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Computations of characteristic classes and genera: A practical toolkit for beginners and practitioners

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Abstract

This paper provides explicit combinatorial expansions of the Chern classes, Pontrjagin classes, Chern character, as well as the Chern character and Chern classes of a tensor product of two vector bundles. One will also find expansions of the Â-genus, L-genus, Todd genus, their twisted forms, as well as relations among the three genera. The computational methods are elementary, but we hope that the clearly displayed and ready-to-use results – which do not seem to be otherwise available – will be useful. This is the first part of research based on the undergraduate project of the second and third authors advised by the first author.

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1 Introduction

The goal of this paper is to perform computations of characteristic classes and genera of vector bundles which yield explicit handy results that would be readily utilizable by mathematicians and physicists not wishing to perform such straightforward but cumbersome computations themselves. The main focus has been on explicitly expanding existing combinatorial formulas that are often lengthy and complicated to compute by hand and that cannot be easily looked up in the literature. The results are tabulated throughout the paper in a way readers can find useful. While this work by no means exhausts the subject, every attempt was made to provide comprehensive listings of computations with clear notation. Moreover, for those willing to do their own computations, they would still find it quite useful to have tables against which to check their calculations. Additionally, we make use of Mathematica and the codes for every expansion are available upon request.

The calculations are elementary and systematic, so why would this be useful? Someone who is willing to go through the task can certainly reproduce them. However, for someone who uses characteristic classes extensively, having to do such elementary but often laborious calculations repeatedly is not the most efficient task. So we provide the list at least for those students and researchers who routinely use such formulas in their research. We believe that being able to reproduce the formulas and having these displayed in front of a reader are two different things. One byproduct of the latter might be allowing for known patterns to be seen visually and perhaps for new ones to be discovered. Indeed, the patterns do reveal specific interrelations that might otherwise not be obvious from the general formulas.

The key objects of study are *fiber bundles*. A fiber bundle is a quadruple (E, M, π, F), where E, M, and F are topological spaces called, respectively, the total space, the base space, and the fiber of the bundle, and $\pi : E \to M$ is a continuous surjection called the bundle projection map. To be a bundle requires local trivialization; that is for every x in E, there is an open neighborhood $U \subset M$ of $\pi(x)$ with a homeomorphism ϕ from the preimage $\pi^{-1}(U)$ to the Cartesian product $U \times F$ such that π agrees with the projection onto the first factor [16] [27]. A smooth fiber bundle is a fiber bundle in the category of smooth manifolds, i.e., the spaces E, M, and F are smooth manifolds and all functions between them are smooth maps. For the purposes of the current paper we will specifically consider real or complex *vector bundles* (E, M, π, V) where the fiber is now a real or complex vector space V respectively. The rank of a vector bundle is defined as $\mathrm{rank}(E) = \dim_{\mathbb{F}} V$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ respectively.

The group of transformations of the fiber V are those automorphisms in $GL(V)$ preserving the inner product associated to V. If the fiber V is a real vector space, then the group of transformations is $O(n)$ (or $SO(n)$), the (special) orthogonal group. On the other hand, if V is complex then the group of transformations is $U(n)$ (or $SU(n)$), the (special) unitary group. Vector bundles can be added and tensored *over the same base*. Given (E_1, M, π, V_1) and (E_2, M, π, V_2), then one constructs the Whitney sum bundle $(E_1 \oplus E_2, M, \pi, V_1 \oplus V_2)$, and the tensor product bundle $(E_1 \otimes E_2, M, \pi, V_1 \otimes V_2)$. It would often be convenient, once the base is understood, to identify the bundle with its total space E or with the map $\pi : E \to M$.

Characteristic classes [3] of a vector bundle $\pi : E \to M$ are certain cohomology classes living in $H^*(M; R)$, for various choices of commutative rings or fields R, that can be used to determine whether a bundle is trivial or not, as well as give some idea about the extent of its non-triviality or "twisting". A bundle is trivial if it is isomorphic to the product of base and fiber. A non-trivial bundle can be intuitively thought of as being "twisted" in comparison to the trivial bundle. Non-trivial vector bundles can have characteristic classes yielding a nontrivial value when evaluated on the base manifold, in contrast with the trivial case where the answer is always zero. Characteristic classes of a given vector bundle depend on the rank of the bundle, whether the vector space fiber is real or complex, as well as on the extent to which the underlying manifold base is topologically nontrivial.

We will consider characteristic classes from a combinatorial point of view, inevitably hiding the rich geometric nature (via Chern-Weil theory) and only touching briefly on the rich algebraic-topological nature (via cohomology of classifying spaces). Extensive theoretical background on characteristic classes and genera can be found in [3][4][5][6][12][27][8][7][22][25], with useful summaries and applications, for instance, in [10][29][2][37][9]. We will only provide enough background to be able to state the results in a meaningful way. One can deduce results for other types of fiber bundles by associating them to vector bundles, although we will not do this explicitly here (see, e.g., [28]).

Genera [30][35] are certain polynomials in characteristic classes that are cobordism invariant, i.e., take on the same value on certain equivalence class of compact manifolds of the same dimension obtained by using boundaries (see [13] for an overview). These are typically computed directly from their defining series expansions, stated in Section 4. Expansions of the three genera of interest, namely the $\widehat{A}$-genus, the Todd genus and the L-genus, are readily available - one may even find accurate formulae up to degree 4 on [36], up to degree 5 (for the Todd genus) in [29], degrees 4-6 in the notes [9], and for the Â-genus up to degree 6 in [14], which also explains in detail how to compute these expansions in Mathematica. Now, for the L-genus up to an impressive degree 14 one should visit Carl McTague's blog [26]. As a matter of fact, Carl McTague wrote a universal Mathematica formula that allows one to compute a genus expansion by taking its characteristic power series and the desired degree as input. The formula was indeed crucial for this paper for obtaining the results in Section 4.

One may find the standard expansions of the three genera in question up to certain (low) degrees. However, aside from supplying expansions to sufficiently high degree in this paper, we also indicate the various simplifications applicable to different types of bundles, genera of complexification, and relations among them. To help with patterns that might arise, we have chosen to write down (large) numerical factors in terms of their prime factorizations. As indicated in Section 2.1 such computations are very important for anomaly cancellations in field theory [2], as well as in string theory in the presence of extra higher structures [32].

The need for such computations to be carried out and made readily available is illustrated well by the fact that Carl McTague wrote the L-genus expansion based on a request by Andrew Ranicki [26]. Besides mathematicians working on topology or geometry, the results should be useful to theoretical physicists as well. Characteristic classes and genera appear in string theory and gauge theory especially in the context of anomalies (see [2][29]). Furthermore, the numerical quantities obtained as a result of evaluating characteristic classes, i.e. the characteristic numbers, also provide some measurement of a physical quantity, such as the charge, in the respective theories.

We have chosen to use *Mathematica* as our tool of choice for programming the functions, as we believe this would enable a wider audience to benefit from this work, due to the program's popularity. Although the computations for high degrees in Mathematica may take longer than a low-level highly specialized programming language like SINGULAR [17], there is nevertheless a clear trade-off with the ease of use. Naturally, the speed of computation depends also on the machine used.

The paper is organized as follows. In Section 2 we provide the necessary background on characteristic classes, focusing mainly on Chern classes and the Chern character in Section 2.1. An important computational tool is the splitting principle which we recall in Section 2.2. Then in 2.3 we present a brief overview of previous approaches and the rationale for using Mathematica, which we demonstrate for the case of Chern classes and Chern character in Section 2.4. The expansions obtained are then grouped into three main sections. Section 3, *Characteristic classes*, tabulates expansions of combinatorial formulas of characteristic classes, particularly, expansions that

	Real bundles	Complex bundles
Classes	Pontrjagin p_i	Chern c_i
Characters	Pontrjagin Ph	Chern ch
Genera	A-genus $\widehat{A}$, L-genus L	Todd Td

Name of structure	Calabi-Yau	complex String	String	Fivebrane	Ninebrane
Condition	$c_1 = 0$	$c_1 = 0 = c_2$	$p_1 = 0$	$p_1 = 0 = p_2$	$p_1 = p_2 = p_3 = 0$

express Chern classes, Chern character and Pontrjagin classes in terms of each other. We provide background material on Pontrjagin classes in Section 3.2, which relates to that of Section 2.1 for the case of Chern classes and character. Section 3 also features a few simplifications of these expressions where relevant. Tensor products of two vector bundles are considered in Section 3.3, where expansions of their Chern character and Chern classes in terms of the Chern characters and classes of the individual factors are given. Section 4 deals with the three main classical genera, the Â-genus, the L-genus, and the Todd genus. After providing some basic background in Section 4.1, in the following three subsections 4.2, 4.3, and 4.3 (respectively), we tabulate the standard expansions of the three genera.

We also do so for their various simplifications, for instance, in the presence of extra structures: complex, Calabi-Yau [11], String [1], Fivebrane [32][33], and Ninebrane [31] structures. Additionally, these three subsections also include complexification expansions. We emphasize that we will be working rationally, so that issues about congruence and divisibility in the corresponding cohomology rings can be avoided.

Finally, in Section 5 we explore relations among the genera in general and also in the existence of the above special structures. The three sets of binary relations between pairs from among the set {Â-genus, Todd-genus, L-genus} are given in Sections 5.1, 5.2, and 5.3.

This paper grew from an undergraduate project of the second and third authors under the guidance of the first author.

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2 Some background and computational tools

2.1 Chern classes and the Chern character

Chern classes.

Chern classes are used to characterize complex vector bundles. Algebraically, they are obtained via the cohomology of classifying spaces. A classifying space BG of a Lie group G is the quotient of a weakly contractible space EG, i.e., a topological space all of whose homotopy groups are trivial, by a proper free action of G. The quadruple (EG, BG, π, G) constitutes a *universal bundle* in that any G principal bundle over a paracompact manifold is isomorphic to a pullback of this bundle (see [27][16]).

When $G = U(n)$, one has the classical result (see [22, Theorem 18]):

Theorem 1. *The ring $H^*(BU(n);\mathbb{Z})$ of integer cohomology classes is isomorphic to the polynomial ring $\mathbb{Z}[c_1, c_2, \cdots, c_n]$, where $c_k \in H^{2k}(BU(n);\mathbb{Z})$.*

The classes c_k are called the universal characteristic classes. The Chern classes of a complex bundle E over M are obtained from the universal Chern classes by pullback via a classifying map $f : M \to BU(n)$, where n =rank(E). These characteristic classes are denoted $c_1(E), c_2(E), \ldots, c_n(E)$. They are collected under one expression, the *total Chern class* $c(E)$ of E;

$$c(E) := 1 + c_1(E) + \cdots + c_n(E).$$

The total Chern class lives in the graded ring $H^*(M;\mathbb{Z})$. In fact, the total Chern class $c(E)$ can be completely characterized by the following properties :

1. *Naturality*: $c(f^*E) = f^*c(E)$, where $f : N \to M$ is a continuous (or smooth) map between topological spaces (or smooth manifolds).

2. *Finiteness*: For a rank k complex bundle E, $c(E) = c_0(E) + c_1(E) + ... + c_k(E)$, $c_i(E) \in H^{2i}(M)$; $c_j(E) = 0$, $j > k$.
3. *Whitney sum:* $c(E \oplus F) = c(E)c(F)$.
4. *Normalization:* $c(L) = 1 + x$, ($x = c_1(L)$, L is canonical line bundle over the complex projective space $\mathbb{C}P^n$).

Chern character

The notion of the Chern character can be thought of as arising from the wish to get a multiplicative combinations of the Chern classes (think exponential vs. logarithm). Its importance arises from the fact that it provides a ring homomorphism from complex K-theory (which classifies complex vector bundles up to equivalence and stability) to ordinary cohomology: $\mathrm{ch} : K(M) \to H^*(M; \mathbb{Z})$.

The Chern character has the following properties:

1. *Naturality*: Let $\pi : E \to M$ be a vector bundle and $f : N \to M$ be a smooth map, then $\mathrm{ch}(f^*E) = f^*\mathrm{ch}(E)$.
2. *Tensor product*: $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$.
3. *Whitney sum*: $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$.

One also defines components ch_i degree $2i$ of the Chern character so that for a complex vector bundle E of rank r we have an expansion

$$\mathrm{ch}(E) = \mathrm{ch}_0(E) + \mathrm{ch}_1(E) + \cdots \mathrm{ch}_r(E) + \cdots ,$$

where $\mathrm{ch}_0(E) = r$ the rank of the bundle. Notice that this is in contrast with the total Chern class which starts with 1. When evaluated on a manifold of complex dimension n, the series terminates at ch_n.

Proofs of these properties are provided in [29] and [37] as well as further discussion of Chern character overall. Note that Chern character has one particularly important application, as they appear in the Atiyah-Singer index theorem [29]. Such computations are very important for anomalies in quantum field theory and string theory (see [2][29][33]). Quite remarkably, the Chern character provides an isomorphism when tensoring with the rationals $\mathrm{ch} : K(X) \otimes \mathbb{Q} \simeq H^{\mathrm{even}}(X; \mathbb{Q})$ (see, e.g. [22][20] for details).

2.2 The Splitting Principle

The *splitting principle* is one of the most important methods of computation in our case. It is widely used in characteristic class computations (see [34] and [21]). This principle is based on the remarkable fact that when it comes to computing characteristic classes, a vector bundle can 'pretend' to be a sum of line bundles [12], [7]. More explicitly, suppose the bundle splits as Whitney sum of line bundles

$$E \cong L_1 \oplus L_2 \oplus \cdots \oplus L_r; , \quad r = \mathrm{rank}(E). \tag{2.1}$$

Then its characteristic classes are immediately computable by knowledge of the characteristic classes of the L_i and by the Whitney sum property indicated in a preceding subsection. The class $x_i = c_1(L_i)$ is called a *Chern root* and can be viewed as a cohomology class in $H^2(M; \mathbb{Z})$, via the isomorphism (2.1). In this case, the total Chern class is given by

$$c(E) = \prod_{i=1}^{r}(1 + x_i)$$

and consequently $c_j(E) = \sigma_j(x_1, \cdots , x_r)$ for $j = 1, \cdots , r$, where $\sigma_j(x_1, \cdots , x_r)$ is the j-th elementary symmetric polynomial.

If E of rank r does not split as a sum of line bundles over M, then the splitting principle states that pulling E back to the projectivization $\mathrm{pr} : \mathbb{P}(E) \to M$ splits over $\mathbb{P}(E)$ as a direct sum of a line bundle, called the tautological line bundle, and a universal quotient bundle Q of rank $r - 1$. By iterating the process, one arrives at the so-called *flag bundle* $Fl(E)$, over which the bundle E splits into a Whitney sum of r line bundles $L_1, \cdots , L_r$. Then the cohomology classes $x_i := c_1(L_i) \in H^2(Fl(E); \mathbb{Z})$, $i = 1, \cdots , r$, are called the Chern roots of E and the Chern classes $c_j(E)$ of E can be identified as well with the elementary symmetric polynomials $\sigma_j(x_1, \cdots , x_r)$; see [7, p. 269].

Particularly, for computations with Chern classes, this method enables one to obtain results for tensor product of two vector bundles. In the context of this paper the splitting principle was most notably used to compute the

jth Chern character in terms of Chern classes (see *Expansion 3*) as well as to expand the jth Chern class and jth Chern character of a tensor product of two vector bundles (see Section 3.3). Next follows a demonstration on the essential mechanism of the splitting principle and how it works in computations of Chern character (see [7] [29]).

Example: Take L_j to be complex line bundles with $1 \leq j \leq r$. We deduce from the Whitney sum property of the Chern character (section 2.1) that for $E = L_1 \oplus L_2 \oplus ... \oplus L_r$,

$$\mathrm{ch}(E) = \mathrm{ch}(L_1) + \mathrm{ch}(L_2) + ... + \mathrm{ch}(L_r). \tag{2.2}$$

Yet, since $\mathrm{ch}(L_j) = \exp(x_j)$, then

$$\mathrm{ch}(E) = \sum_{j=1}^{r} \exp(x_j) \tag{2.3}$$

which is Chern character in terms of elementary symmetric polynomials. Therefore, the Chern character of a vector bundle is calculated via the Whitney sum of r complex line bundles. Note that the x_i are called *Chern roots*. The Chern classes are given in terms of Chern roots by

$$c_i(E) = \sigma_i(x_1, \ldots, x_r) \tag{2.4}$$

where σ_i is the i-th elementary symmetric polynomial, defined as [24] [12] [21]

$$\sigma_i(x_1, x_2, ..., x_r) = \sum_{1 \leq j_1 < j_2 < ... < j_i \leq r} x_{j_1} ... x_{j_i}. \tag{2.5}$$

The first few elementary symmetric polynomials are given as follows

$$\begin{aligned}
\sigma_0(x_1, x_2, ..., x_r) &= 1 \ , \\
\sigma_1(x_1, x_2, ..., x_r) &= \sum_{1 \leq j \leq r} x_j \ , \\
\sigma_2(x_1, x_2, ..., x_r) &= \sum_{1 \leq j < k \leq r} x_j x_k \ ,
\end{aligned}$$

while the r-th (top) polynomial is $\sigma_r(x_1, x_2, ..., x_r) = x_1 x_2 ... x_r$.

