s_{(4,1)} & s_{(3,2)} & s_{(3,1^2)} & s_{(3,2)} & s_{(2,2,1)}
\end{tikzcd}$$

giving:

$$h_{(2^2,1)} = s_{(5)} + 2s_{(4,1)} + 2s_{(3,2)} + s_{(3,1^2)} + s_{(2,2,1)}.$$

The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:



The boxes in the horizontal strip added at the ith stage are filled with i.

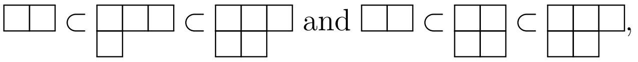
Definition 7.3 (Semistandard tableau)**.** *A semistandard tableau of shape $\lambda = (\lambda_1, \dots, \lambda_l)$ and type $\mu = (\mu_1, \dots, \mu_m)$ is the Young diagram of λ filled with numbers $1, \dots, m$ such that the number i appears μ_i times, the numbers weakly increase along rows, and strictly increase along columns.*

Exercise 7.4. Semistandard tableaux of shape λ and type μ correspond to chains of integer partitions

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip of size μ_i.

Example 7.5. The semistandard tableau of type $(3,2)$ and type $(2,2,1)$ are $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$. They correspond to the chains:



respectively. As illustrated in Example 7.2, the coefficient of $s_{(3,2)}$ in the complete symmetric polynomial $h_{(2,2,1)}$ is the number of semistandard tableau of shape $(3,2)$ and type $(2,2,1)$.

Definition 7.6 (Kostka number)**.** *Given two partitions λ and μ, the Kostka number $K_{\lambda\mu}$ is the number of semistandard tableaux of shape λ and type μ.*

Exercise 7.7. For every partition λ, show that $K_{\lambda\lambda} = 1$.

Definition 7.8 (f-number)**.** *The f-number of a partition λ of n is defined to be the Kostka number $K_{\lambda,(1^n)}$, and is denoted f_λ.*

Exercise 7.9. For a partition λ, let λ^- denote the set of all partitions whose Young diagram can be obtained by removing one box from the Young diagram of λ. For each $\lambda \neq \emptyset$, show that $f_\lambda = \sum_{\mu \in \lambda^-} f_\mu$.

Exercise 7.10. A hook is a partition of the form $h(a,b) = (a+1, 1^b)$. Show that $f_{h(a,b)} = \binom{a+b}{a}$.

In order to understand the expansion of elementary symmetric polynomials, we would need a variant of semistandard tableaux, one where the differences between successive shapes are vertical strips, rather than horizontal strips. However, it has become common practice to *conjugate* partitions instead:

Definition 7.11 (Conjugate of a partition)**.** *The* conjugate *of a partition λ is the partition λ' whose Young diagram is given by:*

$$\lambda' = \{(j,i) \mid (i,j) \in \lambda\}.$$

In other words, the Young diagram of λ' is the reflection of the Young diagram of λ about its principal diagonal.

Clearly $\lambda \mapsto \lambda'$ is an involution. For example, if $\lambda = (2,2,1)$, then $\lambda' = (3,2)$.

Exercise 7.12. Semistandard tableaux of shape λ' and type μ correspond to chains of integer partitions

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a *vertical* strip of size μ_i.

The method for computing elementary and complete polynomials from Schur polynomials illustrated in Examples 7.1 and 7.2 can be expressed as follows:

Theorem 7.13. *The expansion of complete symmetric polynomials in terms of Schur polynomials is given by:*

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda.$$

Dually, the extension of elementary symmetric polynomials in terms of Schur polynomials is given by:

$$e_\mu = \sum_\lambda K_{\lambda'\mu} s_\lambda.$$

8 Triangularity of Kostka Numbers

If partitions $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_m)$ have $K_{\lambda\mu} > 0$, then there exists a semistandard tableau t of shape λ and type μ. Since the columns of t are strictly increasing, all the 1's in t must occur in its first row, so $\lambda_1 \geq \mu_1$. Also, all the 2's must occur in the first two rows (along with all the 1's), so $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. More generally, all the numbers $1, \dots, i$ for $i = 1, \dots, m$ should occur in the first i rows of t. We have:

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \text{ for } i = 1, \dots, m. \tag{8.1}$$

Definition 8.1. *The integer partition λ* dominates *the integer partition μ if $|\lambda| = |\mu|$ and (8.1) holds for $i = 1, \dots, m$. When this happens, write $\lambda \rhd \mu$. This relation defines a partial order on the set of all integer partitions of n for any non-negative integer n.*

Exercise 8.2. Show that (n) is maximal and (1^n) is minimal among all the integer partitions of n. What is the smallest integer n for which the dominance order on partitions of n is not a linear order?

Theorem 8.3 (Triangularity of Kostka Numbers)**.** *Given partitions λ and μ of an integer n, $K_{\lambda\mu} > 0$ if and only if $\lambda \rhd \mu$.*

Proof. It only remains to construct, whenever $\lambda \rhd \mu$, a semistandard tableau of shape λ and type μ. In order to understand the algorithm that follows, it is helpful to keep in mind Example 8.4 below. Since $\lambda \rhd \mu$, $\lambda_1 \geq \mu_1 \geq \mu_m$. Therefore, the Young diagram of λ has at least μ_m cells in its first row, or in other words, it has at least μ_m columns. Choose the largest integer i for which $\lambda_i \geq \mu_m$. Fill the bottom-most box in the λ_{i+1} leftmost columns with m. Also, from the ith row, fill the rightmost $\mu_m - \lambda_{i+1}$ boxes with m. The remaining (unfilled) boxes in the Young diagram of λ now form the Young diagram of the partition

$$\eta = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - \mu_m + \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_l),$$

a partition with $l-1$ parts. Writing $(\eta_1, \dots, \eta_{l-1})$ for the parts of η, note that, since the first $i-1$ parts of η are the same as those of λ, we have:

$$\eta_1 + \cdots + \eta_j \geq \mu_1 + \cdots + \mu_j$$

for $j \leq i-1$. For $j \geq i$, we have

$$\begin{aligned}\eta_1 + \cdots + \eta_j &= \lambda_1 + \cdots + \lambda_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j + \mu_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j.\end{aligned}$$

It follows that $\eta \rhd (\mu_1, \ldots, \mu_{m-1})$. Recursively applying this step to η and $(\mu_1, \ldots, \mu_{m-1})$ gives rise to a semistandard tableau of shape λ and type μ. The base case is where μ has only one part, in which case the dominance condition (8.1) implies that $\lambda = \mu$. ☐

Example 8.4. Consider the case where $\lambda = (7,3,2)$ and $\mu = (4,4,4)$. Then the largest integer i such that $\lambda_i \geq 4$ is 1. Accordingly, we enter 3 into the bottom-most boxes in the three leftmost columns, and also into one rightmost box in the first row:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 3 \\ \hline \ & \ & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2}\end{array}$$

We are left with the problem of finding a semistandard tableau of shape $(6,2)$ and type $(4,4)$. Recursively applying our process to this smaller problem gives:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 2 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2}\end{array},$$

and finally the desired tableau

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2}\end{array}.$$

Theorem 8.5. *The complete symmetric polynomials:*

$$\{h_\mu \mid \mu \text{ is a partition of } d \text{ with at most } n \text{ parts}\}$$

and the elementary symmetric polynomials:

$$\{e_\mu \mid \mu \text{ is a partition of } d \text{ with } \mu_1 \leq n\}$$

form bases of the space of homogeneous symmetric polynomials of degree d in variables $x_1, \ldots, x_n$.

Proof. In view of the triangularity of Kostka numbers (Theorem 8.3) and the fact that $K_{\lambda\lambda} = 1$ (Exercise 7.7) the theorem follows from Theorem 7.13. ☐

9 Schensted's insertion algorithm

Let t be a semistandard tableau, and x be a positive integer. Schensted's insertion algorithm is a method of inserting a box with the number x into t, resulting in a new tableau $\mathrm{INSERT}(t, x)$. Applied repeatedly, it gives a way to convert any word into a tableau. This tableau succinctly expresses some combinatorial properties of the original word.

First consider the case where t has a single row, with entries $a_1 \leq \cdots \leq a_k$. Use $\emptyset$ to denote the empty word. The algorithm ι takes as input the single row t and a letter x, and returns a pair (b, t'), where b' is either the empty word, or a single letter, and t' is a row:

$$\iota(a_1 a_2 \cdots a_k, x) = \begin{cases} (\emptyset, a_1 \cdots a_k x) & \text{if } x \geq a_i \text{ for all } i, \\ (a_j, a_1 \cdots a_{j-1} x a_{j+1} \cdots a_k) & \text{if } j = \min\{r \mid a_r > x\}. \end{cases}$$

In the second case, one says that *x has been inserted into $t = a_1 \cdots a_k$, obtaining $t' = a_1 \cdots a_{j-1} x a_{j+1} \cdots a_k$, and **bumping out** a_j*. Also, it is notationally convenient to write $\iota(t, \emptyset) = (\emptyset, t)$ (when nothing is inserted, t remains unchanged, and nothing is bumped out).

Now suppose t is a tableau, with first row r. Suppose that $\iota(r, x) = (y, r')$. Recursively define $\mathrm{INSERT}(t, x)$ to be the tableau whose first row is r', and remaining rows are the rows of $\mathrm{INSERT}(t', y)$, where t' is the tableau consisting of all but the first row of t.