2.3 Mathematica implementation

In this section we will present a brief overview of previous work done in finding explicit expressions of the characteristic classes and genera. This discussion will be intertwined with explanations of various approaches of computation which were developed and/or used in the literature.

We have kept in mind that the goal is to provide ready-made formulae and code that would be convenient for others to use towards their own ends. Therefore, we prioritized coherence, completeness, accuracy, and usability. Some of results (e.g. L-genus, Chern classes of tensor bundles) reproduce existing ones in other sources in lower degrees and extends them beyond, and so we include for coherence and completeness. Other times we thought it worthwhile to rewrite code that would allow one to make such computations when previous ones were done in what seemed like non-standard programming languages. This is the case, for instance, with the work O. Iena, who developed a library called `chern.lib`, but in a very specialized programming language called SINGULAR[1] in the context of algebraic geometry. While that work is indeed very comprehensive and systematic, the specificity of the SINGULAR environment may be prohibitive for many. We chose *Mathematica* as the programming tool for this paper exactly because it seems to be the most widely used programming environment within the field. Finally, in many cases the seemingly duplicated lists were actually shorter elsewhere and have been extended here (e.g., the Todd genus). In other instances, combinatorial formulas do exist, but we found them to be still quite abstract and not readily utilizable (see, for instance, [23]).

Most of the expressions are truncated around degree 10 but not always so. Some lists of expansions were truncated at a certain degree because their next iteration would have been particularly long. In other instances Mathematica was taking a very long time in producing the next iteration, as was the case with the genera expansions. Consider that to obtain only the 12-th degree for the Todd genus in terms of Pontrjagin classes we had to wait for over 2 hours, whereas the computation of degrees 0 to 10 took around 15 minutes in total. The rest were arbitrarily truncated around degree 9 or 10, trusting that one would be able to make use of the Mathematica function to obtain further results if needed.

[1] See https://www.singular.uni-kl.de/.

2.4 Implementation for Chern classes and Chern character

In [17], via symbolic computations with Chern classes the author works out expansions with the existing formulas, and also demonstrates how alternative ways of computation compare in terms of the computer time taken to solve the problems. That paper is among the fundamental sources upon which the current paper builds. Let us expand on some of the conclusions presented in that paper which are relevant for our case.

For instance, let us consider the observations made in [17] regarding the Chern character computations. For a complex vector bundle of rank r, the sum of the Chern roots raised to the k-th power yields a symmetric polynomial in $x_1, \cdots, x_r$ of degree k, which can be written as a polynomial in $c_1, ..., c_k$. There exists an approach, called the *elimination method*, which allows to work out the computation of $\mathrm{ch}_k(c_1, ..., c_r)$. However, there also exists an alternative and, in fact, faster method which uses *Newton's identities*

$$P_{k+1} = c_1 P_k - c_2 P_{k-1} + ... + (-1)^k (k+1) c_{k+1}, \quad k \in \mathbb{Z}_{\geq 0} \tag{2.6}$$

to compute the polynomials $\mathrm{ch}_k(c_1, ..., c_r) = \frac{1}{k!} P_k(c_1, ..., c_k)$. [2] Explicitly, the first few terms are

$$\begin{aligned}
P_1 &= c_1, \\
P_2 &= c_1 P_1 - 2c_2 = c_1^2 - 2c_2, \\
P_3 &= c_1 P_2 - c_2 P_1 + 3c_3 = c_1^3 - 3c_1 c_2 + 3c_3, \\
P_4 &= c_1 P_3 - c_2 P_2 + 3c_3 P_1 = c_1^4 - 4c_1^2 c_2 + 4c_1 c_3 + 2c_2^2 - 4c_4.
\end{aligned}$$

Since c_k's are integral $\in \mathbb{Z}$, then it means all P_k take values in $\mathbb{Z}$ as well, since they are expressed in terms of the Chern classes without division. It follows that $\mathrm{ch}_k(c_1, ..., c_k) \in \mathbb{Q}[c_1, ..., c_k]$ due to the formula that expresses the $\mathrm{ch}_k(c_1, ..., c_k)$ in terms of the Newton's identities. Also the Newton's identities can be rewritten to define the Chern classes in terms of the Chern character [3]

$$c_{k+1} = \tfrac{1}{k+1}(c_k \cdot \mathrm{ch}_1 - 2! \cdot c_{k-1} \cdot \mathrm{ch}_2 + ... + (-1)^k (k+1)! \cdot \mathrm{ch}_{k+1}). \tag{2.7}$$

According to the conclusions in [17], starting from $k = 8$ the time it takes to compute the components ch_k of the Chern character exponentially increases for the elimination method, whereas there is only slight increase in computer time for the Newton's method. Therefore, Newton's identities prove to be significantly more efficient for higher powers of k [17]. Taking this into account, we used the approach with Newton's identities when expressing jth Chern character in terms of the Chern classes (see *Expansion 1* in Section 3), which we then inverted to write jth Chern class in terms of the Chern character (see *Expansion 3*).

3 Expansions of the Characteristic Classes

The next two sections will present the actual expansions obtained in the course of this paper as well as explanations of the programming approach and at times additional theoretical background.

As indicated in §2.1, characteristic classes arise from the cohomology of classifying spaces BG. They are obtained by pulling back the universal classes via a classifying map $f : X \to BG$, where G is an orthogonal group $O(n)$ or a unitary group $U(n)$ depending on whether the bundle is real or complex, respectively. This approach is summarized in the following classical result (see [22, Theorem 17]).

Theorem 2. *The family of all characteristic classes of rank n real (complex) vector bundles is in one-to-one correspondence with the cohomology ring $H^*(BO(n))$ (respectively, with $H^*(BU(n))$).*

While the Chern classes are cohomology classes assigned to a complex vector bundle, and they are *integral classes*, i.e. taking values in cohomology with $\mathbb{Z}$ coefficients, the Pontrjagin classes are assigned to real vector bundles and can be taken over $\mathbb{Z}$ or the rational numbers $\mathbb{Q}$. Details on the real case will be given in Section 3.2.

Throughout this section we denote the jth Chern class by c_j, the jth Chern character by ch_j, and the jth Pontrjagin class by p_j.

[2] The notation of P indicating the recursive steps should not be confused with the Pontrjagin classes denoted by lower case p.
[3] Note that the above equation had a small error as stated in [17] – the factorial coefficients had been omitted.

3.1 Chern character & Chern classes

Expansion 1. *Computing the ith Chern character in terms of the Chern classes.*

Expansion 1: Chern character in terms of Chern classes

$$
\begin{aligned}
\mathrm{ch}_0 =& r\\
\mathrm{ch}_1 =& c_1\\
\mathrm{ch}_2 =& \tfrac{1}{2}\left[c_1^2 - 2c_2\right]\\
\mathrm{ch}_3 =& \tfrac{1}{2^1}\left[+\tfrac{1^1}{3^1}c_1^3 - 1^1 c_1c_2 + 1^1 c_3\right]\\
\mathrm{ch}_4 =& \tfrac{1}{2^1.3^1}\left[+\tfrac{1^1}{2^2}c_1^4 - 1^1 c_1^2c_2 + 1^1 c_1c_3 + \tfrac{1^1}{2^1}c_2^2 - 1^1 c_4\right]\\
\mathrm{ch}_5 =& \tfrac{1}{2^3.3^1}\left[+\tfrac{1^1}{5^1}c_1^5 - 1^1 c_1^3c_2 + 1^1 c_1^2c_3 + 1^1 c_1c_2^2 - 1^1 c_1c_4 - 1^1 c_2c_3 + 1^1 c_5\right]\\
\mathrm{ch}_6 =& \tfrac{1}{2^2.5^1}\Big[+\tfrac{1^1}{2^2.3^2}c_1^6 - \tfrac{1^1}{2^1.3^1}c_1^4c_2 + \tfrac{1^1}{2^1.3^1}c_1^3c_3 + \tfrac{1^1}{2^2}c_1^2c_2^2 - \tfrac{1^1}{2^1.3^1}c_1^2c_4 - \tfrac{1^1}{3^1}c_1c_2c_3 + \tfrac{1^1}{2^1.3^1}c_1c_5 - \tfrac{1^1}{2^1.3^2}c_2^3\\
&+\tfrac{1^1}{2^1.3^1}c_2c_4 + \tfrac{1^1}{2^2.3^1}c_3^2 - \tfrac{1^1}{2^1.3^1}c_6\Big]\\
\mathrm{ch}_7 =& \tfrac{1}{2^3.3^1.5^1}\Big[+\tfrac{1^1}{2^1.3^1.7^1}c_1^7 - \tfrac{1^1}{2^1.3^1}c_1^5c_2 + \tfrac{1^1}{2^1.3^1}c_1^4c_3 + \tfrac{1^1}{3^1}c_1^3c_2^2 - \tfrac{1^1}{2^1.3^1}c_1^3c_4 - \tfrac{1^1}{2^1}c_1^2c_2c_3 + \tfrac{1^1}{2^1.3^1}c_1^2c_5 - \tfrac{1^1}{2^1.3^1}c_1c_2^3\\
&+\tfrac{1^1}{3^1}c_1c_2c_4 + \tfrac{1^1}{2^1.3^1}c_1c_3^2 - \tfrac{1^1}{2^1.3^1}c_1c_6 + \tfrac{1^1}{2^1.3^1}c_2^2c_3 - \tfrac{1^1}{2^1.3^1}c_2c_5 - \tfrac{1^1}{2^1.3^1}c_3c_4 + \tfrac{1^1}{2^1.3^1}c_7\Big]\\
\mathrm{ch}_8 =& \tfrac{1}{2^2.3^1.7^1}\Big[+\tfrac{1^1}{2^5.3^1.5^1}c_1^8 - \tfrac{1^1}{2^2.3^1.5^1}c_1^6c_2 + \tfrac{1^1}{2^2.3^1.5^1}c_1^5c_3 + \tfrac{1^1}{2^3.3^1}c_1^4c_2^2 - \tfrac{1^1}{2^2.3^1.5^1}c_1^4c_4 - \tfrac{1^1}{3^1.5^1}c_1^3c_2c_3\\
&+\tfrac{1^1}{2^2.3^1.5^1}c_1^3c_5 - \tfrac{1^1}{2^1.3^1.5^1}c_1^2c_2^3 + \tfrac{1^1}{2^2.5^1}c_1^2c_2c_4 + \tfrac{1^1}{2^3.5^1}c_1^2c_3^2 - \tfrac{1^1}{2^2.3^1.5^1}c_1^2c_6 + \tfrac{1^1}{2^2.5^1}c_1c_2^2c_3 -\\
&\tfrac{1^1}{2^1.3^1.5^1}c_1c_2c_5 - \tfrac{1^1}{2^1.3^1.5^1}c_1c_3c_4 + \tfrac{1^1}{2^2.3^1.5^1}c_1c_7 + \tfrac{1^1}{2^4.3^1.5^1}c_2^4 - \tfrac{1^1}{2^2.3^1.5^1}c_2^2c_4 - \tfrac{1^1}{2^2.3^1.5^1}c_2c_3^2 +\\
&\tfrac{1^1}{2^2.3^1.5^1}c_2c_6 + \tfrac{1^1}{2^2.3^1.5^1}c_3c_5 + \tfrac{1^1}{2^3.3^1.5^1}c_4^2 - \tfrac{1^1}{2^2.3^1.5^1}c_8\Big]\\
\mathrm{ch}_9 =& \tfrac{1}{2^5.3^1.7^1}\Big[+\tfrac{1^1}{2^2.3^3.5^1}c_1^9 - \tfrac{1^1}{2^2.3^1.5^1}c_1^7c_2 + \tfrac{1^1}{2^2.3^1.5^1}c_1^6c_3 + \tfrac{1^1}{2^2.5^1}c_1^5c_2^2 - \tfrac{1^1}{2^2.3^1.5^1}c_1^5c_4 - \tfrac{1^1}{2^2.3^1}c_1^4c_2c_3\\
&+\tfrac{1^1}{2^2.3^1.5^1}c_1^4c_5 - \tfrac{1^1}{2^1.3^2}c_1^3c_2^3 + \tfrac{1^1}{3^1.5^1}c_1^3c_2c_4 + \tfrac{1^1}{2^1.3^1.5^1}c_1^3c_3^2 - \tfrac{1^1}{2^2.3^1.5^1}c_1^3c_6 + \tfrac{1^1}{2^1.5^1}c_1^2c_2^2c_3 - \tfrac{1^1}{2^2.5^1}c_1^2c_2c_5\\
&-\tfrac{1^1}{2^2.5^1}c_1^2c_3c_4 + \tfrac{1^1}{2^2.3^1.5^1}c_1^2c_7 + \tfrac{1^1}{2^2.3^1.5^1}c_1c_2^4 - \tfrac{1^1}{2^2.5^1}c_1c_2^2c_4 - \tfrac{1^1}{2^2.5^1}c_1c_2c_3^2 + \tfrac{1^1}{2^1.3^1.5^1}c_1c_2c_6\\
&+\tfrac{1^1}{2^1.3^1.5^1}c_1c_3c_5 + \tfrac{1^1}{2^2.3^1.5^1}c_1c_4^2 - \tfrac{1^1}{2^2.3^1.5^1}c_1c_8 - \tfrac{1^1}{2^2.3^1.5^1}c_2^3c_3 + \tfrac{1^1}{2^2.3^1.5^1}c_2^2c_5 + \tfrac{1^1}{2^1.3^1.5^1}c_2c_3c_4\\
&-\tfrac{1^1}{2^2.3^1.5^1}c_2c_7 + \tfrac{1^1}{2^2.3^2.5^1}c_3^3 - \tfrac{1^1}{2^2.3^1.5^1}c_3c_6 - \tfrac{1^1}{2^2.3^1.5^1}c_4c_5 + \tfrac{1^1}{2^2.3^1.5^1}c_9\Big]
\end{aligned}
$$

Expansion 2. *Chern classes for Calabi-Yau or SU-bundles:* $c_1 = 0$

We have the following nice simplification.

Expansion 2: Chern character for an SU-bundle E in terms of Chern classes

$$
\begin{aligned}
\mathrm{ch}_0 =& r\\
\mathrm{ch}_1 =& 0\\
\mathrm{ch}_2 =& -c_2\\
\mathrm{ch}_3 =& \tfrac{1}{2^1}c_3\\
\mathrm{ch}_4 =& \tfrac{1}{2^2.3^1}\left[c_2^2 - 2c_4\right]\\
\mathrm{ch}_5 =& \tfrac{1}{2^3.3^1}\left[-c_2c_3 + c_5\right]\\
\mathrm{ch}_6 =& \tfrac{1}{2^4.3^2.5^1}\left[-2c_2^3 + 3c_3^2 + 6c_2c_4 - 6c_6\right]\\
\mathrm{ch}_7 =& \tfrac{1}{2^4.3^2.5^1}\left[c_2^2c_3 - c_3c_4 - c_2c_5 + c_7\right]
\end{aligned}
$$

Now we can invert the above relations $\mathrm{ch}_i(c)$ to get $c_j(\mathrm{ch})$ in the following.