Example 9.1. Consider the insertion of 3 into the tableau:

$$t = \begin{array}{|c|c|c|c|c|}\hline 1&3&3&5&8\\ \hline 2&4&6&6\\ \cline{1-4} 3&5&8\\ \cline{1-3} 4\\ \cline{1-1}\end{array}.$$

We have $\iota(13358,3)=(5,13338)$; $\iota(2466,5)=(6,2456)$; $\iota(358,6)=(8,356)$; $\iota(4,8)=(\emptyset,48)$. Thus, $\mathrm{INSERT}(t,3)$ is the tableau:

$$\begin{array}{|c|c|c|c|c|}\hline 1&3&3&3&8\\ \hline 2&4&5&6\\ \cline{1-4} 3&5&6\\ \cline{1-3} 4&8\\ \cline{1-2}\end{array}.$$

In general, it is not possible to recover t and x from $\mathrm{INSERT}(t,x)$, even if we know x. For example, the above tableau can be obtained by inserting 3 into a different tableau:

$$\mathrm{INSERT}\left(\begin{array}{|c|c|c|c|c|}\hline 1&3&3&6&8\\ \hline 2&4&5\\ \cline{1-3} 3&5&6\\ \cline{1-3} 4&8\\ \cline{1-2}\end{array}, 3\right) = \begin{array}{|c|c|c|c|c|}\hline 1&3&3&3&8\\ \hline 2&4&5&6\\ \cline{1-4} 3&5&6\\ \cline{1-3} 4&8\\ \cline{1-2}\end{array}.$$

Clearly, the shape of $\mathrm{INSERT}(t,x)$ is obtained by adding one box to the shape of t. If we know the row r into which the new box was added, and the value of x, then t can be recovered from $\mathrm{INSERT}(t,x)$. This recovery is based on the fact that ι can be inverted: define

$$\partial(a, a_1a_2\cdots a_k) = (a_1\cdots a_{j-1}aa_{j+1}\cdots a_k, a_j),$$

where $j=k$ if $a_i \leq a$ for all $i=1,\dots,k$ and $j=\min\{i \mid a_{i+1} > a\}$. To recover t and x from $s=\mathrm{INSERT}(t,x)$ and r (the number of the row into which the new box was added), delete the last entry of the rth row of s, say x_r. Let u_{r-1} denote the $(r-1)$st row of s. Suppose $\partial(x_r,u_{r-1})=(v_{r-1},x_{r-1})$, replace the $(r-1)$st row of s with v_{r-1}. Continue this process until $\partial(x_2,u_1)=(v_1,x_1)$ is obtained and the first row of s is replaced with v_1. The tableau obtained at the end of this process is t, and $x=x_1$. Write $\mathrm{DELETE}(t,r)=(s,x)$. The preceding discussion shows:

Theorem 9.2. *If* $\mathrm{DELETE}(t,r)=(s,x)$*, then* $\mathrm{INSERT}(s,x)=t$*, and* $\mathrm{shape}(t)$ *is obtained from* $\mathrm{shape}(s)$ *by adding a cell into its* r*th row.*

Exercise 9.3. Verify Theorem 9.2 for the insertions in Example 10.1.

10 Tableaux and Words

Let L_n^* denote the concatenation monoid of all words in the alphabet $L_n=\{1,\dots,n\}$. For any $w=a_1\cdots a_k\in L_n^*$, Schensted's insertion algorithm allows us to associate a unique semistandard tableau $P(w)$ as follows:

- If $w=a$ has only one letter, then $P(a)$ is the single-cell tableau with entry a.
- If $w=ua$, where $u\in L_n^*$ and $a\in\{1,\dots,n\}$, then $P(w)=\mathrm{INSERT}(P(u),a)$.

Example 10.1. If $w=1374433254$, then $P(w)$ is the tableau:

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&3&3&4\\ \hline 3&4&5\\ \cline{1-3} 4\\ \cline{1-1} 7\\ \cline{1-1}\end{array}$$

Given a semistandard tableau t, its reading word w is defined as the sequence of numbers obtained from reading its rows from left to right, starting with the bottom row, and moving up sequentially to the top row. Since the first entry of each row is strictly smaller than the last entry of the row below it, the tableau t can be recovered from w by chopping it up into segments with a cut after each a_i with $a_{i+1}<a_i$ (we say that w has a descent at i). The resulting segments, taken from right to left, form the rows of t.

Example 10.2. The reading word of the tableau t formed at the end of Example 10.1 is: $w=7434512334$. The tableau t is recovered by marking off the descents $w=7|4|345|12334$, and then rearranging the segments into a tableau.

Exercise 10.3. Let w denote the reading word of a tableau t. Show that $P(w)=t$.

Not every word comes from a semistandard tableau; for example, the word 132, when broken up at descents gives rise to $\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}$. We shall say that a word is a tableau if it is the reading word of a semistandard tableau.

Call the word $w = a_1 \cdots a_k$ a *row* if $a_1 \leq \cdots \leq a_k$. Call it a *column* if $a_1 > \cdots > a_k$. Write x^w for the monomial $x_{a_1} x_{a_2} \cdots x_{a_k}$.

Exercise 10.4. Show that, for every positive integer i,

$$h_k(x_1, \ldots, x_n) = \sum_{w \,\in\, L_n^* \text{ is a row of length } k} x^w,$$

and

$$e_k(x_1, \ldots, x_n) = \sum_{w \,\in\, L_n^* \text{ is a column of length } k} x^w.$$

If w_1 and w_2 are words, and $w_1 w_2$ is their concatenation, then $x^{w_1} x^{w_2} = x^{w_1 w_2}$.

This gives rise to an algebra homomorphism called the *evaluation map*:

$$\mathrm{ev} : \mathbf{Z}[L_n^*] \to \mathbf{Z}[x_1, \ldots, x_n]$$

from the monoid algebra of L_n^* onto the ring of polynomials in n variables. In the algebra $\mathbf{Z}[L_n^*]$, define elements

$$\mathbf{H}_k = \sum_{w \,\in\, L_n^* \text{ is a row of length } k} w,$$

$$\mathbf{E}_k = \sum_{w \,\in\, L_n * \text{ is a column of length } k} w$$

for every positive integer k. Then Exercise 10.4 can be restated as the identities:

$$e_k = \mathrm{ev}(\mathbf{E}_k) \text{ and } h_k = \mathrm{ev}(\mathbf{H}_k).$$

The evaluation map has a large kernel; its domain is the free algebra, and it maps onto the polynomial algebra. Its image contains our primary object of interest—the algebra of symmetric polynomials in n variables. In the next few sections, we shall learn about an equivalence relation "$\equiv$" on L_n^*, called *Knuth equivalence*, such that the resulting quotient monoid $\mathrm{Pl}(L_n) := L_n^*/\equiv$ (called the *plactic monoid*) has the property that the subalgebra of $\mathbf{Z}[\mathrm{Pl}(L_n)]$ generated by the elements $\{\mathbf{E}_k\}_{k=1}^\infty$ or the elements $\{\mathbf{H}_k\}_{k=1}^\infty$ is isomorphic to the subalgebra of symmetric polynomials in $\mathbf{Z}[x_1, \ldots, x_n]$ under the evaluation map.

11 The Plactic Monoid

The plactic monoid $\mathrm{Pl}(L_n)$ is the quotient of L_n^* by the equivalence relation generated by the Knuth relations:

$$xzy \equiv zxy \text{ if } x \leq y < z, \tag{$K1$}$$

$$yxz \equiv yzx \text{ if } x < y \leq z. \tag{$K2$}$$

Two words are said to be in the same plactic class if each can be obtained from the other by a sequence of moves of the form $(K1)$ and $(K2)$. Since both sides of the Knuth relations have the same evaluation, it follows that the evaluation map $\mathrm{ev} : \mathbf{Z}[L_n^*] \to \mathbf{Z}[x_1, \ldots, x_n]$ factors through the plactic monoid algebra $\mathbf{Z}[\mathrm{Pl}(L_n)]$. Let E_k denote the image of $\mathbf{E}_k$ and H_k denote the image of $\mathbf{H}_k$ in $\mathbf{Z}[\mathrm{Pl}(L_n)]$.

Exercise 11.1. Take $n = 2$. Show that E_1 and E_2 commute in $\mathbf{Z}[\mathrm{Pl}(L_2)]$. Show that they commute in $\mathbf{Z}[\mathrm{Pl}(L_3)]$.

Exercise 11.2. Define Schützenberger's forgotten relations by:

$$xzy \cong yxz \text{ if } x < y < z, \tag{$F1$}$$

$$zxy \cong yzx \text{ if } x \leq y \leq z. \tag{$F2$}$$

Let $F(L_n)$ denote the monoid $L_n^*/\cong$. Show that the images of $\mathbf{E}_1$ and $\mathbf{E}_2$ commute in $F(L_3)$.

Exercise 11.3. Show that any evaluation-preserving equivalence on L_3^* under which the images of $\mathbf{E}_1$ and $\mathbf{E}_2$ commute must include either the Knuth equivalences or Schützenberger's forgotten equivalences.

Exercise 11.4.

1. Show that, if $\iota(r, x) = (y, r')$ (as in Section 9), then $rx \equiv yr'$.
2. Show that, if $tx \equiv \mathrm{INSERT}(t, x)$. Here tableaux are to be identified with their reading words.
3. Show that, for every $w \in L_n^*$, $w \equiv P(w)$.

12 The plactic Pieri rules

Observe that, if for any tableau t and $x \in L_n$, if $t' = \mathrm{INSERT}(t, x)$, then shape$(t')$ is obtained by adding one box to shape(t).

Lemma 12.1. *Let t be the (the reading word of) a semistandard tableau in L_n^* and x, y be letters in L_n. Let $t' = \mathrm{INSERT}(t, x)$ and $t'' = \mathrm{INSERT}(t', y)$. Let a be the box added to* shape(t) *in order to obtain* shape(t')*, and b be the box added to* shape(t') *to obtain* shape(t'')*. If $x \leq y$, then b lies in a column strictly to the right of the column of a. If $x > y$, then b lies in a row strictly below the row of a.*

Proof. Let $a_1 \cdots a_k$ be the first row of t.