Expansion 3. *Computing the ith Chern class in terms of Chern character.*

Expansion 3: Chern classes in terms of the Chern character

$$
\begin{aligned}
c_0 =& 1\\
c_1 =& \tfrac{1}{1^1}\mathrm{ch}_1\\
c_2 =& \tfrac{1}{2^1}\left[\mathrm{ch}_1^2 - 2^1\mathrm{ch}_2\right]\\
c_3 =& \tfrac{1}{2^1\cdot 3^1}\left[\mathrm{ch}_1^3 - 2^1\cdot 3^1\mathrm{ch}_1\mathrm{ch}_2 + 2^2\cdot 3^1\mathrm{ch}_3\right]\\
c_4 =& \tfrac{1}{2^3\cdot 3^1}\left[\mathrm{ch}_1^4 - 2^2\cdot 3^1\mathrm{ch}_1^2\mathrm{ch}_2 + 2^2\cdot 3^1\mathrm{ch}_2^2 + 2^4\cdot 3^1\mathrm{ch}_1\mathrm{ch}_3 - 2^4\cdot 3^2\mathrm{ch}_4\right]\\
c_5 =& \tfrac{1}{2^3\cdot 3^1\cdot 5^1}\Big[\mathrm{ch}_1^5 - 2^2\cdot 5^1\mathrm{ch}_1^3\mathrm{ch}_2 + 2^2\cdot 3^1\cdot 5^1\mathrm{ch}_1\mathrm{ch}_2^2 + 2^3\cdot 3^1\cdot 5^1\mathrm{ch}_1^2\mathrm{ch}_3 - 2^4\cdot 3^1\cdot 5^1\mathrm{ch}_2\mathrm{ch}_3 - 2^4\cdot 3^2\cdot\\
&5^1\mathrm{ch}_1\mathrm{ch}_4 + 2^6\cdot 3^2\cdot 5^1\mathrm{ch}_5\Big]\\
c_6 =& \tfrac{1}{2^4\cdot 3^2\cdot 5^1}\Big[\mathrm{ch}_1^6 - 2^1\cdot 3^1\cdot 5^1\mathrm{ch}_1^4\mathrm{ch}_2 + 2^2\cdot 3^2\cdot 5^1\mathrm{ch}_1^2\mathrm{ch}_2^2 - 2^3\cdot 3^1\cdot 5^1\mathrm{ch}_2^3 + 2^4\cdot 3^1\cdot 5^1\mathrm{ch}_1^3\mathrm{ch}_3 - 2^5\cdot 3^2\cdot 5^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_3\\
&+ 2^5\cdot 3^2\cdot 5^1\mathrm{ch}_3^2 - 2^4\cdot 3^3\cdot 5^1\mathrm{ch}_1^2\mathrm{ch}_4 + 2^5\cdot 3^3\cdot 5^1\mathrm{ch}_2\mathrm{ch}_4 + 2^7\cdot 3^3\cdot 5^1\mathrm{ch}_1\mathrm{ch}_5 - 2^7\cdot 3^3\cdot 5^2\mathrm{ch}_6\Big]\\
c_7 =& \tfrac{1}{2^4\cdot 3^2\cdot 5^1\cdot 7^1}\Big[\mathrm{ch}_1^7 - 2^1\cdot 3^1\cdot 7^1\mathrm{ch}_1^5\mathrm{ch}_2 + 2^2\cdot 3^1\cdot 5^1\cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_2^2 - 2^3\cdot 3^1\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_2^3 + 2^2\cdot 3^1\cdot 5^1\cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_3\\
&- 2^4\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2\mathrm{ch}_3 + 2^4\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_2^2\mathrm{ch}_3 + 2^5\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_3^2 - 2^4\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_4\\
&+ 2^5\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_4 - 2^6\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_3\mathrm{ch}_4 + 2^6\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_5 - 2^7\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_2\mathrm{ch}_5\\
&- 2^7\cdot 3^3\cdot 5^2\cdot 7^1\mathrm{ch}_1\mathrm{ch}_6 + 2^8\cdot 3^4\cdot 5^2\cdot 7^1\mathrm{ch}_7\Big]\\
c_8 =& \tfrac{1}{2^7\cdot 3^2\cdot 5^1\cdot 7^1}\Big[\mathrm{ch}_1^8 - 2^3\cdot 7^1\mathrm{ch}_1^6\mathrm{ch}_2 + 2^3\cdot 3^1\cdot 5^1\cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_2^2 - 2^5\cdot 3^1\cdot 5^1\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2^3 + 2^4\cdot 3^1\cdot 5^1\cdot 7^1\mathrm{ch}_2^4 + 2^5\cdot 3^1\cdot 7^1\mathrm{ch}_1^5\mathrm{ch}_3\\
&- 2^7\cdot 3^1\cdot 5^1\cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_2\mathrm{ch}_3 + 2^7\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_2^2\mathrm{ch}_3 + 2^7\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_3^2 - 2^8\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_2\mathrm{ch}_3^2\\
&- 2^5\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_4 + 2^7\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2\mathrm{ch}_4 - 2^7\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_2^2\mathrm{ch}_4 - 2^9\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_3\mathrm{ch}_4\\
&+ 2^8\cdot 3^4\cdot 5^1\cdot 7^1\mathrm{ch}_4^2 + 2^9\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_5 - 2^{10}\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_5 + 2^{11}\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_3\mathrm{ch}_5 - 2^9\cdot 3^3\cdot 5^2\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_6\\
&+ 2^{10}\cdot 3^3\cdot 5^2\cdot 7^1\mathrm{ch}_2\mathrm{ch}_6 + 2^{11}\cdot 3^4\cdot 5^2\cdot 7^1\mathrm{ch}_1\mathrm{ch}_7 - 2^{11}\cdot 3^4\cdot 5^2\cdot 7^2\mathrm{ch}_8\Big]\\
c_9 =& \tfrac{1}{2^7\cdot 3^4\cdot 5^1\cdot 7^1}\Big[\mathrm{ch}_1^9 - 2^3\cdot 3^2\mathrm{ch}_1^7\mathrm{ch}_2 + 2^3\cdot 3^3\cdot 7^1\mathrm{ch}_1^5\mathrm{ch}_2^2 - 2^5\cdot 3^2\cdot 5^1\cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_2^3 + 2^4\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_2^4 + 2^4\cdot 3^2\cdot 7^1\mathrm{ch}_1^6\mathrm{ch}_3\\
&- 2^5\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_2\mathrm{ch}_3 + 2^6\cdot 3^4\cdot 5^1\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2^2\mathrm{ch}_3 - 2^7\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_2^3\mathrm{ch}_3 + 2^7\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_3^2\\
&- 2^8\cdot 3^4\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_3^2 + 2^9\cdot 3^3\cdot 5^1\cdot 7^1\mathrm{ch}_3^3 - 2^5\cdot 3^4\cdot 7^1\mathrm{ch}_1^5\mathrm{ch}_4 + 2^7\cdot 3^4\cdot 5^1\cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_2\mathrm{ch}_4\\
&- 2^7\cdot 3^5\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_2^2\mathrm{ch}_4 - 2^8\cdot 3^5\cdot 5^1\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_3\mathrm{ch}_4 + 2^9\cdot 3^5\cdot 5^1\cdot 7^1\mathrm{ch}_2\mathrm{ch}_3\mathrm{ch}_4 + 2^8\cdot 3^6\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_4^2\\
&+ 2^7\cdot 3^4\cdot 5^1\cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_5 - 2^9\cdot 3^5\cdot 5^1\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2\mathrm{ch}_5 + 2^9\cdot 3^5\cdot 5^1\cdot 7^1\mathrm{ch}_2^2\mathrm{ch}_5 + 2^{11}\cdot 3^5\cdot 5^1\cdot 7^1\mathrm{ch}_1\mathrm{ch}_3\mathrm{ch}_5\\
&- 2^{11}\cdot 3^6\cdot 5^1\cdot 7^1\mathrm{ch}_4\mathrm{ch}_5 - 2^9\cdot 3^4\cdot 5^2\cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_6 + 2^{10}\cdot 3^5\cdot 5^2\cdot 7^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_6 - 2^{11}\cdot 3^5\cdot 5^2\cdot 7^1\mathrm{ch}_3\mathrm{ch}_6\\
&+ 2^{10}\cdot 3^6\cdot 5^2\cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_7 - 2^{11}\cdot 3^6\cdot 5^2\cdot 7^1\mathrm{ch}_2\mathrm{ch}_7 - 2^{11}\cdot 3^6\cdot 5^2\cdot 7^2\mathrm{ch}_1\mathrm{ch}_8 + 2^{14}\cdot 3^6\cdot 5^2\cdot 7^2\mathrm{ch}_9\Big]
\end{aligned}
$$

Sometimes it is desirable to consider situations when only even Chern classes are nonzero. For instance, when complexifying a real bundle (see subsection 3.2 below for details). The summary is that for E a *real* vector bundle denote by $E\otimes\mathbb{C}$ its *complexification*. From this, the "odd" Chern classes $c_{2i+1}(E_{\mathbb{R}}\otimes\mathbb{C})$ vanish for $i = 0, 1, \cdots$. Another instance is when the underlying space has cohomology of degree $4k$. With this in mind the above expansion is simplified by vanishing all odd Chern classes to obtain the result below, in which all the corresponding odd degree Chern characters are identically zero.

Expansion 4. *Chern character expansion in terms of only the even Chern classes ($c_{2i+1} = 0$).*

Expansion 4: Total Chern character expansion in terms of only the even Chern classes

$$
\begin{aligned}
\mathrm{ch}_0 =& r\\
\mathrm{ch}_2 =& -c_2\\
\mathrm{ch}_4 =& \tfrac{1}{2^2\cdot 3^1}\left[c_2^2 - 2c_4\right]\\
\mathrm{ch}_6 =& \tfrac{1}{2^3\cdot 3^2\cdot 5^1}\left[-c_2^3 + 3c_2c_4 - 3c_6\right]\\
\mathrm{ch}_8 =& \tfrac{1}{2^6\cdot 3^2\cdot 5^1\cdot 7^1}\left[c_2^4 + 2c_4^2 - 4c_2^2c_4 + 4c_2c_6 - 4c_8\right]
\end{aligned}
$$

3.2 Pontrjagin classes

Pontrjagin classes

In the real case, there are two complications: The first is the presence of torsion in the cohomology of the classifying space $BO(n)$, which forces one to study the generators with $\mathbb{Z}_2$ coefficients and with rational $\mathbb{Q}$ coefficients. For our purposes the latter is enough. The second complication is that the ring for $BO(n)$ will take a slightly different form depending on whether n is even or odd. We will provide the precise statement although we will not be highlighting such differences. Again, the following fundamental statement is classical (see[22, Theorem 19]).

Theorem 3. *The ring $H^*(BSO(2m);\mathbb{Q})$ is isomorphic to the ring of polynomials $\mathbb{Q}[p_1, p_2, \cdots, p_{m-1}, \chi]$, where*

$$p_k \in H^{4k}(BSO(2m);\mathbb{Q}), \chi \in H^{2m}(BSO(2m);\mathbb{Q}) ,$$

while the ring $H^*(BSO(2m+1);\mathbb{Q})$ is isomorphic to the ring of polynomials $\mathbb{Q}[p_1, p_2, \cdots, p_m]$, where

$$p_k \in H^{4k}(BSO(2m+1);\mathbb{Q}) .$$

We will not consider the Euler class χ separately as in many cases it can be determined by the corresponding Pontrjagin class. The generators can be chosen so that similar axioms to those of the Chern classes are satisfied.

Pontrjagin classes have the property (again rationally)

$$p(E \oplus F) = p(E) \cdot p(F). \tag{3.1}$$

Chern and Pontrjagin classes are directly related. As we stated above Chern classes are defined only for complex vector bundles; consequently in order to compare it is necessary to *complexify* (convert from $\mathbb{R} \to \mathbb{C}$) the fiber of the vector bundle E. The corresponding relation is

$$p_j(E) = (-1)^j c_{2j}(E_{\mathbb{C}}). \tag{3.2}$$

Complexification and realification

The complexification of a real vector space V is defined to be $V_{\mathbb{C}} = V \oplus V$, in which a pair (v_1, v_2) can be thought of as a formal sum $v_1 + iv_2$, with multiplication law $(a+bi)(v_1, v_2) = (av_1 - bv_2, bv_1 + av_2)$, where a and b are real.

Let E be a *real* vector bundle and $E \otimes \mathbb{C}$ its complexification. Then as shown in [27, Lemma 15.1], the bundle $E \otimes \mathbb{C}$ is isomorphic to its own conjugate bundle $\overline{E \otimes \mathbb{C}}$. Therefore, the "odd" Chern classes $c_1(E \otimes \mathbb{C})$, $c_3(E \otimes \mathbb{C})$, $\cdots$, are all torsion elements of order 2. As for the "even" Chern classes, $c_{2i}(E \otimes \mathbb{C}) = (-1)^i p_i(E)$, $i = 1, 2, \cdots$, by definition of Pontrjagin classes (see [27, §15, Definition]). On the other hand, if one starts from a *complex* vector bundle E then, as shown in [27, Corollary 15.5] (see also [16, Remark 6.2]) , one has

$$\begin{aligned} c_1(E_{\mathbb{R}} \otimes \mathbb{C}) &= c_1(E \oplus \overline{E}) = c_1(E) + c_1(\overline{E}) = c_1(E) - c_1(E) = 0 \\ c_2(E_{\mathbb{R}} \otimes \mathbb{C}) &= c_2(E \oplus \overline{E}) = c_2(E) + c_1(E)c_1(\overline{E}) = 2c_2(E) - c_1(E)^2 = -p_1(E_{\mathbb{R}}), \end{aligned}$$

the first Pontrjagin class. For higher degrees, this works similarly with the pattern of even/odd persisting.

Likewise, forgetting the complex structure on a complex vector bundle E turns it into a real vector bundle $E_{\mathbb{R}}$. This operation is called the realification of a complex vector bundle E. Note that the composition of realification and complexification is (see [22, Sec. 1.4.1])

$$rcE = E \oplus E \qquad \text{while} \qquad crE = E \oplus \overline{E} .$$

In explicit terms the reader will find Pontrjagin classes in terms of Chern classes presented in section 3 in *Expansion 5.* The above equation will be very important for the practical calculations later on and we will refer back to it frequently.

The next expansions employs the concept of realification, when an initially complex bundle is converted to the real space $E \to E_{\mathbb{R}}$ [27]. The function is implemented according to Corollary 15.5 in [27], which states that for any complex vector bundle E, the Chern classes of $c_i(E)$ determine the Pontrjagin classes $p_k(E_{\mathbb{R}})$ by the formula

$$1 - p_1 + p_2 - ... \pm p_n = (1 - c_1 + c_2 - ... \pm c_n)(1 + c_1 + c_2 + ... + c_n). \tag{3.3}$$

Therefore,

$$p_k(E_{\mathbb{R}}) = c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + ... \pm 2c_1(E)c_{2k-1}(E) \pm 2c_{2k}(E). \tag{3.4}$$

Expansion 5. *Pontrjagin classes of a realified bundle $E_{\mathbb{R}}$ in terms of Chern classes of a complex bundle E.*

Expansion 5: Pontrjagin classes of a realified bundle in terms of Chern classes of a complex bundle

$$
\begin{aligned}
p_0 =& 1\\
p_1 =& c_1^2 - 2c_2\\
p_2 =& c_2^2 - 2c_1c_3 + 2c_4\\
p_3 =& c_3^2 - 2c_2c_4 + 2c_1c_5 - 2c_6\\
p_4 =& c_4^2 - 2c_3c_5 + 2c_2c_6 - 2c_1c_7 + 2c_8\\
p_5 =& c_5^2 - 2c_4c_6 + 2c_3c_7 - 2c_2c_8 + 2c_1c_9 - 2c_{10}\\
p_6 =& c_6^2 - 2c_5c_7 + 2c_4c_8 - 2c_3c_9 + 2c_2c_{10} - 2c_1c_{11} + 2c_{12}\\
p_7 =& c_7^2 - 2c_6c_8 + 2c_5c_9 - 2c_4c_{10} + 2c_3c_{11} - 2c_2c_{12} + 2c_1c_{13} - 2c_{14}\\
p_8 =& c_8^2 - 2c_7c_9 + 2c_6c_{10} - 2c_5c_{11} + 2c_4c_{12} - 2c_3c_{13} + 2c_2c_{14} - 2c_1c_{15} + 2c_{16}
\end{aligned}
$$

We now provide a similar computation using the Chern character of a complex bundle.

Expansion 6. *Pontrjagin classes of realified bundle $E_{\mathbb{R}}$ expressed in terms of Chern character of a bundle E.*

Expansion 6: Pontrjagin classes of realified bundle $E_{\mathbb{R}}$ expressed in terms of Chern character of a bundle E

$$
\begin{aligned}
p_1 =& 2\mathrm{ch}_2\\
p_2 =& 2(\mathrm{ch}_2^2 - 6\mathrm{ch}_4)\\
p_3 =& \tfrac{4}{3}\left(\mathrm{ch}_2^3 - 18\mathrm{ch}_2\mathrm{ch}_4 + 180\mathrm{ch}_6\right)\\
p_4 =& \tfrac{2}{3}\mathrm{ch}_2^4 - 24\mathrm{ch}_2^2\mathrm{ch}_4 + 480\mathrm{ch}_2\mathrm{ch}_6 + 72\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right)\\
p_5 =& \tfrac{4}{15}\mathrm{ch}_2^5 - 16\mathrm{ch}_2^3\mathrm{ch}_4 + 480\mathrm{ch}_2^2\mathrm{ch}_6 + 144\mathrm{ch}_2\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 2880\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right)\\
p_6 =& \tfrac{4}{45}\mathrm{ch}_2^6 - 8\mathrm{ch}_2^4\mathrm{ch}_4 + 320\mathrm{ch}_2^3\mathrm{ch}_6 + 144\mathrm{ch}_2^2\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 5760\mathrm{ch}_2\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right) -\\
&- 288\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right]\\
p_7 =& \tfrac{8}{315}\mathrm{ch}_2^7 - \tfrac{16}{5}\mathrm{ch}_2^5\mathrm{ch}_4 + 160\mathrm{ch}_2^4\mathrm{ch}_6 + 96\mathrm{ch}_2^3\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 5760\mathrm{ch}_2^2\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right) -\\
&- 576\mathrm{ch}_2\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right] + 17280\left(\mathrm{ch}_4^2\mathrm{ch}_6 - 140\mathrm{ch}_6\mathrm{ch}_8 - 504\mathrm{ch}_4\mathrm{ch}_{10} + 720720\mathrm{ch}_{14}\right)\\
p_8 =& \tfrac{2}{315}\mathrm{ch}_2^8 - \tfrac{16}{15}\mathrm{ch}_2^6\mathrm{ch}_4 + 64\mathrm{ch}_2^5\mathrm{ch}_6 + 48\mathrm{ch}_2^4\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 3840\mathrm{ch}_2^3\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right) -\\
&- 576\mathrm{ch}_2^2\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right] + 34560\mathrm{ch}_2\left(\mathrm{ch}_6\mathrm{ch}_4^2 - 140\mathrm{ch}_6\mathrm{ch}_8 - 504\mathrm{ch}_4\mathrm{ch}_{10} + 720720\mathrm{ch}_{14}\right) +\\
&+ 864\left[\mathrm{ch}_4^4 - 840\mathrm{ch}_4^2\mathrm{ch}_8 - 400\mathrm{ch}_4\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right) + 8400\left(7\mathrm{ch}_8^2 + 24\mathrm{ch}_6\mathrm{ch}_{10} - 360360\mathrm{ch}_{16}\right)\right]
\end{aligned}
$$

Sometimes one encounters complex bundles with vanishing c_1 and c_2, or real bundles with vanishing p_1, such as the String structures presented later on in Section 4. Hence, we compute a simplification of the above expansion by vanishing Chern character ch_2. Note that vanishing of ch_1 is pointless, since all odd Chern characters have dimension $4k+2$, whereas the Pontrjagin classes p_k have dimension $4k$ which means that the odd Chern classes are already absent. Note that ch_1 does not appear in the above expressions, but ch_2 does, so we set it to zero.

Expansion 7. *Pontrjagin classes expressed in terms of Chern character with simplification ($\mathrm{ch}_2 = 0$).*

Expansion 7: Pontrjagin classes expressed in terms of Chern character with zero second Chern character

$$
\begin{aligned}
p_1 =& 0\\
p_2 =& -12\mathrm{ch}_4\\
p_3 =& 240\mathrm{ch}_6\\
p_4 =& 72\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right)\\
p_5 =& -2880\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right)\\
p_6 =& -288\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right]\\
p_7 =& 17280\left(\mathrm{ch}_4^2\mathrm{ch}_6 - 140\mathrm{ch}_6\mathrm{ch}_8 - 504\mathrm{ch}_4\mathrm{ch}_{10} + 720720\mathrm{ch}_{14}\right)\\
p_8 =& 864\left[\mathrm{ch}_4^4 - 840\mathrm{ch}_4^2\mathrm{ch}_8 - 400\mathrm{ch}_4\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right) + 8400\left(7\mathrm{ch}_8^2 + 24\mathrm{ch}_6\mathrm{ch}_{10} - 360360\mathrm{ch}_{16}\right)\right]\\
p_9 =& -23040[3\mathrm{ch}_4^3\mathrm{ch}_6 - 2268\mathrm{ch}_4^2\mathrm{ch}_{10} - 1260\mathrm{ch}_4\left(\mathrm{ch}_6\mathrm{ch}_8 - 5148\mathrm{ch}_{14}\right) - 20\left(5\mathrm{ch}_6^3 - 41580\mathrm{ch}_6\mathrm{ch}_{12} - 15876\left(\mathrm{ch}_8\mathrm{ch}_{10} - 97240\mathrm{ch}_{18}\right)\right)]
\end{aligned}
$$

Now consider the restriction to $\mathrm{ch}_2 = \mathrm{ch}_4 = 0$:

Expansion 8. *Pontrjagin classes expressed in terms of Chern character with zero* ch_2 *and* ch_4.

Expansion 8: Pontrjagin classes expressed in terms of Chern character with zero ch_2 and ch_4

$$
\begin{aligned}
p_1 &= 0\\
p_2 &= 0\\
p_3 &= 240\mathrm{ch}_6\\
p_4 &= 72(140\mathrm{ch}_8)\\
p_5 &= -2880\,(-252\mathrm{ch}_{10})\\
p_6 &= -288\left[-100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right]\\
p_7 &= 17280\,(-140\mathrm{ch}_6\mathrm{ch}_8 + 720720\mathrm{ch}_{14})\\
p_8 &= 864\left[8400\left(7\mathrm{ch}_8^2 + 24\mathrm{ch}_6\mathrm{ch}_{10} - 360360\mathrm{ch}_{16}\right)\right]\\
p_9 &= -23040[-20\Big(5\mathrm{ch}_6^3 - 41580\mathrm{ch}_6\mathrm{ch}_{12} - 15876\,(\mathrm{ch}_8\mathrm{ch}_{10} - 97240\mathrm{ch}_{18})\Big)]
\end{aligned}
$$

Observations: Note that we do get the Ninebrane class as ch_6 with the expected numerical factor $2 \cdot 5!$. For p_4 the factor is $2 \cdot 7!$. For p_5 it is $2 \cdot 9!$ etc. This pattern then gives

$$p_i = 2(2i+1)!\mathrm{ch}_{2i} \qquad \text{when } \mathrm{ch}_{2j} = 0\ ,\ j < i\ .$$

This allows one to read off the obstructions in the Whitehead tower up to a factor of two appearing in alternating degrees (see [32][33][31]).

Now we invert the above expressions.

Expansion 9. *Chern character in terms of Pontrjagin classes.*

Expansion 9: Chern character in terms of Pontrjagin classes

$$
\begin{aligned}
\mathrm{ch}_0 &= r\\
\mathrm{ch}_2 &= \tfrac{1}{2}p_1\\
\mathrm{ch}_4 &= \tfrac{1}{2^3.3^1}[p_1^2 - 2p_2]\\
\mathrm{ch}_6 &= \tfrac{1}{2^4.3^2.5^1}[p_1^3 - 3p_1p_2 + 3p_3]\\
\mathrm{ch}_8 &= \tfrac{1}{2^7.3^2.5^1.7^1}[p_1^4 + 2p_2^2 - 4p_1^2p_2 + 4p_1p_3 - 4p_4]\\
\mathrm{ch}_{10} &= \tfrac{1}{2^8.3^4.5^2.7^1}[p_1^5 + 5p_1p_2^2 - 5p_1^3p_2 + 5p_1^2p_3 - 5p_2p_3 - 5p_1p_4 + 5p_5]\\
\mathrm{ch}_{12} &= \tfrac{1}{2^{10}.3^5.5^2.7^1.11^1}[p_1^6 - 2p_2^3 + 9p_1^2p_2^2 + 3p_3^2 - 6p_1^4p_2 + 6p_1^3p_3 - 12p_1p_2p_3 - 6p_1^2p_4 + 6p_2p_4 + 6p_1p_5 - 6p_6]\\
\mathrm{ch}_{14} &= \tfrac{1}{2^{11}.3^5.5^2.7^2.11^1.13^1}[p_1^7 - 7p_1p_2^3 + 14p_1^3p_2^2 + 7p_1p_3^2 - 7p_1^5p_2 + 7p_1^4p_3 - 21p_1^2p_2p_3 + 7p_2^2p_3 - 7p_1^3p_4 + 14p_1p_2p_4\\
&\quad - 7p_3p_4 + 7p_1^2p_5 - 7p_2p_5 - 7p_1p_6 + 7p_7]
\end{aligned}
$$

Sometimes it is desirable to study vector bundles with a specific rank. The following sums ch_j's from the previous expansion up to the rank of the vector bundle E.

Expansion 10. *Total Chern character in terms of Pontrjagin classes for a rank r bundle E.*

Expansion 10: Total Chern character for a rank r bundle E in terms of Pontrjagin classes

$$
\begin{aligned}
\mathrm{ch}_0(E) &= r\\
\mathrm{ch}_2(E) &= \tfrac{1}{2^1}p_1\\
\mathrm{ch}_4(E) &= \tfrac{1}{2^3.3^1}\left[p_1^2 - 2p_2\right]\\
\mathrm{ch}_6(E) &= \tfrac{1}{2^4.3^2.5^1}\left[p_1^3 - 3p_1p_2 + 3p_3\right]\\
\mathrm{ch}_8(E) &= \tfrac{1}{2^7.3^2.5^1.7^1}[p_1^4 + 2p_2^2 - 4p_1^2p_2 + 4p_1p_3 - 4p_4]\\
\mathrm{ch}_{10}(E) &= \tfrac{1}{2^8.3^4.5^2.7^1}[p_1^5 + 5p_1p_2^2 - 5p_1^3p_2 + 5p_1^2p_3 - 5p_2p_3 - 5p_1p_4 + 5p_5]
\end{aligned}
$$

3.3 Tensor product

Tensor product

In [17] computations of Chern classes of a tensor product of two vector bundles are also performed. From the theoretical point of view these computations are possible due to the splitting principle (see Section 2.2). Let us consider an example on how to compute Chern classes of a tensor product of two vector bundles (see [34]).

Example: Let E be a vector bundle and L a line bundle. How to obtain Chern classes of $E \otimes L$ in terms of E and L? Consider, first case when E is a line bundle, then

$$c(E \otimes L) = 1 + c_1(E \otimes L) = 1 + c_1(E) + c_1(L).$$

If E is decomposable into line bundles, $E = L_1 \oplus ... \oplus L_r$ where r is the rank of the vector bundle, then by linearity it follows that $c(E) = (1 + c_1(L_1))...(1 + c_1(L_r))$. Further, we have $E \otimes L = (L_1 \otimes L) \oplus ... \oplus (L_r \otimes L)$, hence

$$\begin{aligned} c(E \otimes L) &= (1 + c_1(L_1) + c_1(L))...(1 + c_1(L_r) + c_1(L)) \\ &= 1 + (c_1(E) + rc_1(L)) + \left(c_2(E) + (r-1)c_1(E)c_1(L) + \binom{r}{2} c_1(L)^2\right) + ... \end{aligned}$$

In fact, the result of the above equation holds even without the assumption that the vector bundle is decomposable or split. In general, for any vector bundle E and line bundle L the expression just stated shows the breakdown of the Chern class of a tensor product of two bundles into known Chern classes of the two elements by the splitting principle. The general formula for the tensor product of two vector bundles is

$$c(E \otimes F) = \prod_{1 \le i \le r,\ 1 \le j \le s} (1 + a_i + b_j),$$

where a_i and b_j are the Chern roots of vector bundles E and F, respectively.

Regarding computerized calculations, there are three methods for computing $c(E \otimes F)$: Elimination method, Lascoux formula [19], and multiplicity of the Chern character method [17]. The latter uses the tensor product property $\text{ch}(E \otimes F) = \text{ch}(E) \cdot \text{ch}(F)$, together with Newton's identities. It appears to be the most efficient approach as demonstrated in [17]. Hence, it is quite natural that we followed this method in reproducing the expansions in Section 3.3. In [17] the reader will find explicit formulas of Chern classes of a tensor product of two vector bundles up to degree four. [4]

We also would like to mention L. Manivel's paper [23], which describes formulae that may assist one in obtaining the same results. However, the paper itself does not contain explicit expansions of Chern classes of tensor bundles and more work is needed to extract them. One could even argue that such expressions are still rather abstract and not very efficient [18].

Throughout this section r denotes the rank of the bundle E_r and its Chern classes are c_i, whereas R denotes the rank of the bundle F_R and its Chern classes are C_i.

Chern character of a tensor product

Essentially, the way we perform the computations below is by first expanding the Chern character as

$$\begin{aligned} &\text{ch}_0(E \otimes F) + \text{ch}_1(E \otimes F) + ... \\ = \ &[\text{ch}_0(E) + \text{ch}_1(E) + \text{ch}_2(E) + ...][\text{ch}_0(F) + \text{ch}_1(F) + \text{ch}_2(F) + ...] \ , \end{aligned}$$

and then match the total degrees on the right hand side with the degree j of $\text{ch}_j(E \otimes F)$ on the left hand side and write them out sequentially.

[4] However, we have to remark that the expansion of $c_3(E \otimes F)$ has an error on the third line - ranks of F and E are swapped in two of the parenthesis.

Expansion 11. *Chern character of a tensor product of two complex bundles E_r and F_R.*

Expansion 11: Chern character of tensor product of two vector bundles

$$\begin{aligned}
\mathrm{ch}_0(E_r \otimes F_R) =& rR \\
\mathrm{ch}_1(E_r \otimes F_R) =& Rc_1 + rC_1 \\
\mathrm{ch}_2(E_r \otimes F_R) =& \tfrac{1}{2^1}\left[R(c_1^2 - 2c_2) + 2c_1C_1 + r(C_1^2 - 2C_2)\right] \\
\mathrm{ch}_3(E_r \otimes F_R) =& \tfrac{1}{2^1 \cdot 3^1}\left[R(c_1^3 - 3c_1c_2 + 3c_3) + 3\left(c_1^2 - 2c_2\right)C_1 + 3c_1\left(C_1^2 - 2C_2\right) + r\left(C_1^3 - 3C_1C_2 + 3C_3\right)\right] \\
\mathrm{ch}_4(E_r \otimes F_R) =& \tfrac{1}{2^3 \cdot 3^1}[R(c_1^4 + 2c_2^2 - 4c_1^2c_2 + 4c_1c_3 - 4c_4) + r(C_1^4 + 2C_2^2 - 4C_1^2C_2 + 4C_1C_3 - 4C_4) \\
& + 4C_1(c_1^3 - 3c_1c_2 + 3c_3) + 4c_1\left(C_1^3 - 3C_1C_2 + 3C_3\right) + 6c_1^2C_1^2 - 12(c_1^2C_2 + c_2C_1^2) + 24c_2C_2] \\
\mathrm{ch}_5(E_r \otimes F_R) =& \tfrac{1}{2^3 \cdot 3^1 \cdot 5^1}\Big[R[c_1^5 - 5c_1^3c_2 + 5c_1^2c_3 - 5c_2c_3 + 5c_1(c_2^2 - c_4) + 5c_5] + r[C_1^5 - 5C_1^3C_2 + 5C_1^2C_3 - 5C_2C_3 + 5C_1(C_2^2 \\
& - C_4) + 5C_5] + 10(c_1^3C_1^2 + c_1^2C_1^3) + 30(c_3C_1^2 + c_1^2C_3) + 5(c_1^4C_1 + c_1C_1^4) + 10(c_2^2C_1 + c_1C_2^2) - 20(c_1^3C_2 \\
& + c_2C_1^3) - 20(c_4C_1 + c_1C_4) - 30(c_1^2C_1C_2 + c_1c_2C_1^2) - 20(c_1^2c_2C_1 + c_1C_1^2C_2) + 20(c_1c_3C_1 + c_1C_1C_3) \\
& - 60(c_3C_2 + c_2C_3) + 60(c_2C_1C_2 + c_1c_2C_2)\Big]
\end{aligned}$$

Simplifications

We now provide the simplifications, imposing conditions appropriate for the tangential structure desired. These include instances when the structured bundle can be viewed as the lift of the tangent bundle (e.g. Calabi-Yau, complex version of String structure, etc.). Hence we are tensoring the tangent bundle with an auxiliary bundle such that either or both may admit an extra structure.

Expansion 12. *Chern character of a tensor product of two bundles E_r and F_R, where E_r is a special unitary bundle* $\mathrm{SU}(r)$ *with $c_1 = 0$.*

Expansion 12: Chern character of tensor product of a vector bundle with an SU-bundle

$$\begin{aligned}
\mathrm{ch}_0(E_r \otimes F_R) =& rR \\
\mathrm{ch}_1(E_r \otimes F_R) =& rC_1 \\
\mathrm{ch}_2(E_r \otimes F_R) =& -Rc_2 + \tfrac{1}{2^1}r\left(C_1^2 - 2C_2\right) \\
\mathrm{ch}_3(E_r \otimes F_R) =& \tfrac{1}{2^1 \cdot 3^1}[3Rc_3 - 6c_2C_1 + r\left(C_1^3 - 3C_1C_2 + 3C_3\right)] \\
\mathrm{ch}_4(E_r \otimes F_R) =& \tfrac{1}{2^3 \cdot 3^1}[2R(c_2^2 - 2c_4) + r(C_1^4 + 2C_2^2 - 4C_1^2C_2 + 4C_1C_3 - 4C_4) + 12C_1c_3 - 12c_2C_1^2 + 24c_2C_2] \\
\mathrm{ch}_5(E_r \otimes F_R) =& \tfrac{1}{2^3 \cdot 3^1 \cdot 5^1}\Big[5R(-c_2c_3 + c_5) + r[C_1^5 - 5C_1^3C_2 + 5C_1^2C_3 - 5C_2C_3 + 5C_1(C_2^2 - C_4) + 5C_5] \\
& + 10C_1(3c_3C_1 + c_2^2 - 2c_2C_1^2 - 2c_4) - 60(c_3C_2 + c_2C_3) + 60c_2C_1C_2\Big] \\
\mathrm{ch}_6(E_r \otimes F_R) =& \tfrac{1}{2^4 \cdot 3^2 \cdot 5^1}\Big[r(C_1^6 - 2C_2^3 + 9C_1^2C_2^2 + 3C_3^2 - 6C_1^4C_2 + 6C_1^3C_3 - 12C_1C_2C_3 + 6C_2C_4 - 6C_1^2C_4 + 6C_1C_5 - 6C_6) \\
& + R(-2c_2^3 + 6c_2c_4 + 3c_3^2 - 6c_6) + 30\left(-2c_4C_1^2 + c_5C_1 + 4c_4C_2 + c_2^2(C_1^2 - 2C_2)\right) \\
& + 60c_3(C_1^3 - 3C_1C_2 + 3C_3) - 30c_2(C_1^4 + c_3C_1 + 2C_2^2 - 4C_1^2C_2 + 4C_1C_3 - 4C_4)\Big]
\end{aligned}$$