Suppose $x \leq y$. Consider first the case where $a_k \leq y$. If $a_k \leq x \leq y$, then the result is obvious. If $x < a_k$ and $y \geq a_k$, then b lies in the $(k+1)$st column, whereas x bumps some letter $x' \leq y$ to a lower row. This letter cannot come to rest in the $(k+1)$st column because that would violate the fact that columns increase strictly in a semistandard tableau.

Now consider the case where $a_k > y$. Then x bumps out x' and y bumps out y' with $x' \leq y'$. The problem is now reduced to the tableau obtained by removing the top row of t, allowing for the application of induction. In the base case (where the original tableau t is a row), a and b are the first and second boxes in the second row of t''.

Now suppose $x > y$. If $x \geq a_k$, then x first comes to rest at the end of the first row in t', but then y bumps some element of the first row of t' up to a lower row in t''. So a lies in the first row and b in a lower row. If $x \leq a_k$, then the elements x' and y' bumped out from the first row by x and y respectively again satisfy $x' > y'$, allowing for an inductive argument. ☐

Theorem 12.2 (Plactic Pieri rules)**.** *Let $\mathrm{Tab}_n(\lambda)$ denote the set of all semistandard tableaux of shape λ and entries in L_n. Let $R_k(L_n)$ denote the set of rows of length k in L_n^*. Then the map $(t, r) \mapsto P(tr)$ defines a bijection:*

$$\mathrm{Tab}_n(\lambda) \times R_k(L_n) \to \coprod_{\mu/\lambda \text{ is a horizontal strip of size } k} \mathrm{Tab}_n(\mu).$$

Let $C_k(L_n)$ denote the set of columns of length k in L_n^*. Then the map $(t, c) \mapsto P(tc)$ defines a bijection:

$$\mathrm{Tab}_n(\lambda) \times C_k(L_n) \to \coprod_{\mu/\lambda \text{ is a vertical strip of size } k} \mathrm{Tab}_n(\mu).$$

Proof. Lemma 12.1 implies that shape$(P(wr))$ is obtained from shape$(P(w))$ by adding a horizontal strip, and that shape$(P(wc))$ is obtained from shape$(P(w))$ by adding a vertical strip. The bijectivity can be shown by repeated use of the DELETE algorithm (because we know which box from μ has to be removed at each step). ☐

Exercise 12.3. Define an element of $\mathbf{Z}[L_n^*]$ by $\mathbf{S}_\lambda = \sum_{t \in \mathrm{Tab}_n(\lambda)} t$. Let S_λ denote the image of $\mathbf{S}_\lambda$ in $\mathbf{Z}[\mathrm{Pl}(L_n)]$. Use Theorem 12.2 to show that:

$$S_\lambda H_k = \sum_{\mu/\lambda \text{ is a horizontal strip of size } k} S_\mu,$$

$$S_\lambda E_k = \sum_{\mu/\lambda \text{ is a vertical strip of size } k} S_\mu.$$

Exercise 12.4. Show that E_k and E_l (also H_k and H_l) commute in $\mathbf{Z}[\mathrm{Pl}(L_n)]$ for all positive integers n, k and l.

Corollary 12.5 (Kostka's definition of Schur polynomials)**.** *For every partition λ and every positive integer n, we have:*

$$s_\lambda(x_1, \ldots, x_n) = \sum_{t \in \mathrm{Tab}_n(\lambda)} x^t.$$

Proof. Note that $\mathrm{ev}(\mathbf{S}_\lambda) = \sum_{t \in \mathrm{Tab}_n(\lambda)} x^t$. Then $\mathrm{ev}(\mathbf{S}_{(n)}) = h_n$, and $\mathrm{ev}(\mathbf{S}_{(1^n)}) = e_n$, just like the Schur polynomials. Moreover, since the evaluation map $\mathbf{Z}[L_n^*]$ factors through $\mathbf{Z}[\mathrm{Pl}(L_n^*)]$, Exercise 12.3 implies that the polynomials $\mathrm{ev}(\mathbf{S}_\lambda)$ satisfy the Pieri rule, just like the Schur polynomials. This suffices for the identities of Theorem 7.13 to hold with s_λ replaced by $\mathrm{ev}(\mathbf{S}_\lambda)$. By the triangularity properties of Kostka numbers (Section 8), these identities uniquely determine the Schur polynomials, therefore $S_\lambda = s_\lambda$ for every partition λ. ☐

13 The Lindström-Gessel-Viennot Lemma

Let R be a commutative ring. Let S be any set of points, and $v : S \times S \to R$ be any function (we think of v as a *weight function*). Given $s, t \in S$, a path in S from s to t is is a sequence $\omega = (s = s_0, s_1, \dots, s_k = t)$ of distinct points in S. We denote this by $\omega : s \to t$. The weight of the path ω is defined to be:

$$v(\omega) = v(s_0, s_1)v(s_1, s_2) \cdots v(s_{k-1}, s_k).$$

Definition 13.1 (Crossing paths). *Two paths $\omega = (s_0, \dots, s_k)$ and $\eta = (t_0, \dots, t_l)$ are said to cross if $s_i = t_j$ for some $0 \leq i \leq k$ and $0 \leq j \leq l$.*

Definition 13.2 (Crossing condition). *Given a set S of points, a weight function $v : S \times S \to R$, and a points $A_1, \dots, A_n$, and $B_1, \dots, B_n$, we say that the* crossing condition *is satisfied if, whenever $1 \leq i < j \leq n$ and $1 \leq i' < j' \leq n$, and $\omega : i \to j'$ and $\eta : j \to i'$ are paths such that $v(\omega) \neq 0$ and $v(\eta) \neq 0$, then the paths ω and η cross.*

Fix points $A_1, \dots, A_n$ and $B_1, \dots, B_n$ in S, and define an $n \times n$ matrix (a_{ij}) by:

$$a_{ij} = \sum_{\omega : A_i \to B_j} v(\omega).$$

Theorem 13.3 (Lindström-Gessel-Viennot Lemma). *Assume that the crossing condition (Definition 13.2) holds. Then the determinant of the matrix (a_{ij}) defined above is given by:*

$$\det(a_{ij}) = \sum_{\omega_i : A_i \to B_i} v(\omega_1) \cdots v(\omega_n), \tag{13.1}$$

where the sum is over all n-tuples $(\omega_1, \dots, \omega_n)$ of pairwise non-crossing paths $\omega_i : A_i \to B_i$.

Proof. Let P be the set of all n-tuples of paths of the form:

$$\bar{\omega} = (\omega_i : A_i \to B_{w(i)}, i = 1, \dots, n), \tag{13.2}$$

where w is a permutation of $\{1, \dots, n\}$. Define the weight of $\bar{\omega} \in P$ by $v(\bar{\omega}) = \prod_{i=1}^{n} v(\omega_i)$ and its sign by $\epsilon(\bar{\omega}) = \epsilon(w)$. Then the determinant on the left hand side of (13.1) expands to the sum:

$$\sum_{\bar{\omega} \in P} \epsilon(\bar{\omega}) v(\bar{\omega}). \tag{13.3}$$

The cancelling involution $I : P \to P$ is defined by *swapping the first crossing of the first path that crosses another path*: given $\bar{\omega}$ as in (13.2), if the paths are pairwise non-crossing, then $\bar{\omega}$ is a fixed point for I. In this case, the crossing condition implies that w is the identity permutation. Otherwise, take the least i such that the path $\omega_i = (s_0, \dots, s_k)$ crosses another path, and then the least j such that $\omega_j = (t_0, \dots, t_l)$ crosses ω_i. Let m be the smallest number such that a point s_m of ω_i lies in the path ω_j, say $s_m = t_r$. Let $I(\bar{\omega})$ be the family of paths obtained from $\bar{\omega}$ by modifying ω_i and ω_j to ω_i' and ω_j' as follows:

$$\omega_i' = (s_0, \dots, s_m, t_{r+1}, \dots, t_l),$$
$$\omega_j' = (t_0, \dots, t_r, s_{m+1}, \dots, s_k).$$

Clearly, $v(I(\bar{\omega})) = v(\bar{\omega})$ and $\epsilon(I(\bar{\omega})) = -\epsilon(\bar{\omega})$. It is not hard to see that I is an involution. This involution cancels out all the terms in (13.3) except those that occur on the right hand side of (13.1). □

14 The Jacobi-Trudi Identities

We have seen that the Kostka numbers can be used to express complete and elementary symmetric polynomials in terms of Schur polynomials. The reverse operation—that of expressing Schur polynomials in terms of complete or elementary symmetric polynomials—is done by the Jacobi-Trudi identities:

Theorem 14.1 (Jacobi-Trudi identities). *For every integer partition $\lambda = (\lambda_1, \dots, \lambda_l)$ form the $l \times l$ matrices with (i, j)th entry $h_{\lambda_i - i + j}$ and $e_{\lambda_i' - i + j}$ respectively. Then*

$$s_\lambda = \det(h_{\lambda_i - i + j}) = \det(e_{\lambda_j' + i - j}).$$

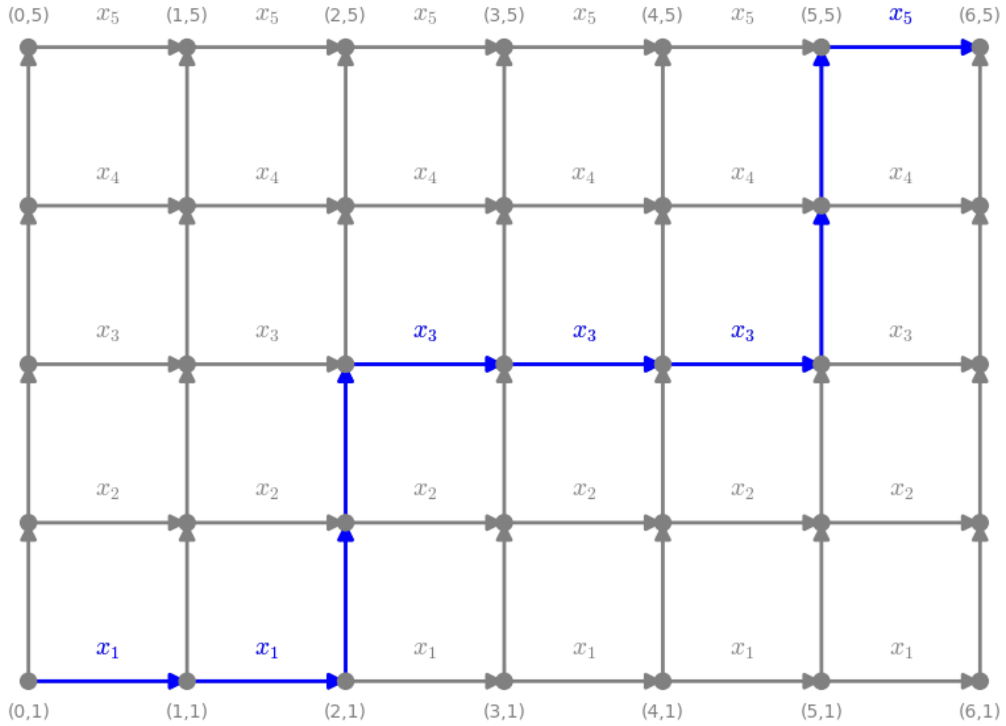


Fig. 1: A path from $(0,1)$ to $(6,5)$ whose weight is the monomial $x_1^2x_3^3x_5$ in $h_6(x_1,\ldots,x_5)$.

Proof. The Jacobi-Trudi identities can be proved using the Lindström-Gessel-Viennot lemma (Theorem 13.3). For the first identity take S to be the positive cone in the integer lattice:

$$S = \{(i,j) \mid i \geq 0,\ j > 0 \text{ are integers}\}.$$

Set the weight $v((i,j),(i+1,j))$ of each rightward horizontal edge to be x_j for $j=1,\ldots,n$, the weight of each upward vertical edge $v((i,j),(i,j+1))$ to be 1 for all $j=1,\ldots,n-1$. The remaining weights are all zero.

Lemma 14.2. *For all integers $i>0$ and $k \geq 0$, we have:*

$$\sum_{\omega:(i,1)\to(i+k,n)} v(\omega) = h_k(x_1,\ldots,x_n).$$

Proof. Only rightward or upward steps have non-zero weights. So every path with non-zero weight is composed of unit upward and rightward steps. A path with non-zero weight from $(i,1)$ to $(i+k,n)$ must have exactly k rightward steps, say in rows $1 \leq j_1 \leq j_2 \cdots \leq j_k \leq n$. The weight of such a path is $x_{j_1}\cdots x_{j_k}$, and hence, the sum of the weights of all such paths is $h_k(x_1,\ldots,x_n)$. For an example, see Fig. 1. ☐

Given $\lambda = (\lambda_1,\ldots,\lambda_l)$, and working with n variables $x_1,\ldots,x_n$, let $A_i = (l-i,1)$ and $B_i = (\lambda_i + l - i, n)$ for $i=1,\ldots,l$. Then by Lemma 14.2,

$$\sum_{\omega:A_i\to B_j} v(\omega) = h_{\lambda_j+i-j}.$$

So the left-hand-side of the first Jacobi-Trudi identity is the left-hand-side of the Lindström-Gessel-Viennot lemma. The right hand side of the Lindström-Gessel-Viennot lemma consists of a sequence of non-crossing paths $(\omega_1,\ldots,\omega_n)$, where $\omega_i : A_i \to B_i$. Reading the row numbers of the horizontal steps in ω_i gives a weakly increasing sequence of integers $1 \leq k_1 \leq \cdots \leq k_{\lambda_i} \leq n$. Enter these numbers into the ith row of the Young diagram of λ for $i=1,\ldots,n$. Since the paths are non-crossing, the jth rightward step of ω_i must be strictly higher than the jth rightward step of ω_{i+1}. This means that the columns of the resulting numbering are strictly increasing, resulting in a semistandard tableau of shape λ (for an example, see Figure 2). Thus, it follows from the Lindström-Gessel-Viennot lemma that

$$\det(h_{\lambda_j+i-j}) = \sum_{t\in\mathrm{Tab}(\lambda)} x^t.$$

For the second Jacobi-Trudi identity take $S = \{(i,j) \mid i \geq 0,\ j \geq 0\}$.

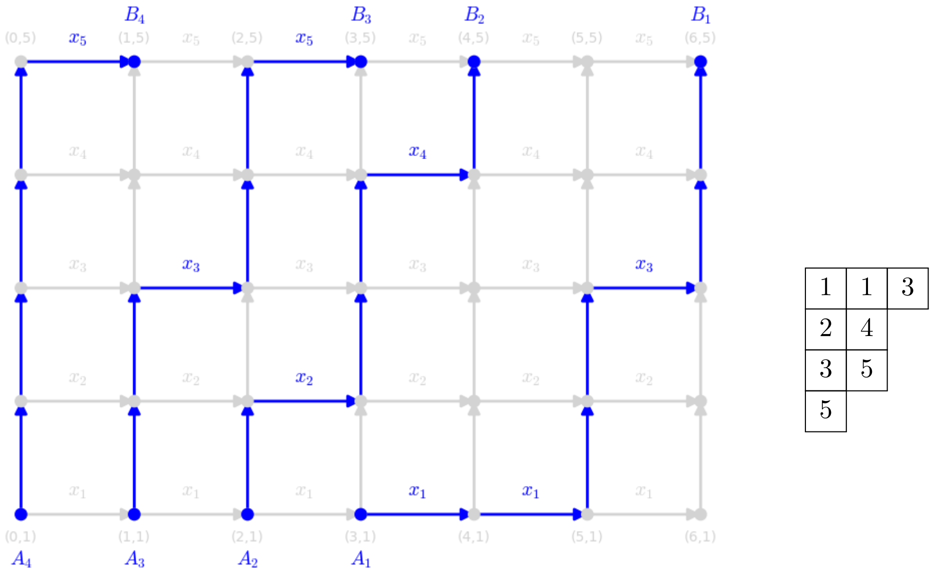


Fig. 2: Non-crossing paths and corresponding tableau.

Define the weight of each upward vertical edge $v((i,j),(i,j+1))$ to be 1 (as before) and the weight of a diagonal edge in the upper-right direction $v((i,j-1),(i+1,j))$ to be x_j; all other weights are zero. For the new weights, the analog of Lemma 14.2 is:

Lemma 14.3. *For all integers $i > 0$ and $k > 0$, we have:*

$$\sum_{\omega:(i,0)\to(i+k,n)} v(\omega) = e_k(x_1,\dots,x_n).$$

Proof. Every path with non-zero weights consists of unit upward or upper-rightward diagonal steps. A path with non-zero weight from $(i,0)$ to $(i+k,n)$ must have n such steps, of which k must be diagonal. If the steps ending in rows $1 \le j_1 < \ldots < j_k \le n$ are the diagonal steps, then the path has weight $x_{j_1}\cdots x_{j_k}$. For an example of such a path, see Fig. 3. Summing over all possible paths gives $e_k(x_1,\dots,x_n)$. □

Suppose that the conjugate partition of λ is $\lambda' = (\lambda'_1,\dots,\lambda'_k)$. In order to apply the Lindström-Gessel-Viennot lemma to obtain the second Jacobi-Trudi identity, take $A_i = (k-i,0)$ and $B_i = (\lambda'_i + k - i, n)$ for $i = 1,\dots,k$. Then by Lemma 14.3,

$$\sum_{\omega_i:A_i\to B_j} v(\omega) = e_{\lambda'_j+i-j}.$$

So the left-hand-side of the second Jacobi-Trudi identity is the left-hand-side of the Lindström-Gessel-Viennot lemma.