Expansion 13. *Chern character of a tensor product of two bundles E_r and F_R, where $c_1 = C_1 = 0$, i.e. the bundles are of type $SU(r)$ and $SU(R)$, respectively.*

Expansion 13: Chern character of tensor product of two SU-bundles

$$\mathrm{ch}_0(E_r \otimes F_R) = rR$$
$$\mathrm{ch}_1(E_r \otimes F_R) = 0$$
$$\mathrm{ch}_2(E_r \otimes F_R) = -Rc_2 - rC_2$$
$$\mathrm{ch}_3(E_r \otimes F_R) = \frac{1}{2^1}\left[Rc_3 + rC_3\right]$$
$$\mathrm{ch}_4(E_r \otimes F_R) = \frac{1}{2^2.3^1}\left[R(c_2^2 - 2c_4) + 12c_2C_2 + r\left(C_2^2 - 2C_4\right)\right]$$
$$\mathrm{ch}_5(E_r \otimes F_R) = \frac{1}{2^3.3^1}\left[R(c_5 - c_2c_3) - 12(c_3C_2 + c_2C_3) + r(C_5 - C_2C_3)\right]$$
$$\mathrm{ch}_6(E_r \otimes F_R) = \frac{1}{2^4.3^2.5^1}[R(-2c_2^3 + 3c_3^2 + 6c_2c_4 - 6c_6) + r(-2C_2^3 + 3C_3^2 + 6C_2C_4 - 6C_6) + 60(-c_2^2C_2 + 2c_4C_2 + 3c_3C_3 + 2c_2C_4 - c_2C_2^2)]$$
$$\mathrm{ch}_7(E_r \otimes F_R) = \frac{1}{2^4.3^2.5^1}\Big[R(c_2^2c_3 - c_2c_5 - c_3c_4 + c_7) + r(C_2^2C_3 - C_2C_5 - C_3C_4 + C_7) + 30\left\{C_3c_2^2 + c_3C_2^2 + c_3C_2c_2 + C_3C_2c_2 - (C_5c_2 + c_5C_2) - 2(c_4C_3 + c_3C_4)\right\}\Big]$$
$$\mathrm{ch}_8(E_r \otimes F_R) = \frac{1}{2^6.3^2.5^1.7^1}[R(c_2^4 - 4c_2c_3^2 - 4c_2^2c_4 + 4c_2c_6 + 4c_3c_5 + 2c_4^2 - 4c_8) + r(C_2^4 - 4C_2C_3^2 - 4C_2^2C_4 + 4C_2C_6 + 4C_3C_5 + 2C_4^2 - 4C_8) + 140c_2^2C_2^2 + 56(c_2^3C_2 + c_2C_2^3) - 84(c_3^2C_2 + c_2C_3^2) - 420(c_3C_2C_3 + c_2c_3C_3) - 280(c_2^2C_4 + c_4C_2^2) - 168(c_2c_4C_2 + c_2C_2C_4) + 168(c_6C_2 + C_6c_2) + 420(c_5C_3 + c_3C_5) + 560c_4C_4]$$

Now we take E_r and F_R to both be complex String bundles, i.e., $c_1 = c_2 = 0$ and $C_1 = C_2 = 0$.

Expansion 14. *Chern character of tensor product of complex String bundles.*

Expansion 14: Chern character of tensor product of complex String bundles

$$\mathrm{ch}_0(E_r \otimes F_R) = rR$$
$$\mathrm{ch}_1(E_r \otimes F_R) = 0$$
$$\mathrm{ch}_2(E_r \otimes F_R) = 0$$
$$\mathrm{ch}_3(E_r \otimes F_R) = \frac{1}{2^1}\left[Rc_3 + rC_3\right]$$
$$\mathrm{ch}_4(E_r \otimes F_R) = -\frac{1}{2^1.3^1}\left[Rc_4 + rC_4\right]$$
$$\mathrm{ch}_5(E_r \otimes F_R) = \frac{1}{2^3.3^1}\left[Rc_5 + rC_5\right]$$
$$\mathrm{ch}_6(E_r \otimes F_R) = \frac{1}{2^4.3^2.5^1}\left[R(3c_3^2 - 6c_6) + r(3C_3^2 - 6C_6) + 180c_3C_3\right]$$
$$\mathrm{ch}_7(E_r \otimes F_R) = \frac{1}{2^4.3^2.5^1}\left[R(-c_3c_4 + c_7) + r(-C_3C_4 + C_7) - 60(c_4C_3 + c_3C_4)\right]$$
$$\mathrm{ch}_8(E_r \otimes F_R) = \frac{1}{2^6.3^2.5^1.7^1}\left[R(4c_3c_5 + 2c_4^2 - 4c_8) + r(4C_3C_5 + 2C_4^2 - 4C_8) + 420(c_5C_3 + c_3C_5) + 560c_4C_4\right]$$

Chern classes of a tensor product

The Chern classes, unlike the Chern character, are not multiplicative. Thus, we combine Expansion 11 with Expansion 3 to write Chern classes of a tensor product.

Expansion 15. *Chern classes of a tensor product of two bundles E_r and F_R.*

Expansion 15: Chern classes of tensor product of two vector bundles

$$c_0(E_r \otimes F_R) = 1$$
$$c_1(E_r \otimes F_R) = Rc_1 + rC_1$$
$$c_2(E_r \otimes F_R) = \frac{1}{2^1}\left[R(R-1)c_1^2 + r(r-1)C_1^2 + 2(Rc_2 + rC_2) + 2(rR-1)c_1C_1\right]$$
$$c_3(E_r \otimes F_R) = \frac{1}{2^1.3^1}[(Rc_1 + rC_1)^3 - 3(Rc_1 + rC_1)\left(Rc_1^2 - 2Rc_2 + 2c_1C_1 + rC_1^2 - 2rC_2\right) + 2R\left(c_1^3 - 3c_1c_2 + 3c_3\right) + 2r(C_1^3 - 3C_1C_2 + 3C_3) + 6\left(c_1^2 - 2c_2\right)C_1 + 6c_1\left(C_1^2 - 2C_2\right)]$$
$$c_4(E_r \otimes F_R) = \frac{1}{2^3.3^1}\Big[(Rc_1 + rC_1)^4 + 3\left(R(c_1^2 - 2c_2) + 2c_1C_1 + r\left(C_1^2 - 2C_2\right)\right)^2 - 6(Rc_1 + rC_1)^2\Big(R(c_1^2 - 2c_2) + 2c_1C_1 + r\left(C_1^2 - 2C_2\right)\Big) + 8(Rc_1 + rC_1)\Big[R\left(c_1^3 - 3c_1c_2 + 3c_3\right) + 3(c_1^2 - 2c_2)C_1 + 3c_1\left(C_1^2 - 2C_2\right) + r\left(C_1^3 - 3C_1C_2 + 3C_3\right)\Big] - 6(Rc_1^4 + rC_1^4) - 12(2c_1C_1^3 + 3c_1^2C_1^2 + 2c_1^3C_1) - 12(Rc_2^2 + 12c_2C_2 + rC_2^2) + 24(rC_1^2C_2 + Rc_1^2c_2 + 3c_1^2C_2 + 3c_2C_1^2 + 3c_1c_2C_1 + 3c_1C_1C_2) + 24(-rC_1C_3 - 3c_3C_1 - Rc_1c_3 - 3c_1C_3) + 24(rC_4 + Rc_4)\Big]$$

Simplifications

Expansion 16. *Chern classes of a tensor product of two bundles E_r and F_R, where $c_1 = C_1 = 0$, i.e. the bundles are of type $SU(r)$ and $SU(R)$, respectively.*

Expansion 16: Chern classes of tensor product of two SU-bundles

$$
\begin{aligned}
c_0(E_r \otimes F_R) =& 1\\
c_1(E_r \otimes F_R) =& 0\\
c_2(E_r \otimes F_R) =& Rc_2 + rC_2\\
c_3(E_r \otimes F_R) =& Rc_3 + rC_3\\
c_4(E_r \otimes F_R) =& \tfrac{1}{2^1}[R(R-1)c_2^2 + 2(Rc_4 + rC_4) + 2(rR-6)c_2C_2 + r(r-1)C_2^2]\\
c_5(E_r \otimes F_R) =& [C_5 r + Rc_5 + (Rr-12)(c_3C_2 + C_3c_2) + r(r-1)C_2C_3 + R(R-1)c_2c_3]\\
c_6(E_r \otimes F_R) =& \tfrac{1}{2^1 \cdot 3^1}\Big[R(R^2-3R+2)c_2^3 + r(r^2-3r+2)C_2^3 + 3\left(rR^2 - (r+12)R + 20\right)c_2^2C_2 + 3\left(Rr^2 - (R+12)r + 20\right)c_2C_2^2\\
&+ 6R(R-1)c_2c_4 + 6r(r-1)C_2C_4 + 6(rR-20)(c_4C_2 + C_4c_2)\\
&+ 6(rR-30)c_3C_3 + 3R(R-1)c_3^2 + 3r(r-1)C_3^2 + 6(Rc_6 + rC_6)\Big]
\end{aligned}
$$

Observation: It is now possible to group the terms in such as way as to make the symmetry: $r \leftrightarrow R$ and $c_i \leftrightarrow C_i$ manifest.

Now we take both E_r and F_R to be complex String bundles.

Expansion 17. *Chern classes of tensor product of complex String bundles, i.e., $c_1 = c_2 = 0$ and $C_1 = C_2 = 0$.*

Expansion 17: Chern classes of tensor product of complex String bundles

$$
\begin{aligned}
c_0(E_r \otimes F_R) =& 1\\
c_1(E_r \otimes F_R) =& 0\\
c_2(E_r \otimes F_R) =& 0\\
c_3(E_r \otimes F_R) =& Rc_3 + rC_3\\
c_4(E_r \otimes F_R) =& Rc_4 + rC_4\\
c_5(E_r \otimes F_R) =& Rc_5 + rC_5\\
c_6(E_r \otimes F_R) =& (rR-30)c_3C_3 + \tfrac{1}{2}R(R-1)c_3^2 + \tfrac{1}{2}r(r-1)C_3^2 + (Rc_6 + rC_6)\\
c_7(E_r \otimes F_R) =& R(R-1)c_3c_4 + (rR-60)(c_3C_4 + c_4C_3) + r(r-1)C_3C_4 + (Rc_7 + rC_7)
\end{aligned}
$$

4 Expansions of the Genera

4.1 Genera

We will provide some basic theoretical background on genera (see [12][27][13]). A genus (singular of "genera") is a certain combinatorial formula that involves characteristic classes. It is a functor on cobordism. Two n-dimensional manifolds M_1 and M_2 are cobordant if there is an oriented manifold W of dimension $n+1$ whose boundary is the disjoint union of M_1 and M_2, i.e., $\partial W = M_1 \coprod M_2$. The cobordism relation defines a ring Ω whose elements are the cobordism classes of manifolds, where addition is induced from disjoint union $\coprod$ and the product is induced from Cartesian product $\times$ of spaces.

An R-genus is a homomorphism from the cobordism ring Ω to a commutative ring with unit R. Further structures can be put on the manifolds so that the corresponding ring Ω gets decorated accordingly. The codomain ring is usually the rationals $\mathbb{Q}$ or the integers $\mathbb{Z}$. So, a genus g assigns a number $g(M)$ to each manifold M such that the following hold true (see [12] [27])

1. *Additivity*: $g(M \coprod N) = g(M) + g(N)$ where $\coprod$ is disjoint union,
2. *Multiplicativity*: $g(M \times N) = g(M)g(N)$,
3. *Triviality*: $g(M) = 0$ if M is a boundary of some other manifold.

We will find the following useful for the computations. One calls a sequence of polynomials $K_1, K_2, ...$ in variables $s_1, s_2, ..$ *multiplicative* if there exists a factorization (see [12])

$$1 + s_1 t + s_2 t^2 + ... = (1 + y_1 t + y_2 t^2 + ...)(1 + z_1 t + z_2 t^2 + ...)$$
$$\Rightarrow \sum_j K_j(s_1, s_2, ...)t^j = \sum_j K_j(y_1, y_2, ...)t^j \sum_k K_k(z_1, z_2, ...)t^k.$$

A multiplicative sequence $K = 1 + K_1 + K_2 + ...$ is then given by

$$K(s_1, s_2, s_3...) = Q(y_1)Q(y_2)Q(y_3)... \tag{4.1}$$

where s_i is the i-th elementary symmetric function of y_j and $Q(u)$ is a formal power series in u which starts with 1. The above combinatorial formula (4.1) will be written in terms of the Pontrjagin and Chern classes once the manifold under investigation admits the corresponding structure (i.e. real or oriented for Pontrjagin classes and complex for Chern classes).

Indeed, the genus g of an oriented manifold M corresponding to Q is written as $g(M) = K(s_1, s_2, s_3, ...)$ where s_i are Pontrjagin classes p_i of M. The power series Q are the characteristic power series of the genus g. A similar description holds for the Chern classes. More precisely, by definition (see e.g. [5, §21.2], [27, §19, Definition]), for a smooth, compact, oriented manifold M (of dimension, say $4n$), the *genus* $K(M)$ associated with a multiplicative sequence $\{K_j(p_1, \cdots, p_j)\}$ is defined by

$$K(M) := K_n(p_1(M), p_2(M), \cdots, p_n(M))[M],$$

where $p_i(M)$ $(i = 1, 2, \cdots)$ are the Pontrjagin classes of M, and $[M]$ denotes the *fundamental class* of M. It is (usually) a rational number.

Next we will combinatorially define the three genera of interest. Strictly speaking, these are not genera; they are formal power series in $x_1, x_2, \cdots$, and define multiplicative sequences $\{\hat{A}_j(p_1, \cdots, p_j)\}$, $\{L_j(p_1, \cdots, p_j)\}$ when expressed in terms of elementary symmetric functions p_j in $x_1^2, x_2^2, \cdots$, and similarly for $\{\mathrm{Td}_j(c_1, \cdots, c_j)\}$ in terms of the c_i's. Furthermore, as defined in [5, §23.1], for a real vector bundle $E \to M$, the cohomology class $\hat{\mathcal{A}}(E) \in H^*(M; R) = \prod_{j\geq 0} H^j(M; R)$ (here $R = \mathbb{Q}$ or $\mathbb{R}$) is defined to be

$$\hat{\mathcal{A}}(E) := \sum_{j=0}^{\infty} \hat{A}_j(p_1(E), \cdots, p_j(E)).$$

If E is the tangent bundle TM of a differentiable manifold M, then we set $\hat{\mathcal{A}}(TM) = \hat{\mathcal{A}}(M)$. The $\hat{A}(M)$ of a compact oriented differentiable manifold M is given by

$$\hat{A}(M) := \hat{\mathcal{A}}(M)[M].$$

1. The Â-genus

The *$\hat{A}$-genus* is defined combinatorially via the power series expansion

$$\begin{aligned}\hat{A}(x_1, \cdots, x_k) &= \prod_{j=1}^{k} \frac{x_j/2}{\sinh(x_j/2)} \\ &= \prod_{j=1}^{k} \left(1 + \sum_{n\geq 1} (-1)^n \tfrac{2^{2n-1}-1}{2^{2n-1}(2n)!} B_n x_j^{2n}\right)\end{aligned}$$

where B_n are the Bernoulli numbers given as

$$B_1 = \tfrac{1}{6},\ B_2 = \tfrac{1}{30},\ B_3 = \tfrac{1}{42},\ B_4 = \tfrac{1}{30},\ B_5 = \tfrac{5}{66},\ \cdots.$$

The Â-genus is an even function of x_j which can be expanded in the Pontrjagin classes p_i. One of the properties of the Â-genus is that it satisfies the following

$$\hat{A}(E \oplus F) = \hat{A}(E) \cdot \hat{A}(F).$$

The next section has more on how to expand Â-genus, while *Expansion 18* in Section 4 states the actual expressions.

The importance of the Â-genus arises in the formulation of the Atiyah-Singer index theorem, where it appears as the expression whose evaluation on a manifold M is the index of the Dirac operator associated with the Spin bundle on M. As such, it takes integer values for Spin manifolds of dimension $4k$. The Â-genus also arises in determining whether a Spin manifold M admits a metric of positive scalar curvature, whose existence implies the vanishing of the Â-genus of the tangent bundle of M, i.e. $\hat{A}(M) = 0$ (see [20]).

2. The L-genus

The *Hirzebruch L-polynomial* or *L-genus* is defined by

$$L(x_1, \cdots, x_k) = \prod_{j=1}^{k} \frac{\sqrt{x_j}}{\tanh \sqrt{x_j}} \tag{4.2}$$

$$= \prod_{j=1}^{k} \left(1 + \sum_{n \geq 1} (-1)^{n-1} \tfrac{2^{2n}}{(2n)!} B_n x_j^n\right). \tag{4.3}$$

$L(x)$ can be expressed in terms of the Pontrjagin classes and enjoys the same property as the Â-genus

$$L(E \oplus F) = L(E) \cdot L(F).$$

The L-genus expressions are listed in *Expression 27* in Section 4.