The right hand side of the Lindström-Gessel-Viennot lemma consists of a sequence of non-crossing paths $(\omega_1,\dots,\omega_n)$, where $\omega_i : A_i \to B_i$. Reading the row numbers where the upper-rightward steps in ω_i terminate gives a strictly increasing sequence of integers $1 \le j_1 < \cdots < j_{\lambda'_i} \le n$. Enter these numbers into the ith column of the Young diagram of λ. Since the paths are non-crossing, the jth upper-rightward step of ω_i must be no lower than the jth upper-rightward step of ω_{i+1}. This means that the rows of the resulting numbering are weakly increasing, resulting in a semistandard tableau of shape λ (for an example, see Figure 4). Thus, it follows from the Lindström-Gessel-Viennot lemma that

$$\det(e_{\lambda_j+i-j}) = \sum_{t\in\mathrm{Tab}(\lambda')} x^t,$$

proving the second Jacobi-Trudi identity. □

15 Skew-Schur Polynomials

In the proof of the first Jacobi-Trudi identity for s_λ, where $\lambda = (\lambda_1,\dots,\lambda_l)$, we used:

$$A_i = (l-i,1) \quad , \quad B_i = (\lambda_i + l - i, n)$$

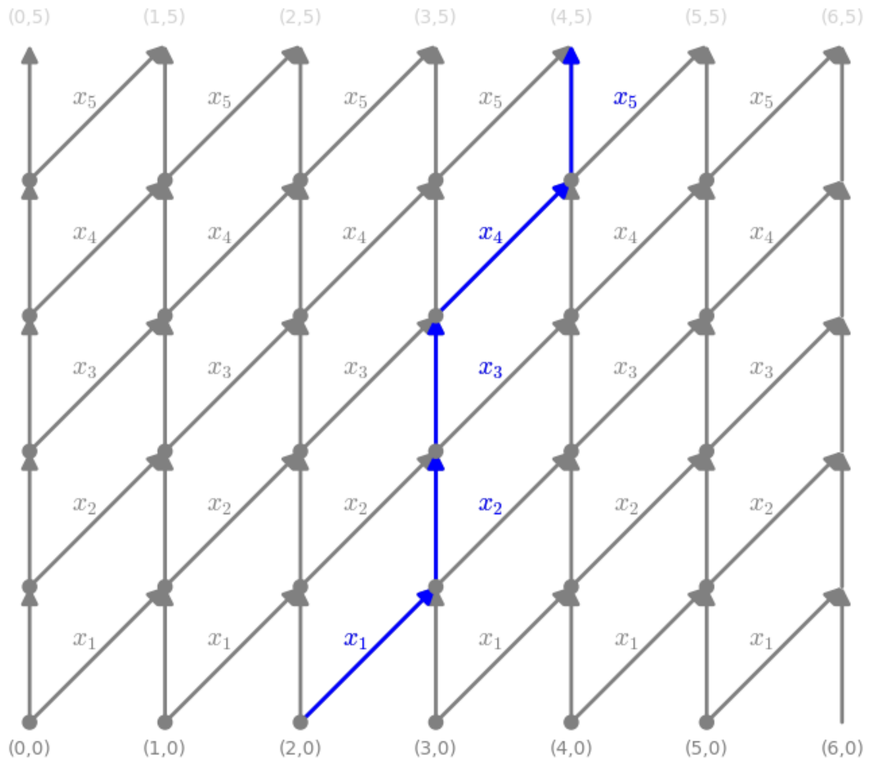


Fig. 3: A path from $(2,0)$ to $(4,5)$ whose weight is the monomial x_1x_4 in $e_2(x_1,\ldots,x_5)$.

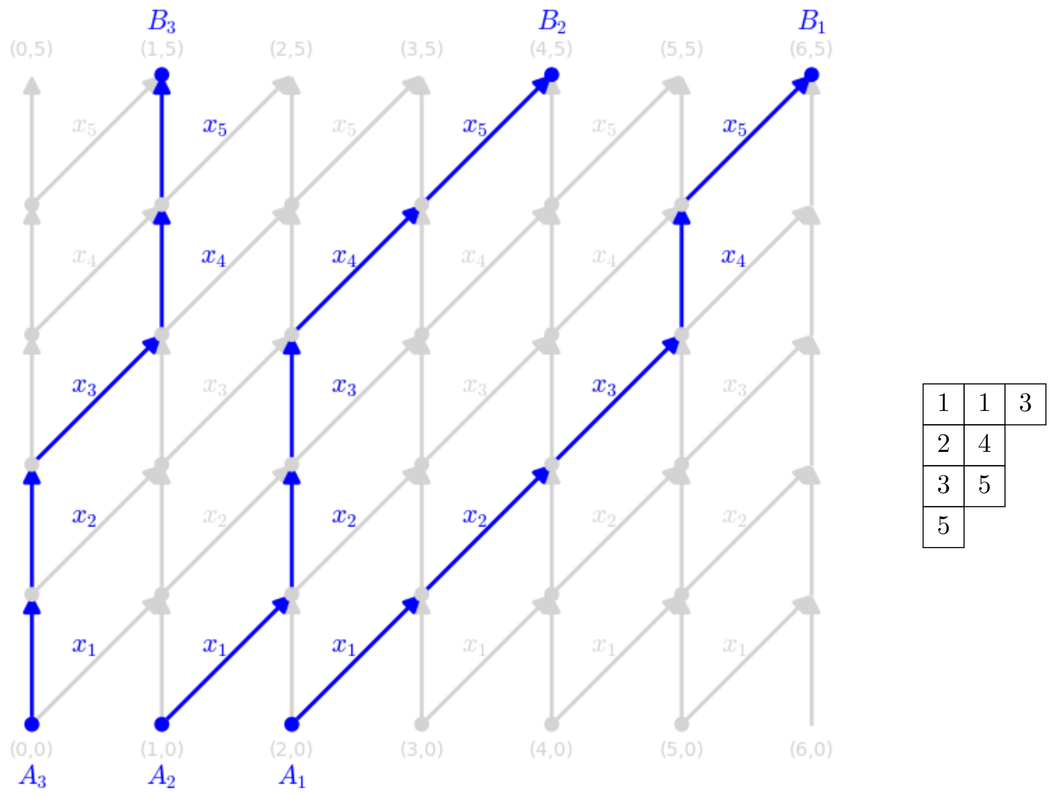


Fig. 4: Non-crossing paths and corresponding tableau.

for $i = 1, \dots, l$. Now suppose $\mu = (\mu_1, \dots, \mu_l)$ is a partition (possibly padded with zero's so that it has the same number of parts as λ) such that $\mu \subset \lambda$, then we can take:

$$\begin{aligned} A_i &= (\mu_i + l - i, 1), \\ B_i &= (\lambda_i + l - i, n). \end{aligned}$$

Consider a collection of non-crossing paths $\omega_i : A_i \to B_i$, $i = 1, \dots, l$. The path ω_i has $\lambda_i - \mu_i$ horizontal steps. If these steps occur in rows $1 \leq k_1 \leq \dots \leq k_{\lambda_i - \mu_i} \leq n$, then enter the integers $k_1, \dots, k_{\lambda_i - \mu_i}$ into the ith row of the skew-shape. As in the proof of the first Jacobi-Trudi identity, this results in a semistandard tableau of skew-shape λ/μ. For an example, see Figure 5.

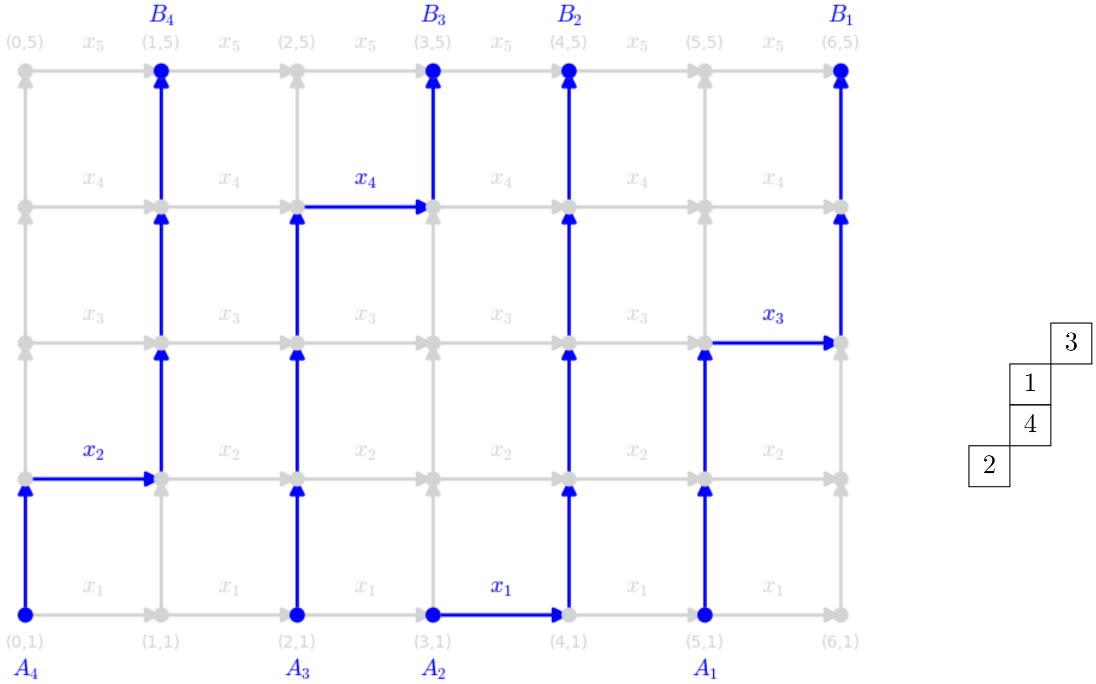


Fig. 5: Non-crossing paths and the corresponding skew-tableau.

Therefore the symmetric polynomial

$$s_{\lambda/\mu}(x_1, \dots, x_n) = \det(h_{\lambda_j - \mu_j + i - j}), \tag{15.1}$$

by the Lindström-Gessel-Viennot, is also given by

$$s_{\lambda/\mu}(x_1, \dots, x_n) = \sum_{t \in \mathrm{Tab}_n(\lambda/\mu)} x^t. \tag{15.2}$$

The polynomials $s_{\lambda/\mu}$ generalize the Schur polynomials and are called *skew-Schur polynomials.* A modification of the proof of the second Jacobi-Trudi identity gives:

$$s_{\lambda/\mu}(x_1, \dots, x_n) = \det(e_{\lambda'_j - \mu'_j + i - j}), \tag{15.3}$$

Exercise 15.1. Expand the skew-Schur polynomial $s_{(2,1)/(1)}(x_1, x_2, x_3)$ in the basis of Schur polynomials.