One major highlight about the L-genus is that its integral over a closed $4k$-dimensional space X calculates the signature $\sigma(X)$ of X, hence is an integer which is an invariant of the oriented homotopy type (see [27, Corollary 19.6]). Being an integer is a distinguishing property from the $\hat{A}$-genus, which is only an integer for Spin manifolds.

3. The Todd genus

The Todd genus is associated with complex vector bundles and is defined by the series

$$T(x_1, \cdots, x_k) = \prod_j \frac{x_j}{1 - e^{-x_j}}. \tag{4.4}$$

If the *Todd genus* is expanded in powers of x_j, then the expression looks like

$$\begin{aligned} \mathrm{Td}(E) &= \prod_j \left(1 + \tfrac{1}{2} x_j + \sum_{k \geq 1} (-1)^{k-1} \tfrac{1}{(2k)!} B_k x_j^{2k}\right) \\ &= 1 + \tfrac{1}{2} \sum_j x_j + \tfrac{1}{12} \sum_j x_j^2 + \tfrac{1}{4} \sum_{j\ k} x_j x_k + \ldots \\ &= 1 + \tfrac{1}{2} c_1(E) + \tfrac{1}{12} [c_1(E)^2 + c_2(E)] + \ldots, \end{aligned}$$

where we use the identification between the Chern classes and the elementary symmetric functions in the Chern roots. As with the other two genera, the Todd genus also satisfies

$$\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \cdot \mathrm{Td}(F).$$

The reader will find the Todd genus written out explicitly in *Expansion 36*.

The earliest successful use of the Todd genus has been in the classical Hirzebruch-Riemann-Roch formula computing the holomorphic Euler characteristic $\sum_i (-1)^i \dim_{\mathbb{C}}(X, \mathcal{O}_X(E))$, where E is any holomorphic vector bundle on a compact complex manifold (see [12]). The Todd genus also appears in the complex version of the Atiyah-Singer index theorem.

The genera expansions are important because they capture information about the vector bundles. The Â-genus and L-genus are relevant for real spaces, while the Todd genus is relevant for complex spaces. Details can be found in [9][12], while geometric perspectives are provided, e.g. in [29][28][37].

4.2 Â-genus

The Â-genus is computed by plugging the characteristic power series of Â-genus $\frac{x_j/2}{\sinh(x_j/2)}$ (see Section 4) into the function below originally developed by C. McTague [26].

Expansion 18. *Â-genus in terms of Pontrjagin classes.*

Expansion 18: Â-genus in terms of Pontrjagin classes

$$\hat{A}_0 = 1$$

$$\hat{A}_1 = \frac{1}{2^{3}\cdot 3^{1}}\Big[-1^{1}p_1\Big]$$

$$\hat{A}_2 = \frac{1}{2^{5}\cdot 3^{2}\cdot 5^{1}}\Big[\frac{7^{1}}{2^{2}}p_1^{2} - 1^{1}p_2\Big]$$

$$\hat{A}_3 = \frac{1}{2^{6}\cdot 3^{3}\cdot 5^{1}\cdot 7^{1}}\Big[-\frac{31^{1}}{2^{4}}p_1^{3} + \frac{11^{1}}{2^{2}}p_1p_2 - 1^{1}p_3\Big]$$

$$\hat{A}_4 = \frac{1}{2^{6}\cdot 3^{3}\cdot 5^{2}\cdot 7^{1}}\Big[\frac{127^{1}}{2^{9}}p_1^{4} - \frac{113^{1}}{2^{6}\cdot 3^{1}}p_1^{2}p_2 + \frac{1^{1}}{3^{1}}p_1p_3 + \frac{13^{1}}{2^{5}\cdot 3^{1}}p_2^{2} - \frac{1^{1}}{2^{3}}p_4\Big]$$

$$\begin{aligned}\hat{A}_5 = \frac{1}{2^{10}\cdot 3^{4}\cdot 5^{1}\cdot 11^{1}}\Big[&-\frac{73^{1}}{2^{8}\cdot 3^{1}}p_1^{5} + \frac{29^{1}\cdot 37^{1}}{2^{5}\cdot 3^{1}\cdot 5^{1}\cdot 7^{1}}p_1^{3}p_2 - \frac{61^{1}}{2^{3}\cdot 5^{1}\cdot 7^{1}}p_1^{2}p_3 - \frac{311^{1}}{2^{4}\cdot 3^{1}\cdot 5^{1}\cdot 7^{1}}p_1p_2^{2} + \frac{53^{1}}{2^{2}\cdot 3^{1}\cdot 5^{1}\cdot 7^{1}}p_1p_4 + \frac{1^{1}}{2^{1}\cdot 5^{1}}p_2p_3\\ &- \frac{1^{1}}{3^{1}\cdot 7^{1}}p_5\Big]\end{aligned}$$

$$\begin{aligned}\hat{A}_6 = \frac{1}{2^{12}\cdot 3^{5}\cdot 5^{2}\cdot 7^{2}\cdot 11^{1}\cdot 13^{1}}\Big[&\frac{23^{1}\cdot 89^{1}\cdot 691^{1}}{2^{10}\cdot 3^{1}\cdot 5^{1}}p_1^{6} - \frac{1540453^{1}}{2^{8}\cdot 3^{1}\cdot 5^{1}}p_1^{4}p_2 + \frac{29^{1}\cdot 1249^{1}}{2^{3}\cdot 3^{1}\cdot 5^{1}}p_1^{3}p_3 + \frac{19^{1}\cdot 4013^{1}}{2^{6}\cdot 3^{1}}p_1^{2}p_2^{2} - \frac{16759^{1}}{2^{4}\cdot 5^{1}}p_1^{2}p_4 - \frac{3491^{1}}{2^{1}\cdot 5^{1}}p_1p_2p_3\\ &+ \frac{23^{1}\cdot 53^{1}}{2^{1}\cdot 5^{1}}p_1p_5 - \frac{19^{1}\cdot 211^{1}}{2^{4}\cdot 5^{1}}p_2^{3} + \frac{73^{1}\cdot 79^{1}}{2^{2}\cdot 3^{1}\cdot 5^{1}}p_2p_4 + \frac{19^{1}\cdot 37^{1}}{3^{1}\cdot 5^{1}}p_3^{2} - \frac{691^{1}}{3^{1}\cdot 5^{1}}p_6\Big]\end{aligned}$$

$$\begin{aligned}\hat{A}_7 = \frac{1}{2^{11}\cdot 3^{4}\cdot 7^{1}\cdot 13^{1}}\Big[&-\frac{8191^{1}}{2^{14}\cdot 3^{2}\cdot 5^{2}\cdot 11^{1}}p_1^{7} + \frac{37^{1}\cdot 31121^{1}}{2^{12}\cdot 3^{3}\cdot 5^{3}\cdot 7^{1}\cdot 11^{1}}p_1^{5}p_2 - \frac{67^{1}\cdot 127^{1}}{2^{10}\cdot 5^{3}\cdot 7^{1}\cdot 11^{1}}p_1^{4}p_3 - \frac{9161^{1}}{2^{10}\cdot 3^{1}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}}p_1^{3}p_2^{2} + \frac{23^{1}\cdot 127^{1}}{2^{8}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}}p_1^{3}p_4\\ &+ \frac{179^{1}\cdot 317^{1}}{2^{7}\cdot 3^{3}\cdot 5^{3}\cdot 7^{1}\cdot 11^{1}}p_1^{2}p_2p_3 - \frac{2543^{1}}{2^{6}\cdot 3^{2}\cdot 5^{3}\cdot 7^{1}\cdot 11^{1}}p_1^{2}p_5 + \frac{109^{1}\cdot 307^{1}}{2^{8}\cdot 3^{3}\cdot 5^{3}\cdot 7^{1}\cdot 11^{1}}p_1p_2^{3} - \frac{67^{1}}{2^{6}\cdot 5^{3}\cdot 11^{1}}p_1p_2p_4 - \frac{97^{1}}{3^{3}\cdot 5^{3}\cdot 7^{1}\cdot 11^{1}}p_1p_3^{2}\\ &+ \frac{101^{1}}{2^{4}\cdot 3^{3}\cdot 5^{3}\cdot 7^{1}}p_1p_6 - \frac{23^{1}\cdot 233^{1}}{2^{6}\cdot 3^{3}\cdot 5^{3}\cdot 7^{1}\cdot 11^{1}}p_2^{2}p_3 + \frac{1^{1}}{2^{4}\cdot 3^{3}\cdot 11^{1}}p_2p_5 + \frac{283^{1}}{2^{4}\cdot 3^{2}\cdot 5^{3}\cdot 7^{1}\cdot 11^{1}}p_3p_4 - \frac{1^{1}}{2^{2}\cdot 3^{2}\cdot 5^{2}\cdot 11^{1}}p_7\Big]\end{aligned}$$

$$\begin{aligned}\hat{A}_8 = \frac{1}{2^{15}\cdot 3^{5}\cdot 5^{2}\cdot 7^{1}\cdot 17^{1}}\Big[&\frac{31^{1}\cdot 151^{1}\cdot 3617^{1}}{2^{16}\cdot 3^{2}\cdot 5^{2}\cdot 11^{1}\cdot 13^{1}}p_1^{8} - \frac{2241667^{1}}{2^{12}\cdot 3^{1}\cdot 5^{2}\cdot 11^{1}\cdot 13^{1}}p_1^{6}p_2 + \frac{3661841^{1}}{2^{7}\cdot 3^{3}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_1^{5}p_3 + \frac{71^{1}\cdot 268823^{1}}{2^{11}\cdot 3^{3}\cdot 5^{2}\cdot 11^{1}\cdot 13^{1}}p_1^{4}p_2^{2}\\ &- \frac{3941363^{1}}{2^{9}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_1^{4}p_4 - \frac{317^{1}\cdot 4129^{1}}{2^{4}\cdot 3^{3}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_1^{3}p_2p_3 + \frac{26921^{1}}{2^{5}\cdot 3^{2}\cdot 5^{2}\cdot 11^{1}\cdot 13^{1}}p_1^{3}p_5 - \frac{29^{1}\cdot 41^{1}\cdot 227^{1}}{2^{8}\cdot 3^{3}\cdot 5^{1}\cdot 11^{1}\cdot 13^{1}}p_1^{2}p_2^{3}\\ &+ \frac{38923^{1}}{2^{6}\cdot 3^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_1^{2}p_2p_4 + \frac{907^{1}}{2^{2}\cdot 3^{1}\cdot 5^{2}\cdot 7^{1}\cdot 13^{1}}p_1^{2}p_3^{2} - \frac{197033^{1}}{2^{4}\cdot 3^{3}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_1^{2}p_6 + \frac{9091^{1}}{2^{3}\cdot 3^{3}\cdot 5^{1}\cdot 11^{1}\cdot 13^{1}}p_1p_2^{2}p_3\\ &- \frac{23339^{1}}{2^{3}\cdot 3^{3}\cdot 5^{2}\cdot 11^{1}\cdot 13^{1}}p_1p_2p_5 - \frac{39887^{1}}{2^{1}\cdot 3^{3}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_1p_3p_4 + \frac{1063^{1}}{2^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_1p_7 + \frac{11^{1}\cdot 1249^{1}}{2^{8}\cdot 3^{3}\cdot 5^{1}\cdot 7^{1}\cdot 13^{1}}p_2^{4} - \frac{275593^{1}}{2^{5}\cdot 3^{3}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_2^{2}p_4\\ &- \frac{31^{1}}{3^{2}\cdot 5^{2}\cdot 11^{1}}p_2p_3^{2} + \frac{11299^{1}}{2^{1}\cdot 3^{3}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_2p_6 + \frac{11^{1}\cdot 181^{1}}{2^{2}\cdot 3^{3}\cdot 5^{2}\cdot 7^{1}\cdot 13^{1}}p_3p_5 + \frac{73^{1}\cdot 199^{1}}{2^{4}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_4^{2} - \frac{3617^{1}}{2^{2}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_8\Big]\end{aligned}$$

$$\begin{aligned}\hat{A}_9 = \frac{1}{2^{13}\cdot 3^{7}\cdot 5^{2}\cdot 7^{2}\cdot 19^{1}}\Big[&-\frac{43867^{1}\cdot 131071^{1}}{2^{21}\cdot 3^{2}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{9} + \frac{47^{1}\cdot 907^{1}\cdot 7949^{1}}{2^{17}\cdot 3^{1}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 17^{1}}p_1^{7}p_2 - \frac{9397^{1}\cdot 33317^{1}}{2^{15}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{6}p_3\\ &- \frac{83^{1}\cdot 3688543^{1}}{2^{16}\cdot 3^{1}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}}p_1^{5}p_2^{2} + \frac{5303^{1}\cdot 44519^{1}}{2^{14}\cdot 3^{2}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{5}p_4 + \frac{601^{1}\cdot 4429813^{1}}{2^{13}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{4}p_2p_3 + \frac{601^{1}\cdot 4429813^{1}}{2^{13}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{4}p_2p_3\\ &- \frac{47756197^{1}}{2^{12}\cdot 3^{2}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{4}p_5 + \frac{1625000107^{1}}{2^{13}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{3}p_2^{3} - \frac{613^{1}\cdot 688087^{1}}{2^{11}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{3}p_2p_4 - \frac{23^{1}\cdot 653^{1}}{2^{9}\cdot 5^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{3}p_3^{2}\\ &+ \frac{4569683^{1}}{2^{9}\cdot 3^{2}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{3}p_6 - \frac{59^{1}\cdot 163^{1}\cdot 8377^{1}}{2^{11}\cdot 3^{2}\cdot 5^{2}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{2}p_2^{2}p_3 + \frac{1511459^{1}}{2^{9}\cdot 3^{1}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 17^{1}}p_1^{2}p_2p_5 + \frac{23^{1}\cdot 263^{1}\cdot 3187^{1}}{2^{9}\cdot 3^{1}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{2}p_3p_4\\ &- \frac{23^{1}\cdot 86573^{1}}{2^{6}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1^{2}p_7 - \frac{37857689^{1}}{2^{13}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1p_2^{4} + \frac{11389153^{1}}{2^{10}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1p_2^{2}p_4 + \frac{199^{1}\cdot 4231^{1}}{2^{7}\cdot 3^{1}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1p_2p_3^{2}\\ &- \frac{3301303^{1}}{2^{6}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1p_2p_6 - \frac{127^{1}\cdot 2029^{1}}{2^{7}\cdot 3^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1p_3p_5 - \frac{311^{1}\cdot 41263^{1}}{2^{9}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1p_4^{2} + \frac{467^{1}\cdot 4969^{1}}{2^{7}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_1p_8\\ &+ \frac{15013651^{1}}{2^{9}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_2^{3}p_3 - \frac{991^{1}\cdot 1123^{1}}{2^{8}\cdot 3^{2}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_2^{2}p_5 - \frac{43^{1}\cdot 42239^{1}}{2^{7}\cdot 3^{1}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_2p_3p_4 + \frac{41^{1}\cdot 557^{1}}{2^{3}\cdot 3^{2}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_2p_7\\ &- \frac{61^{1}\cdot 3659^{1}}{2^{5}\cdot 3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_3^{3} + \frac{13829^{1}}{3^{2}\cdot 5^{2}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_3p_6 + \frac{43^{1}\cdot 4091^{1}}{2^{6}\cdot 3^{2}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_4p_5 - \frac{43867^{1}}{2^{5}\cdot 3^{2}\cdot 5^{1}\cdot 7^{1}\cdot 11^{1}\cdot 13^{1}\cdot 17^{1}}p_9\Big]\end{aligned}$$

Â-genus with simplifications.

One might be interested in bundles with higher notions of symmetries beyond the usual SO or SU structures; for example, String or p_1-structures ($p_1 = 0$), Fivebrane or p_2-structures ($p_1 = p_2 = 0$) [32][33], and Ninebrane or p_3-structures ($p_1 = p_2 = p_3 = 0$) [31].

The expansions in this subsection were obtained by using two Mathematica formulas, both of which are modifications of C. McTague's code [26]. The two new functions deal with genera simplification and only differ in how they set which characteristic classes would vanish based on user's input.

Expansion 19. *$\hat{A}$-genus with String or p_1-structure ($p_1 = 0$).*

Expansion 19: Â-genus of String or p_1-structure manifolds

$$\hat{A}_0 = 1$$

$$\hat{A}_1 = 0$$

$$\hat{A}_2 = \frac{1}{2^5.3^2.5^1}\Big[-1^1 p_2\Big]$$

$$\hat{A}_3 = \frac{1}{2^6.3^3.5^1.7^1}\Big[-1^1 p_3\Big]$$

$$\hat{A}_4 = \frac{1}{2^9.3^3.5^2.7^1}\Big[\frac{13^1}{2^2.3^1}p_2^2 - 1^1 p_4\Big]$$

$$\hat{A}_5 = \frac{1}{2^{10}.3^4.5^1.11^1}\Big[\frac{1^1}{2^1.5^1}p_2p_3 - \frac{1^1}{3^1.7^1}p_5\Big]$$

$$\hat{A}_6 = \frac{1}{2^{12}.3^5.5^3.7^2.11^1.13^1}\Big[-\frac{19^1\cdot 211^1}{2^4}p_2^3 + \frac{73^1\cdot 79^1}{2^2.3^1}p_2p_4 + \frac{19^1\cdot 37^1}{3^1}p_3^2 - \frac{691^1}{3^1}p_6\Big]$$

$$\hat{A}_7 = \frac{1}{2^{13}.3^6.7^1.11^1.13^1}\Big[-\frac{23^1\cdot 233^1}{2^4.3^1.5^3.7^1}p_2^2p_3 + \frac{1^1}{2^2.3^1}p_2p_5 + \frac{283^1}{2^2.5^3.7^1}p_3p_4 - \frac{1^1}{5^2}p_7\Big]$$

$$\hat{A}_8 = \frac{1}{2^{15}.3^7.5^3.7^1.17^1}\Big[\frac{11^1\cdot 1249^1}{2^8.3^1.7^1.13^1}p_2^4 - \frac{275593^1}{2^5.3^1.5^1.7^1.11^1.13^1}p_2^2p_4 - \frac{31^1}{5^1.11^1}p_2p_3^2 + \frac{11299^1}{2^1.3^1.5^1.7^1.11^1.13^1}p_2p_6$$
$$+ \frac{11^1\cdot 181^1}{2^2.3^1.5^1.7^1.13^1}p_3p_5 + \frac{73^1\cdot 199^1}{2^4.5^1.7^1.11^1.13^1}p_4^2 - \frac{3617^1}{2^2.5^1.7^1.11^1.13^1}p_8\Big]$$

$$\hat{A}_9 = \frac{1}{2^{13}.3^8.5^3.7^3.11^1.13^1.17^1.19^1}\Big[\frac{15013651^1}{2^9.3^1.5^1}p_2^3p_3 - \frac{991^1\cdot 1123^1}{2^8.3^1}p_2^2p_5 - \frac{43^1\cdot 42239^1}{2^7.5^1}p_2p_3p_4 + \frac{41^1\cdot 557^1}{2^3.3^1}p_2p_7 - \frac{61^1\cdot 3659^1}{2^5.3^1.5^1}p_3^3$$
$$+ \frac{13829^1}{3^1.5^1}p_3p_6 + \frac{43^1\cdot 4091^1}{2^6.3^1}p_4p_5 - \frac{43867^1}{2^5.3^1}p_9\Big]$$

Expansion 20. *$\hat{A}$-genus of p_2-structure manifolds ($p_2 = 0$).*

Expansion 20: Â-genus with p_2-structure

$$\hat{A}_0 = 1$$

$$\hat{A}_1 = \frac{1}{2^3.3^1}\Big[-1^1 p_1\Big]$$

$$\hat{A}_2 = \frac{1}{2^7.3^2.5^1}\Big[7^1 p_1^2\Big]$$

$$\hat{A}_3 = \frac{1}{2^6.3^3.5^1.7^1}\Big[-\frac{31^1}{2^4}p_1^3 - 1^1 p_3\Big]$$

$$\hat{A}_4 = \frac{1}{2^6.3^3.5^2.7^1}\Big[\frac{127^1}{2^9}p_1^4 + \frac{1^1}{3^1}p_1p_3 - \frac{1^1}{2^3}p_4\Big]$$

$$\hat{A}_5 = \frac{1}{2^{10}.3^4.5^1.11^1}\Big[-\frac{73^1}{2^8.3^1}p_1^5 - \frac{61^1}{2^3.5^1.7^1}p_1^2p_3 + \frac{53^1}{2^2.3^1.5^1.7^1}p_1p_4 - \frac{1^1}{3^1.7^1}p_5\Big]$$

$$\hat{A}_6 = \frac{1}{2^{12}.3^5.5^3.7^2.11^1.13^1}\Big[\frac{23^1\cdot 89^1\cdot 691^1}{2^{10}.3^1}p_1^6 + \frac{29^1\cdot 1249^1}{2^3.3^1}p_1^3p_3 - \frac{16759^1}{2^4}p_1^2p_4 + \frac{23^1\cdot 53^1}{2^1}p_1p_5 + \frac{19^1\cdot 37^1}{3^1}p_3^2 - \frac{691^1}{3^1}p_6\Big]$$

$$\hat{A}_7 = \frac{1}{2^{11}.3^4.5^2.7^1.13^1}\Big[-\frac{8191^1}{2^{14}.3^2.11^1}p_1^7 - \frac{67^1\cdot 127^1}{2^{10}.5^1.7^1.11^1}p_1^4p_3 + \frac{23^1\cdot 127^1}{2^8.3^2.7^1.11^1}p_1^3p_4 - \frac{2543^1}{2^6.3^2.5^1.7^1.11^1}p_1^2p_5 - \frac{97^1}{3^3.5^1.7^1.11^1}p_1p_3^2$$
$$+ \frac{101^1}{2^4.3^3.5^1.7^1}p_1p_6 + \frac{283^1}{2^4.3^2.5^1.7^1.11^1}p_3p_4 - \frac{1^1}{2^2.3^2.11^1}p_7\Big]$$

$$\hat{A}_8 = \frac{1}{2^{16}.3^5.5^4.7^1.13^1.17^1}\Big[\frac{31^1\cdot 151^1\cdot 3617^1}{2^{15}.3^2.11^1}p_1^8 + \frac{3661841^1}{2^6.3^3.7^1.11^1}p_1^5p_3 - \frac{3941363^1}{2^8.3^2.7^1.11^1}p_1^4p_4 + \frac{26921^1}{2^4.3^2.11^1}p_1^3p_5 + \frac{907^1}{2^1.3^1.7^1}p_1^2p_3^2$$
$$- \frac{197033^1}{2^3.3^3.7^1.11^1}p_1^2p_6 - \frac{39887^1}{3^3.7^1.11^1}p_1p_3p_4 + \frac{1063^1}{2^1.7^1.11^1}p_1p_7 + \frac{11^1\cdot 181^1}{2^1.3^3.7^1}p_3p_5 + \frac{73^1\cdot 199^1}{2^3.3^2.7^1.11^1}p_4^2 - \frac{3617^1}{2^1.3^2.7^1.11^1}p_8\Big]$$

$$\hat{A}_9 = \frac{1}{2^{13}.3^7.5^2.7^2.13^1.17^1.19^1}\Big[-\frac{43867^1\cdot 131071^1}{2^{21}.3^2.5^1.7^1.11^1}p_1^9 - \frac{9397^1\cdot 33317^1}{2^{15}.5^2.7^1.11^1}p_1^6p_3 + \frac{5303^1\cdot 44519^1}{2^{14}.3^2.5^1.7^1.11^1}p_1^5p_4 - \frac{47756197^1}{2^{12}.3^2.5^1.7^1.11^1}p_1^4p_5$$
$$- \frac{23^1\cdot 653^1}{2^9.5^1}p_1^3p_3^2 + \frac{4569683^1}{2^9.3^2.5^1.7^1.11^1}p_1^3p_6 + \frac{23^1\cdot 263^1\cdot 3187^1}{2^9.3^1.5^2.7^1.11^1}p_1^2p_3p_4 - \frac{23^1\cdot 86573^1}{2^6.3^2.5^2.7^1.11^1}p_1^2p_7 - \frac{127^1\cdot 2029^1}{2^7.3^2.7^1.11^1}p_1p_3p_5$$
$$- \frac{311^1\cdot 41263^1}{2^9.3^2.5^2.7^1.11^1}p_1p_4^2 + \frac{467^1\cdot 4969^1}{2^7.3^2.5^2.7^1.11^1}p_1p_8 - \frac{61^1\cdot 3659^1}{2^5.3^2.5^2.7^1.11^1}p_3^3 + \frac{13829^1}{3^2.5^2.7^1.11^1}p_3p_6 + \frac{43^1\cdot 4091^1}{2^6.3^2.5^1.7^1.11^1}p_4p_5 - \frac{43867^1}{2^5.3^2.5^1.7^1.11^1}p_9\Big]$$

Expansion 21. *$\hat{A}$-genus of p_3-structure manifolds ($p_3 = 0$).*

Expansion 21: Â-genus with p_3-structure

$$\hat{A}_0 = 1$$

$$\hat{A}_1 = \frac{1}{2^3 \cdot 3^1}\Big[-1^1 p_1 \Big]$$

$$\hat{A}_2 = \frac{1}{2^5 \cdot 3^2 \cdot 5^1}\Big[\frac{7^1}{2^2} p_1^2 - 1^1 p_2 \Big]$$

$$\hat{A}_3 = \frac{1}{2^8 \cdot 3^3 \cdot 5^1 \cdot 7^1}\Big[-\frac{31^1}{2^2} p_1^3 + 11^1 p_1 p_2 \Big]$$

$$\hat{A}_4 = \frac{1}{2^9 \cdot 3^3 \cdot 5^2 \cdot 7^1}\Big[\frac{127^1}{2^6} p_1^4 - \frac{113^1}{2^3 \cdot 3^1} p_1^2 p_2 + \frac{13^1}{2^2 \cdot 3^1} p_2^2 - 1^1 p_4 \Big]$$

$$\hat{A}_5 = \frac{1}{2^{10} \cdot 3^5 \cdot 5^1 \cdot 11^1}\Big[-\frac{73^1}{2^8} p_1^5 + \frac{29^1 \cdot 37^1}{2^5 \cdot 5^1 \cdot 7^1} p_1^3 p_2 - \frac{311^1}{2^4 \cdot 5^1 \cdot 7^1} p_1 p_2^2 + \frac{53^1}{2^2 \cdot 5^1 \cdot 7^1} p_1 p_4 - \frac{1^1}{7^1} p_5 \Big]$$

$$\hat{A}_6 = \frac{1}{2^{12} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[\frac{23^1 \cdot 89^1 \cdot 691^1}{2^{10} \cdot 3^1 \cdot 5^1} p_1^6 - \frac{1540453^1}{2^8 \cdot 3^1 \cdot 5^1} p_1^4 p_2 + \frac{19^1 \cdot 4013^1}{2^6 \cdot 3^1} p_1^2 p_2^2 - \frac{16759^1}{2^4 \cdot 5^1} p_1^2 p_4 + \frac{23^1 \cdot 53^1}{2^1 \cdot 5^1} p_1 p_5 - \frac{19^1 \cdot 211^1}{2^4 \cdot 5^1} p_2^3$$
$$+ \frac{73^1 \cdot 79^1}{2^2 \cdot 3^1 \cdot 5^1} p_2 p_4 - \frac{691^1}{3^1 \cdot 5^1} p_6 \Big]$$

$$\hat{A}_7 = \frac{1}{2^{13} \cdot 3^4 \cdot 7^1 \cdot 13^1}\Big[-\frac{8191^1}{2^{12} \cdot 3^2 \cdot 5^2 \cdot 11^1} p_1^7 + \frac{37^1 \cdot 31121^1}{2^{10} \cdot 3^3 \cdot 5^3 \cdot 7^1 \cdot 11^1} p_1^5 p_2 - \frac{9161^1}{2^8 \cdot 3^1 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_1^3 p_2^2 + \frac{23^1 \cdot 127^1}{2^6 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_1^3 p_4 - \frac{2543^1}{2^4 \cdot 3^2 \cdot 5^3 \cdot 7^1 \cdot 11^1} p_1^2 p_5$$
$$+ \frac{109^1 \cdot 307^1}{2^6 \cdot 3^3 \cdot 5^3 \cdot 7^1 \cdot 11^1} p_1 p_2^3 - \frac{67^1}{2^4 \cdot 5^3 \cdot 11^1} p_1 p_2 p_4 + \frac{101^1}{2^2 \cdot 3^3 \cdot 5^3 \cdot 7^1} p_1 p_6 + \frac{1^1}{2^2 \cdot 3^3 \cdot 11^1} p_2 p_5 - \frac{1^1}{3^2 \cdot 5^2 \cdot 11^1} p_7 \Big]$$

$$\hat{A}_8 = \frac{1}{2^{16} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 13^1 \cdot 17^1}\Big[\frac{31^1 \cdot 151^1 \cdot 3617^1}{2^{15} \cdot 3^2 \cdot 5^2 \cdot 11^1} p_1^8 - \frac{2241667^1}{2^{11} \cdot 3^1 \cdot 5^2 \cdot 11^1} p_1^6 p_2 + \frac{71^1 \cdot 268823^1}{2^{10} \cdot 3^3 \cdot 5^2 \cdot 11^1} p_1^4 p_2^2 - \frac{3941363^1}{2^8 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_1^4 p_4 + \frac{26921^1}{2^4 \cdot 3^2 \cdot 5^2 \cdot 11^1} p_1^3 p_5$$
$$- \frac{29^1 \cdot 41^1 \cdot 227^1}{2^7 \cdot 3^3 \cdot 5^1 \cdot 11^1} p_1^2 p_2^3 + \frac{38923^1}{2^5 \cdot 3^2 \cdot 7^1 \cdot 11^1} p_1^2 p_2 p_4 - \frac{197033^1}{2^3 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_1^2 p_6 - \frac{23339^1}{2^2 \cdot 3^3 \cdot 5^2 \cdot 11^1} p_1 p_2 p_5 + \frac{1063^1}{2^1 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_1 p_7 + \frac{11^1 \cdot 1249^1}{2^7 \cdot 3^3 \cdot 5^1 \cdot 7^1} p_2^4$$
$$- \frac{275593^1}{2^4 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_2^2 p_4 + \frac{11299^1}{3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_2 p_6 + \frac{73^1 \cdot 199^1}{2^3 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_4^2 - \frac{3617^1}{2^1 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_8 \Big]$$

$$\hat{A}_9 = \frac{1}{2^{16} \cdot 3^7 \cdot 5^3 \cdot 7^3 \cdot 11^1 \cdot 19^1}\Big[-\frac{43867^1 \cdot 131071^1}{2^{18} \cdot 3^2 \cdot 13^1 \cdot 17^1} p_1^9 + \frac{47^1 \cdot 907^1 \cdot 7949^1}{2^{14} \cdot 3^1 \cdot 5^1 \cdot 17^1} p_1^7 p_2 - \frac{83^1 \cdot 3688543^1}{2^{13} \cdot 3^1 \cdot 5^1 \cdot 13^1} p_1^5 p_2^2 + \frac{5303^1 \cdot 44519^1}{2^{11} \cdot 3^2 \cdot 13^1 \cdot 17^1} p_1^5 p_4 - \frac{47756197^1}{2^9 \cdot 3^2 \cdot 13^1 \cdot 17^1} p_1^4 p_5$$
$$+ \frac{1625000107^1}{2^{10} \cdot 3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1^3 p_2^3 - \frac{613^1 \cdot 688087^1}{2^8 \cdot 3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1^3 p_2 p_4 + \frac{4569683^1}{2^6 \cdot 3^2 \cdot 13^1 \cdot 17^1} p_1^3 p_6 + \frac{1511459^1}{2^6 \cdot 3^1 \cdot 5^1 \cdot 17^1} p_1^2 p_2 p_5 - \frac{23^1 \cdot 86573^1}{2^3 \cdot 3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1^2 p_7$$
$$- \frac{37857689^1}{2^{10} \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1 p_2^4 + \frac{11389153^1}{2^7 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1 p_2^2 p_4 - \frac{3301303^1}{2^3 \cdot 3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1 p_2 p_6 - \frac{311^1 \cdot 41263^1}{2^6 \cdot 3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1 p_4^2 + \frac{467^1 \cdot 4969^1}{2^4 \cdot 3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1 p_8$$
$$- \frac{991^1 \cdot 1123^1}{2^5 \cdot 3^2 \cdot 13^1 \cdot 17^1} p_2^2 p_5 + \frac{41^1 \cdot 557^1}{3^2 \cdot 13^1 \cdot 17^1} p_2 p_7 + \frac{43^1 \cdot 4091^1}{2^3 \cdot 3^2 \cdot 13^1 \cdot 17^1} p_4 p_5 - \frac{43867^1}{2^2 \cdot 3^2 \cdot 13^1 \cdot 17^1} p_9 \Big]$$

One may go further and impose vanishing of multiple Pontrjagin classes at once. This is demonstrated in a few of the next expansions.