16 Giambelli's Identity

Definition 16.1 (Frobenius coordinates)**.** *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition. Its* Durfee rank *d is defined to be the largest integer i such that (i, i) lies in the Young diagram of λ. Let α_i denote the number of cells in the ith row, that lie strictly to the right of (i, i) in the Young diagram of λ. Similarly let β_i denote the number of cells in the ith column that lie strictly below (i, i). Clearly $\alpha_1 > \dots > \alpha_d$, $\beta_1 > \dots > \beta_d$, and the Young diagram of λ can be recovered from the data $(\alpha|\beta) = (\alpha_1, \dots, \alpha_d|\beta_1, \dots, \beta_d)$, which are called the* Frobenius coordinates *of λ*[1]*.*

---

[1] While constructing the character tables of symmetric groups, Frobenius used these coordinates to index the irreducible representation, while he used the ordinary coordinates to index the conjugacy classes.

Example 16.2. The hook partition $(a+1, 1^b)$ has Frobenius coordinates $(a|b)$. Hook partitions are precisely those partitions which have Durfee rank 1. The partition with Frobenius coordinates $(5, 2, 1|4, 3, 0)$ is $(6, 4, 4, 2, 2)$. If λ has Frobenius coordinates $(\alpha|\beta)$, then its conjugate λ' has Frobenius coordinates $(\beta|\alpha)$. The size of a partition with Durfee rank d and Frobenius coordinates $(\alpha|\beta)$ is $d + |\alpha| + |\beta|$.

Schur polynomials of hook partitions can be calculated using Exercise 6.5, which, when written is terms of Frobenius coordinates, becomes:

$$s_{(a|b)} = \sum_{l=0}^{b} (-1)^l h_{a+l+1} e_{b-l}. \tag{16.1}$$

Theorem 16.3 (Giambelli's formula)**.** *For a partition* $(\alpha_1, \dots, \alpha_d|\beta_1, \dots, \beta_d)$ *in Frobenius coordinates,*

$$s_{(\alpha|\beta)} = \det(s_{(\alpha_j|\beta_i)})_{d\times d}. \tag{16.2}$$

Note that the determinant on the right consists of hook-partition Schur polynomials, which are given by (16.1).

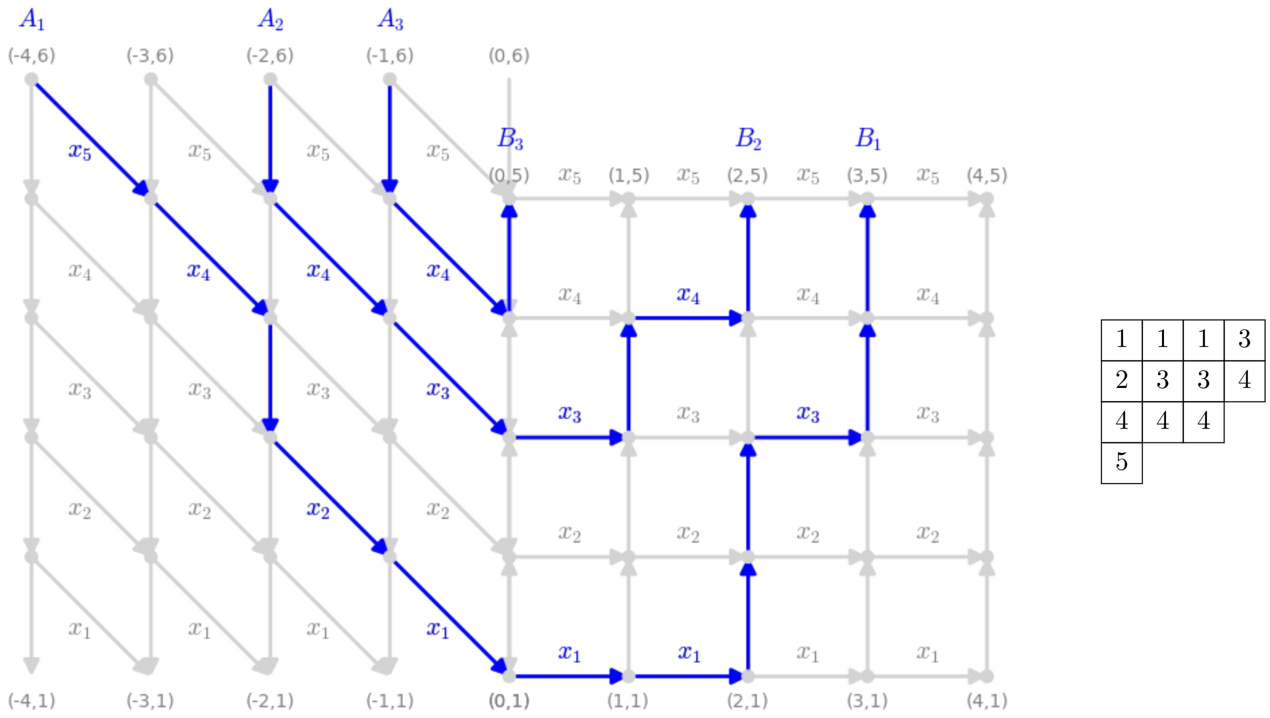
Example 16.4. The Schur polynomial for $\lambda = (4, 4, 3, 1) = (3, 2, 0|3, 1, 0)$ can be computed as:

$$s_{(3,2,1|3,1,0)} = \det \begin{pmatrix} s_{(3|3)} & s_{(2|3)} & s_{(0|3)} \\ s_{(3|1)} & s_{(2|1)} & s_{(0|1)} \\ s_{(3|0)} & s_{(2|0)} & s_{(0|0)} \end{pmatrix}.$$

Proof. Giambelli's identity can be proved using the Lindström-Gessel-Viennot Lemma. Let $\lambda = (\alpha_1, \dots, \alpha_d|\beta_1, \dots, \beta_d)$ be given. Working with n variables, set

$$S = \{(i, j) \mid 1 \le j \le n \; i \ge 0\} \cup \{(-i, j) \mid 1 \le j \le n + 1 \; i > 0\}.$$

Define a weight function as follows (see Fig. 6):



1	1	1	3
2	3	3	4
4	4	4	
5			

Fig. 6: Non-crossing paths and corresponding tableau.

$$\begin{aligned} v((-i, j+1), v(-i, j)) &= 1 \text{ for all } i > 0, 0 \le j \le n, \\ v((-i, j+1), v(-(i-1), j)) &= x_j \text{ for all } i > 0, j \ge 1, \\ v((i, j), (i+1, j)) &= x_j \text{ for all } i \ge 0, 1 \le j \le n, \\ v((i, j), (i, j+1)) &= 1 \text{ for all } i \ge 0, 1 \le j \le n. \end{aligned}$$

All other weights are set to zero. A path from $(-(b-1), n+1)$ to (a, n) has weight x^t, where t is a semistandard tableau of shape $(a|b)$ in $x_1, \dots, x_n$. For instance, the path in Fig. 6 from $(-4, 6)$ to $(3, 5)$ corresponds to the tableau $\begin{array}{|c|c|c|c|}\hline 1&1&1&3\\ \hline 2\\ \cline{1-1} 4\\ \cline{1-1} 5\\ \cline{1-1}\end{array}$.

Thus, setting $A_i = (-(\beta_i - 1), n+1)$ and $B_i = (\alpha_i, n)$ for $i = 1, \dots, d$, the (i, j)th entry of the determinant on the right hand side of (16.2) can be written as:

$$\sum_{\omega : A_i \to B_j} v(\omega). \tag{16.3}$$

It is not hard to see that the non-crossing path configurations $\{\omega_i : A_i \to B_i\}$ correspond to semistandard tableaux of shape $(\alpha|\beta)$ (for an example, see Fig. 6). □

17 Greene's Theorem

Given a word $w = (a_1, \dots, a_k) \in L_n^*$, a subword is a word of the form $w' = a_{i_1} \cdots a_{i_r}$, where $1 \le i_1 < \cdots < i_r \le k$. This section is concerned with the enumeration of subwords which are rows or columns. For the purposes of such enumeration, given $1 \le j_1 < \cdots < j_r \le k$, $w'' = a_{j_1} \cdots a_{j_r}$ will be considered to be a different subword from w' *even if* $w' = w''$ as words, unless the indices $j_1, \dots, j_r$ coincide with the indices $i_1, \dots, i_r$. Two subwords of w will be said to be disjoint if their indexing sets are disjoint.

Example 17.1. The word $w = 111$ has three subwords of length two, all of them equal to 11. No two of these subwords are disjoint. However, each of them is disjoint from a subword of w of length one.

Definition 17.2 (Greene invariants)**.** *Given $w \in L_n^*$, for each integer $k \ge 0$, let $l_k(w)$ denote the maximum cardinality of a union of k pairwise disjoint weakly increasing subwords of w. Let $l'_k(w)$ denote the maximum cardinality of a union of k pairwise disjoint strictly decreasing subwords of w.*

Example 17.3. If $w = 2133$, then $l_1(w) = 3$, $l_k(w) = 4$ for all $k \ge 2$. Also, $l'_1(w) = 2$, $l'_2(w) = 3$, and $l'_k(w) = 4$ for all $k \ge 3$.

Theorem 17.4 (Greene's Theorem)**.** *Given a word w, define a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ by $\lambda_k = l_k(w) - l_{k-1}(w)$ for each $k \ge 1$. Then λ is the shape of $P(w)$. Moreover, if $\mu'_k = l'_k(w) - l'_{k-1}(w)$ for each $k \ge 1$, then $\mu = (\mu_1, \mu_2, \dots)$ is the partition conjugate to λ.*

Proof. Greene's theorem follows by putting together two relatively simple observations—the first is that the Greene invariants $l_k(w)$ and $l'_k(w)$ remain unchanged when either of the Knuth relations $(K1)$ and $(K2)$ is applied to w. The second is that when w is the reading word of a semistandard tableau, then Greene's theorem holds.

To see the first, suppose that w is of the form u_1xzyu_2, with $x \le y < z$, and arbitrary $u_1, u_2 \in L_n^*$. Applying a Knuth transformation of the form $(K1)$, w transforms to $w' = u_1zxyu_2$. Any weakly increasing subword of w' is also a weakly increasing subword of w, so $l_k(w') \le l_k(w)$. On the other hand, if v is a weakly increasing subword of w of the form v_1xzv_2 (where v_i is a subword of u_i for $i = 1, 2$) it will not necessarily remain a weakly increasing subword of w'. However, v_1xyv_2 *is* a weakly increasing subword of w. If a collection of k weakly increasing subwords of w contains v_1xzv_2 and another weakly increasing subword $v'_1yv'_2$, replacing them by $v_1yzv'_2$ and v'_1xv_2 gives a collection of k weakly increasing subwords of w' of the same cardinality. It follows that $l_k(w') \ge l_k(w)$ also holds. Thus the Knuth transformation $(K1)$ preserves the Greene invariants $l_k(w)$. Similar arguments can be used to show that both $(K1)$ and $(K2)$ preserve all the Greene invariants $l_k(w)$ and $l'_k(w)$.

Now suppose w is the reading word of a tableau of shape λ. Then the top k rows of w form a union of k pairwise disjoint weakly increasing subwords of total size $\lambda_1 + \cdots + \lambda_k$. Also, if v is a weakly increasing subword of w, then the fact that the columns of t are strictly increasing (and that the rows are read from *bottom to top*) implies that v cannot contain more than one element from each column of w. Therefore, any collection of k pairwise disjoint weakly increasing subwords of w can have at most k entries in each column of w. Thus no union of k pairwise disjoint weakly increasing subwords of w can have cardinality more than $\lambda_1 + \cdots + \lambda_k$. Therefore, $l_k(w) = \lambda_1 + \cdots + \lambda_k$.

Similarly, the leftmost k columns of w (read bottom to top) form a union of k pairwise disjoint strictly decreasing subwords of w, and any such union can only contain k elements from each row. It follows that if λ' is the partition conjugate to λ, then $l'_k(w) = \lambda'_1 + \cdots + \lambda'_k$. □

18 The Robinson-Schensted-Knuth Correspondences

For an $m \times n$ matrix $A = (a_{ij})$, the column word u_A, row word v_A and their duals $\overleftarrow{u}_A$ and $\overleftarrow{v}_A$ are defined as follows:

$$\begin{aligned}
u_A &= 1^{a_{11}}2^{a_{12}} \cdots n^{a_{1n}}\ 1^{a_{21}}2^{a_{22}} \cdots n^{a_{2n}}\ \cdots\ 1^{a_{m1}}2^{a_{m2}} \cdots n^{a_{mn}} \\
v_A &= 1^{a_{11}}2^{a_{21}} \cdots m^{a_{m1}}\ 1^{a_{12}}2^{a_{22}} \cdots m^{a_{m2}}\ \cdots\ 1^{a_{1n}}2^{a_{2n}} \cdots m^{a_{mn}} \\
\overleftarrow{u}_A &= n^{a_{1n}} \cdots 2^{a_{12}}1^{a_{11}}\ n^{a_{2n}} \cdots 2^{a_{22}}1^{a_{21}}\ n^{a_{mn}} \cdots 2^{a_{m2}}1^{a_{11}} \\
\overleftarrow{v}_A &= m^{a_{m1}} \cdots 2^{a_{21}}1^{a_{11}}\ m^{a_{m2}} \cdots 2^{a_{22}}1^{a_{12}}\ \cdots\ m^{a_{mn}} \cdots 2^{a_{2n}} \cdots 1^{a_{1n}}
\end{aligned}$$

Definition 18.1 (Robinson-Schensted-Knuth Correspondences). *Define functions from integer matrices onto pairs of semistandard tableaux by:*

$$\begin{aligned}
\mathrm{RSK}(A) &= (P(u_A), P(v_A)), \\
\mathrm{RSK}^*(A) &= (P^*(u_A), P(\overleftarrow{v}_A)).
\end{aligned}$$

Lemma 18.2. *For every integer matrix A with non-negative entries, the tableaux $P(u_A)$ and $P(v_A)$ have the same shape. For every zero-one matrix A, the tableaux $P^*(u_A)$ and $P(\overleftarrow{v}_A)$ have the same shape.*

Proof. The proof is an application of Greene's theorem (Theorem 17.4). Any weakly increasing subword of u_A comes from reading the column number of a sequence of entries $(i_1, j_1), \ldots, (i_r, j_r)$ with repetitions of up to $a_{i_1 j_1}, \ldots, a_{i_r j_r}$ respectively, with $i_1 \leq \cdots \leq i_r$ and $j_1 \leq \ldots \leq j_r$. If A is an $m \times n$ matrix, its entries are indexed by the rectangular lattice $P_{mn} = \{(i,j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ which may be regarded as a poset under $(i,j) \leq (i',j')$ if $i \leq i'$ and $j \leq j'$. It follows that

$$l_k(u_A) = \max_{C_k} \left\{ \sum_{(i,j) \in C_k} a_{ij} \right\}$$

where the maximum is over all subsets C_k of P_{mn} which can be written as a union of k chains in the partially ordered set P_{mn}. This description of the shape of $P(u_A)$ is invariant under interchanging the rows and columns of A, and therefore also the shape of $P(v_A)$.

For the dual RSK correspondence, if A is a 0-1 matrix, note that a strictly increasing subword of u_A comes from reading the column number of a sequence of entries $(i_1, j_1), \ldots, (i_r, j_r)$ with entries equal to 1, and with $i_1 \leq \cdots \leq i_r$ and $j_1 < \ldots < j_r$. Define a new partial order P_{mn} by $(i,j) < (i',j')$ if $i \leq i'$ and $j < j'$. It follows that

$$l_k^*(u_A) = \max_{C_k} \left\{ \sum_{(i,j) \in C_k} a_{ij} \right\}$$

where the maximum is over all subsets $C_k \subset P_{mn}$ which can be written as a union of k chains in the new partial order. On the other hand, the row numbers in the sequence of entries $(i_1, j_1), \ldots, (i_r, j_r)$ form a weakly increasing subword of $\overleftarrow{v}_A$ if and only if $i_1 \leq \cdots \leq i_r$, and since the entries must come from distinct rows (since $\overleftarrow{v}_A$ reads each row in reverse order and all entries are 0 or 1), so $j_1 < \cdots < j_r$. Thus $l_k(\overleftarrow{v}_A) = l_k^*(u_A)$, so $P^*(u_A)$ and $P(\overleftarrow{v}_A)$ have the same shape. □

Theorem 18.3 (Knuth's theorem). *Let $\mathbf{M}_{\mu\nu}$ denote the set of integer matrices with non-negative entries, row sums $(\mu_1, \ldots, \mu_m)$, column sums $(\nu_1, \ldots, \nu_n)$. Let $\mathrm{Tab}(\lambda, \mu)$ denote the set of semistandard tableaux of shape λ and type μ. Then* RSK *gives rise to a bijection:*

$$\mathbf{M}_{\mu\nu} \xrightarrow{\sim} \coprod_{\lambda} \mathrm{Tab}(\lambda, \nu) \times \mathrm{Tab}(\lambda, \mu). \tag{18.1}$$

Similarly let $\mathbf{N}_{\mu\nu}$ denote the set of zero-one matrices with row sums $(\mu_1, \ldots, \mu_m)$ and column sums $(\nu_1, \ldots, \nu_n)$. Then RSK* *gives rise to a bijection:*

$$\mathbf{N}_{\mu\nu} \xrightarrow{\sim} \coprod_{\lambda} \mathrm{Tab}^*(\lambda, \nu) \times \mathrm{Tab}(\lambda, \mu). \tag{18.2}$$

Proof. We will show that RSK is a bijection:

$$\mathbf{M}_{m \times n} \xrightarrow{\sim} \coprod_{\lambda} \mathrm{Tab}_n(\lambda) \times \mathrm{Tab}_m(\lambda).$$

For the definition it is clear that this bijection will map the left hand side of (18.1) onto its right hand side.

Let A' be the matrix consisting of the first $m-1$ rows of A. Let $r = 1^{a_{m1}}2^{a_{m2}}\cdots n^{a_{mn}}$ be the column word of the last row of A.

By inducting on the number of rows of A (the base case of one-row matrices is easy), we have bijections:

$$\mathbf{M}_{m\times n} \leftrightarrow \mathbf{M}_{(m-1)\times n} \times R(L_n) \leftrightarrow \coprod_{\lambda} \mathrm{Tab}_n(\lambda) \times \mathrm{Tab}_{m-1}(\lambda) \times R(L_n) \tag{18.3}$$

given by

$$A \leftrightarrow (A', r) \leftrightarrow (P(u_{A'}), P(v_{A'}), r).$$

Here $R(L_n)$ denotes the set of all rows (weakly increasing words) in L_n^*.

Define a function

$$\mathrm{Tab}_n(\lambda) \times \mathrm{Tab}_{m-1}(\lambda) \times R(L_n) \to \coprod_{\mu} \mathrm{Tab}_n(\mu) \times \mathrm{Tab}_m(\mu) \tag{18.4}$$

by

$$(t_1, t_2, r) \mapsto (P(t_1 r), t_2^{\uparrow m}),$$

where $t_2^{\uparrow m}$ is the unique tableau with the same shape as $P(t_1 r)$ such that if all the boxes containing m are removed from it, then t_2 is obtained.

It turns out that the above function is invertible. Given tableaux $(t_1', t_2') \in \mathrm{Tab}_n(\mu) \times \mathrm{Tab}_m(\mu)$. Let t_2 be the tableau obtained from t_2' by removing all the boxes containing m. Let λ be the corresponding shape. Obviously μ/λ is a horizontal strip. Applying the inverse of the first bijection in Theorem 12.2 recovers t_1 and r.

Combining the bijections (18.3) and (18.4) gives rise to the RSK correspondence, which is therefore also a bijection.

The proof for the bijectivity of RSK* is similar (although with a few twists) and is left as an interesting exercise to the reader. ☐

Exercise 18.4. [The Burge Correspondence] Define

$$\mathrm{BUR}(A) = (P^*(\overleftarrow{u}_A), P^*(\overleftarrow{v}_A)).$$

Show that BUR is a bijection $\mathbf{M}_{\mu\nu} \tilde{\to} \coprod_{\lambda} \mathrm{Tab}^*(\lambda,\nu) \times \mathrm{Tab}^*(\lambda,\mu)$.

19 The Littlewood-Richardson Rule

Lemma 19.1. *Given a partition λ, fix any $t_\lambda \in \mathrm{Tab}_m(\lambda)$. Then*

$$\sum_{\{A_{m\times n} \mid P(v_A) = t_\lambda\}} x^{u_A} = s_\lambda(x_1, \ldots, x_n).$$

Proof. If $P(v_A) = t_\lambda$, a tableau of shape λ, $P(u_A)$ is also a semistandard tableau of shape λ. Moreover, for every semistandard tableau $t \in \mathrm{Tab}_n(\lambda)$, by Knuth's theorem (Theorem 18.3), there exists a unique $m\times n$ integer matrix A such that $\mathrm{RSK}(A) = (t, t_\lambda)$. In other words, among matrices with $P(v_A) = t_\lambda$, there exists a unique matrix such that $u_A \equiv t$. The lemma now follows from Kostka's definition of Schur polynomials (Corollary 12.5). ☐

Theorem 19.2 (Littlewood-Richardson Rule)**.** *Let α, β and λ be partitions. Let t_β be any semistandard tableau in* $\mathrm{Tab}_b(\beta)$ *for some integer b. Let $c^\lambda_{\alpha\beta}$ denote the number of semistandard skew-tableaux of shape λ/α whose reading word is Knuth-equivalent to t_β, the unique tableau of shape and type β. Then*

$$s_\alpha s_\beta = \sum_\lambda c^\lambda_{\alpha\beta} s_\lambda.$$

Proof. Let $t_\alpha \in \mathrm{Tab}_a(\alpha)$ for some integer a. By Lemma 19.1, we have:

$$s_\alpha s_\beta = \sum_{\{A_{a\times n} \mid P(v_A) = t_\alpha\}} x^{u_A} \sum_{\{B_{b\times n} \mid P(v_B) = t_\beta\}} x^{u_B} = \sum_{C_{\alpha\beta}} x^{u_C},$$

where

$$C_{\alpha\beta} = \{\tbinom{A}{B} \mid P(v_A) = t_\alpha \text{ and } P(v_B) = t_\beta\}.$$

Define a monoid homomorphism $\pi_a : L_{a+b}^* \to L_a^*$ by taking $\pi_a(w)$ to be the word obtained by discarding all letters in w that are not in $\{1, \dots, a\}$. Also define $\pi_b : L_{a+b}^* \to L_b^*$ by taking $\pi_b(w)$ to be the word obtained discarding all letters in w that are not in $\{a+1, \dots, a+b\}$, and then replacing the letter $a+i$ by i. It is not hard to see that π_a and π_b preserve the Knuth relations $(K1)$ and $(K2)$. Let $C = \binom{A}{B}$. The words v_A and v_B are obtained from v_C as follows:

$$v_A = \pi_a(v_C) \text{ and } v_B = \pi_b(v_C).$$

So $P(v_A) = P(\pi_a(v_C)) \equiv \pi_a(P(v_C))$. Since π_A is a restriction to the first few letters of L_{a+b}, $\pi_a(P(v_C))$ is still a tableau. So $P(v_A) = \pi_a(P(v_C))$. Also, $P(v_B) \equiv \pi_b(P(v_c))$. Therefore

$$C_{\alpha\beta} = \{C \mid \pi_a(P(v_C)) = t_\alpha \text{ and } \pi_b(P(v_C)) \equiv t_\beta\} = \coprod_t \{C \mid P(v_C) = t\},$$

the disjoint union being over

$$\{t \in \mathrm{Tab}(L_{a+b}) \mid \pi_a(t) = t_\alpha \text{ and } \pi_b(t) \equiv t_\beta\},$$

Here $\mathrm{Tab}(L_n)$ is used to denote the set of all semistandard tableaux in L_n^*. Let t be a tableau t in the above set of shape λ. The entries of this tableau in the cells of α are completely fixed by the condition $\pi_a(t) = t_\alpha$. Subtracting a from the remaining entries gives rise to a skew-tableau of shape λ/α whose reading word is equivalent to t_β. The number of such tableaux is, by definition, $c_{\alpha\beta}^\lambda$. Therefore,

$$s_\alpha s_\beta = \sum_\lambda \sum_{\{t \in \mathrm{Tab}_b(\lambda/\alpha) \mid t \equiv t_\beta\}} \sum_{\{C \mid P(v_C) = t\}} x^{u_C} = \sum_\lambda c_{\alpha\beta}^\lambda s_\lambda \qquad \text{[using Lemma 19.1]}$$

as required. ☐

Definition 19.3 (Yamanouchi Word). *A word $w \in L_n^*$ is called a Yamanouchi word if x^u is a monomial with weakly increasing powers for every suffix u of w.*

Lemma 19.4. *A word w is a Yamanouchi word of type λ if and only if its plactic class contains the unique semistandard tableau t_λ^0 of shape λ and type λ.*

Proof. Check that Knuth relations preserve *Yamanouchiness*. The only Yamanouchi tableau of type λ also has shape λ. ☐

In view of Lemma 19.4, the Littlewood-Richardson rule becomes:

Theorem 19.5. *For partition α, β, λ, $c_{\alpha\beta}^\lambda$ is the number of semistandard tableaux of shape λ/α and type β whose reading word is a Yamanouchi word.*

Exercise 19.6. If $c_{\alpha\beta}^\lambda > 0$ then $\lambda \supset \alpha$ and $\lambda \supset \beta$.

Exercise 19.7. Suppose $\lambda = (\lambda_1, \dots, \lambda_l)$, with $\lambda_1 \leq m$. Let $\check{\lambda} = (m - \lambda_l, \dots, m - \lambda_1)$, and let Λ denote the partition $(l, \dots, l)$ (with m repetitions of l). Show that $c_{\lambda\mu}^\Lambda = 1$.

20 Skew-Schur polynomials and the Littlewood-Richardson Rule

Littlewood-Richardson coefficients also answer the question of expanding skew-Schur polynomials in terms of Schur polynomials:

Theorem 20.1 (Expansion of skew-Schur polynomials). *Let λ, α and β be partitions such that $\lambda \supset \alpha$. Then*

$$s_{\lambda/\alpha}(x_1, \dots, x_n) = \sum_\beta c_{\alpha\beta}^\lambda s_\beta(x_1, \dots, x_n).$$

Proof. We have:

$$\begin{aligned} s_{\lambda/\alpha}(x_1, \dots, x_n) &= \sum_{t \in \mathrm{Tab}_n(\lambda/\alpha)} x^t = \sum_\beta \sum_{t_\beta \in \mathrm{Tab}_n(\beta)} \sum_{\{t \in \mathrm{Tab}_n(\lambda/\alpha) \mid P(t) = t_\beta\}} x^t \\ &= \sum_\beta c_{\alpha\beta}^\lambda \sum_{t_\beta \in \mathrm{Tab}_n(\beta)} x^t \\ &= \sum_\beta c_{\alpha\beta}^\lambda s_\beta, \end{aligned}$$

as required. ☐

Exercise 20.2. For partitions $\alpha \subset \lambda$, show that $K_{\lambda/\alpha,\mu} = \sum_{\beta} c^{\lambda}_{\alpha\beta} K_{\beta\mu}$. Here $K_{\lambda/\alpha}$ is the number of skew-tableaux of shape λ/α and type μ.

21 Sources

The arrangement of topics in these notes loosely follows the first Chapter of Manivel's book [7]. The use of labeled abaci to prove Schur polynomial identities is from Loehr [5]. A nice exposition of the LGV lemma can be found in Viennot's 2016 lectures at The Institute of Mathematical Sciences, Chennai [11]. The LGV-lemma proof of the Giambelli identity has been published by Stembridge [10]. The treatment of the RSK correspondence and the Littlewood-Richardson rule is guided by Lascoux and Schützenberger[4], and Lascoux, Leclerc and Thibon [3]. However, I have not seen the precise definition of RSK and RSK* that I used in these notes published elsewhere. My use of integer matrices to prove the Littlewood-Richardson rule does not appear to be widespread. Exercise 18.4 describes a correspondence introduced by Burge in [1]. Besides the above references, readers who wish to go deeper into the subject may consult Fulton's book on Young tableaux [2], Chapter 7 and Fomin's appendix in the second volume of Stanley's book on Enumerative Combinatorics [9]. The relationship between the RSK correspondence, its dual, symmetric functions, and representation theory is the subject of my own book on Representation Theory [8], which is written at a relatively elementary level. Finally, there is no single book that has had a greater impact on the theory of symmetric functions than Macdonald's classic [6].

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