Expansion 22. *$\hat{A}$-genus of Fivebrane manifolds ($p_1 = p_2 = 0$).*

Expansion 22: Â-genus with Fivebrane structure

$$\hat{A}_3 = \frac{1}{2^6 \cdot 3^3 \cdot 5^1 \cdot 7^1}\Big[-1^1 p_3 \Big]$$

$$\hat{A}_4 = \frac{1}{2^9 \cdot 3^3 \cdot 5^2 \cdot 7^1}\Big[-1^1 p_4 \Big]$$

$$\hat{A}_5 = \frac{1}{2^{10} \cdot 3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1}\Big[-1^1 p_5 \Big]$$

$$\hat{A}_6 = \frac{1}{2^{12} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[19^1 \cdot 37^1 p_3^2 - 691^1 p_6 \Big]$$

$$\hat{A}_7 = \frac{1}{2^{13} \cdot 3^6 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1}\Big[\frac{283^1}{2^2 \cdot 5^1 \cdot 7^1} p_3 p_4 - 1^1 p_7 \Big]$$

$$\hat{A}_8 = \frac{1}{2^{17} \cdot 3^7 \cdot 5^4 \cdot 7^2 \cdot 13^1 \cdot 17^1}\Big[\frac{11^1 \cdot 181^1}{3^1} p_3 p_5 + \frac{73^1 \cdot 199^1}{2^2 \cdot 11^1} p_4^2 - \frac{3617^1}{11^1} p_8 \Big]$$

$$\hat{A}_9 = \frac{1}{2^{13} \cdot 3^9 \cdot 5^3 \cdot 7^3 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1}\Big[-\frac{61^1 \cdot 3659^1}{2^5 \cdot 5^1} p_3^3 + \frac{13829^1}{5^1} p_3 p_6 + \frac{43^1 \cdot 4091^1}{2^6} p_4 p_5 - \frac{43867^1}{2^5} p_9 \Big]$$

Expansion 23. *Â-genus of Ninebrane manifolds ($p_1 = p_2 = p_3 = 0$).*

Expansion 23: Â-genus with Ninebrane structure

$$\hat{A}_4 = \frac{1}{2^9.3^3.5^2.7^1}\Big[-1^1 p_4\Big]$$
$$\hat{A}_5 = \frac{1}{2^{10}.3^5.5^1.7^1.11^1}\Big[-1^1 p_5\Big]$$
$$\hat{A}_6 = \frac{1}{2^{12}.3^6.5^3.7^2.11^1.13^1}\Big[-691^1 p_6\Big]$$
$$\hat{A}_7 = \frac{1}{2^{13}.3^6.5^2.7^1.11^1.13^1}\Big[-1^1 p_7\Big]$$
$$\hat{A}_8 = \frac{1}{2^{17}.3^7.5^4.7^2.11^1.13^1.17^1}\Big[\frac{73^1\cdot 199^1}{2^2}p_4^2 - 3617^1 p_8\Big]$$
$$\hat{A}_9 = \frac{1}{2^{18}.3^9.5^3.7^3.11^1.13^1.17^1.19^1}\Big[\frac{43^1\cdot 4091^1}{2^1}p_4 p_5 - 43867^1 p_9\Big]$$

As in the case of Chern classes, situations might arise where one only has cohomology classes of degree $8k$, as all those of degree $8k+4$ vanish; hence, the next expansion. Observe that if further odd Pontrjagin classes would be set to zero, then all odd Â-genus would equal zero too.

Expansion 24. *Â-genus simplification - vanishing of all odd Pontrjagin classes ($p_{2i+1} = 0$).*

This necessarily makes the odd degree Â-genus vanish.

Expansion 24: Â-genus with only even Pontrjagin classes

$$\hat{A}_0 = 1$$
$$\hat{A}_2 = \frac{1}{2^5.3^2.5^1}\Big[-1^1 p_2\Big]$$
$$\hat{A}_4 = \frac{1}{2^9.3^3.5^2.7^1}\Big[\frac{13^1}{2^2.3^1}p_2^2 - 1^1 p_4\Big]$$
$$\hat{A}_6 = \frac{1}{2^{12}.3^5.5^3.7^2.11^1.13^1}\Big[-\frac{19^1\cdot 211^1}{2^4}p_2^3 + \frac{73^1\cdot 79^1}{2^2.3^1}p_2 p_4 - \frac{691^1}{3^1}p_6\Big]$$
$$\hat{A}_8 = \frac{1}{2^{16}.3^7.5^3.7^2.13^1.17^1}\Big[\frac{11^1\cdot 1249^1}{2^7.3^1}p_2^4 - \frac{275593^1}{2^4.3^1.5^1.11^1}p_2^2 p_4 + \frac{11299^1}{3^1.5^1.11^1}p_2 p_6 + \frac{73^1\cdot 199^1}{2^3.5^1.11^1}p_4^2 - \frac{3617^1}{2^1.5^1.11^1}p_8\Big]$$

Note that if an ith expansion of a genus is zero, then it is often omitted in this paper.

4.2.1 Â-genus and complexification

Here we are using Equation 3.2, which relates Pontrjagin and Chern class, to convert the L-genus or Â-genus from Pontrjagin to Chern classes.

Expansion 25. *Â-genus expansion in terms of Chern classes of complexification.*

Note that the denominators in the following expressions are exactly those showing up in the corresponding term in the expansion via Pontrjagin classes, Expansion 18, since the relation between the p's and the c's only involves a minus sign and no numerical factors.

Expansion 25: Â-genus expansion in terms of Chern classes of complexification

$$\hat{A}_1 = \frac{1}{2^3\cdot 3^1}\Big[c_2\Big]$$

$$\hat{A}_2 = \frac{1}{2^5\cdot 3^2\cdot 5^1}\Big[\frac{7^1}{2^2}c_2^2 - 1^1 c_4\Big]$$

$$\hat{A}_3 = \frac{1}{2^6\cdot 3^3\cdot 5^1\cdot 7^1}\Big[\frac{31^1}{2^4}c_2^3 - \frac{11^1}{2^2}c_2c_4 + 1^1 c_6\Big]$$

$$\hat{A}_4 = \frac{1}{2^6\cdot 3^3\cdot 5^2\cdot 7^1}\Big[\frac{127^1}{2^9}c_2^4 - \frac{113^1}{2^6\cdot 3^1}c_2^2c_4 + \frac{1^1}{3^1}c_2c_6 + \frac{13^1}{2^5\cdot 3^1}c_4^2 - \frac{1^1}{2^3}c_8\Big]$$

$$\hat{A}_5 = \frac{1}{2^{10}\cdot 3^4\cdot 5^1\cdot 11^1}\Big[\frac{73^1}{2^8\cdot 3^1}c_2^5 - \frac{29^1\cdot 37^1}{2^5\cdot 3^1\cdot 5^1\cdot 7^1}c_2^3c_4 + \frac{61^1}{2^3\cdot 5^1\cdot 7^1}c_2^2c_6 + \frac{311^1}{2^4\cdot 3^1\cdot 5^1\cdot 7^1}c_2c_4^2 - \frac{53^1}{2^2\cdot 3^1\cdot 5^1\cdot 7^1}c_2c_8 - \frac{1^1}{2^1\cdot 5^1}c_4c_6 + \frac{1^1}{3^1\cdot 7^1}c_{10}\Big]$$

$$\hat{A}_6 = \frac{1}{2^{12}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11^1\cdot 13^1}\Big[\frac{23^1\cdot 89^1\cdot 691^1}{2^{10}\cdot 3^1\cdot 5^1}c_2^6 - \frac{1540453^1}{2^8\cdot 3^1\cdot 5^1}c_2^4c_4 + \frac{29^1\cdot 1249^1}{2^3\cdot 3^1\cdot 5^1}c_2^3c_6 + \frac{19^1\cdot 4013^1}{2^6\cdot 3^1}c_2^2c_4^2 - \frac{16759^1}{2^4\cdot 5^1}c_2^2c_8 - \frac{3491^1}{2^1\cdot 5^1}c_2c_4c_6$$
$$+ \frac{23^1\cdot 53^1}{2^1\cdot 5^1}c_2c_{10} - \frac{19^1\cdot 211^1}{2^4\cdot 5^1}c_4^3 + \frac{73^1\cdot 79^1}{2^2\cdot 3^1\cdot 5^1}c_4c_8 + \frac{19^1\cdot 37^1}{3^1\cdot 5^1}c_6^2 - \frac{691^1}{3^1\cdot 5^1}c_{12}\Big]$$

$$\hat{A}_7 = \frac{1}{2^{11}\cdot 3^4\cdot 7^1\cdot 13^1}\Big[\frac{8191^1}{2^{14}\cdot 3^2\cdot 5^2\cdot 11^1}c_2^7 - \frac{37^1\cdot 31121^1}{2^{12}\cdot 3^3\cdot 5^3\cdot 7^1\cdot 11^1}c_2^5c_4 + \frac{67^1\cdot 127^1}{2^{10}\cdot 5^3\cdot 7^1\cdot 11^1}c_2^4c_6 + \frac{9161^1}{2^{10}\cdot 3^1\cdot 5^2\cdot 7^1\cdot 11^1}c_2^3c_4^2 - \frac{23^1\cdot 127^1}{2^8\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1}c_2^3c_8$$
$$- \frac{179^1\cdot 317^1}{2^7\cdot 3^3\cdot 5^3\cdot 7^1\cdot 11^1}c_2^2c_4c_6 + \frac{2543^1}{2^6\cdot 3^2\cdot 5^3\cdot 7^1\cdot 11^1}c_2^2c_{10} - \frac{109^1\cdot 307^1}{2^8\cdot 3^3\cdot 5^3\cdot 7^1\cdot 11^1}c_2c_4^3 + \frac{67^1}{2^6\cdot 5^3\cdot 11^1}c_2c_4c_8 + \frac{97^1}{3^3\cdot 5^3\cdot 7^1\cdot 11^1}c_2c_6^2$$
$$- \frac{101^1}{2^4\cdot 3^3\cdot 5^3\cdot 7^1}c_2c_{12} + \frac{23^1\cdot 233^1}{2^6\cdot 3^3\cdot 5^3\cdot 7^1\cdot 11^1}c_4^2c_6 - \frac{1^1}{2^4\cdot 3^3\cdot 11^1}c_4c_{10}$$
$$- \frac{283^1}{2^4\cdot 3^2\cdot 5^3\cdot 7^1\cdot 11^1}c_6c_8 + \frac{1^1}{2^2\cdot 3^2\cdot 5^2\cdot 11^1}c_{14}\Big]$$

$$\hat{A}_8 = \frac{1}{2^{15}\cdot 3^5\cdot 5^2\cdot 7^1\cdot 17^1}\Big[\frac{31^1\cdot 151^1\cdot 3617^1}{2^{16}\cdot 3^2\cdot 5^2\cdot 11^1\cdot 13^1}c_2^8 - \frac{2241667^1}{2^{12}\cdot 3^1\cdot 5^2\cdot 11^1\cdot 13^1}c_2^6c_4 + \frac{3661841^1}{2^7\cdot 3^3\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_2^5c_6 + \frac{71^1\cdot 268823^1}{2^{11}\cdot 3^3\cdot 5^2\cdot 11^1\cdot 13^1}c_2^4c_4^2$$
$$- \frac{3941363^1}{2^9\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_2^4c_8 - \frac{317^1\cdot 4129^1}{2^4\cdot 3^3\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_2^3c_4c_6 + \frac{26921^1}{2^5\cdot 3^2\cdot 5^2\cdot 11^1\cdot 13^1}c_2^3c_{10} - \frac{29^1\cdot 41^1\cdot 227^1}{2^8\cdot 3^3\cdot 5^1\cdot 11^1\cdot 13^1}c_2^2c_4^3$$
$$+ \frac{38923^1}{2^6\cdot 3^2\cdot 7^1\cdot 11^1\cdot 13^1}c_2^2c_4c_8 + \frac{907^1}{2^2\cdot 3^1\cdot 5^2\cdot 7^1\cdot 13^1}c_2^2c_6^2 - \frac{197033^1}{2^4\cdot 3^3\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_2^2c_{12} + \frac{9091^1}{2^3\cdot 3^3\cdot 5^1\cdot 11^1\cdot 13^1}c_2c_4^2c_6$$
$$- \frac{23339^1}{2^3\cdot 3^3\cdot 5^2\cdot 11^1\cdot 13^1}c_2c_4c_{10} - \frac{39887^1}{2^1\cdot 3^3\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_2c_6c_8 + \frac{1063^1}{2^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_2c_{14} + \frac{11^1\cdot 1249^1}{2^8\cdot 3^3\cdot 5^1\cdot 7^1\cdot 13^1}c_4^4 - \frac{275593^1}{2^5\cdot 3^3\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_4^2c_8$$
$$- \frac{31^1}{3^2\cdot 5^2\cdot 11^1}c_4c_6^2 + \frac{11299^1}{2^1\cdot 3^3\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_4c_{12} + \frac{11^1\cdot 181^1}{2^2\cdot 3^3\cdot 5^2\cdot 7^1\cdot 13^1}c_6c_{10} + \frac{73^1\cdot 199^1}{2^4\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_8^2 - \frac{3617^1}{2^2\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_{16}\Big]$$

$$\hat{A}_9 = \frac{1}{2^{13}\cdot 3^7\cdot 5^2\cdot 7^2\cdot 19^1}\Big[\frac{43867^1\cdot 131071^1}{2^{21}\cdot 3^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^9 - \frac{47^1\cdot 907^1\cdot 7949^1}{2^{17}\cdot 3^1\cdot 5^2\cdot 7^1\cdot 11^1\cdot 17^1}c_2^7c_4 + \frac{9397^1\cdot 33317^1}{2^{15}\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^6c_6 + \frac{83^1\cdot 3688543^1}{2^{16}\cdot 3^1\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1}c_2^5c_4^2$$
$$- \frac{5303^1\cdot 44519^1}{2^{14}\cdot 3^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^5c_8 - \frac{601^1\cdot 4429813^1}{2^{13}\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^4c_4c_6 + \frac{47756197^1}{2^{12}\cdot 3^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^4c_{10}$$
$$- \frac{1625000107^1}{2^{13}\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^3c_4^3 + \frac{613^1\cdot 688087^1}{2^{11}\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^3c_4c_8 + \frac{23^1\cdot 653^1}{2^9\cdot 5^1\cdot 13^1\cdot 17^1}c_2^3c_6^2 - \frac{4569683^1}{2^9\cdot 3^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^3c_{12}$$
$$+ \frac{59^1\cdot 163^1\cdot 8377^1}{2^{11}\cdot 3^2\cdot 5^2\cdot 11^1\cdot 13^1\cdot 17^1}c_2^2c_4^2c_6 - \frac{1511459^1}{2^9\cdot 3^1\cdot 5^2\cdot 7^1\cdot 11^1\cdot 17^1}c_2^2c_4c_{10} - \frac{23^1\cdot 263^1\cdot 3187^1}{2^9\cdot 3^1\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^2c_6c_8 + \frac{23^1\cdot 86573^1}{2^6\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2^2c_{14}$$
$$+ \frac{37857689^1}{2^{13}\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2c_4^4 - \frac{11389153^1}{2^{10}\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2c_4^2c_8 - \frac{199^1\cdot 4231^1}{2^7\cdot 3^1\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2c_4c_6^2 + \frac{3301303^1}{2^6\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2c_4c_{12}$$
$$+ \frac{127^1\cdot 2029^1}{2^7\cdot 3^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2c_6c_{10} + \frac{311^1\cdot 41263^1}{2^9\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2c_8^2 - \frac{467^1\cdot 4969^1}{2^7\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_2c_{16} - \frac{15013651^1}{2^9\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_4^3c_6$$
$$+ \frac{991^1\cdot 1123^1}{2^8\cdot 3^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_4^2c_{10} + \frac{43^1\cdot 42239^1}{2^7\cdot 3^1\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_4c_6c_8 - \frac{41^1\cdot 557^1}{2^3\cdot 3^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_4c_{14} + \frac{61^1\cdot 3659^1}{2^5\cdot 3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_6^3$$
$$- \frac{13829^1}{3^2\cdot 5^2\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_6c_{12} - \frac{43^1\cdot 4091^1}{2^6\cdot 3^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_8c_{10} + \frac{43867^1}{2^5\cdot 3^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1\cdot 17^1}c_{18}\Big]$$

Expansion 26. *Â-genus in terms of Chern classes with complex String structure ($c_2 = 0$).*

Expansion 26: Â-genus in terms of Chern classes with complex String structure

$$\hat{A}_2 = \frac{1}{2^5\cdot 3^2\cdot 5^1}\Big[-1^1 c_4\Big]$$

$$\hat{A}_3 = \frac{1}{2^6\cdot 3^3\cdot 5^1\cdot 7^1}\Big[c_6\Big]$$

$$\hat{A}_4 = \frac{1}{2^9\cdot 3^3\cdot 5^2\cdot 7^1}\Big[\frac{13^1}{2^2\cdot 3^1}c_4^2 - 1^1 c_8\Big]$$

$$\hat{A}_5 = \frac{1}{2^{10}\cdot 3^4\cdot 5^1\cdot 11^1}\Big[-\frac{1^1}{2^1\cdot 5^1}c_4c_6 + \frac{1^1}{3^1\cdot 7^1}c_{10}\Big]$$

$$\hat{A}_6 = \frac{1}{2^{12}\cdot 3^5\cdot 5^3\cdot 7^2\cdot 11^1\cdot 13^1}\Big[-\frac{19^1\cdot 211^1}{2^4}c_4^3 + \frac{73^1\cdot 79^1}{2^2\cdot 3^1}c_4c_8 + \frac{19^1\cdot 37^1}{3^1}c_6^2 - \frac{691^1}{3^1}c_{12}\Big]$$

$$\hat{A}_7 = \frac{1}{2^{13}\cdot 3^6\cdot 7^1\cdot 11^1\cdot 13^1}\Big[\frac{23^1\cdot 233^1}{2^4\cdot 3^1\cdot 5^3\cdot 7^1}c_4^2c_6 - \frac{1^1}{2^2\cdot 3^1}c_4c_{10} - \frac{283^1}{2^2\cdot 5^3\cdot 7^1}c_6c_8 + \frac{1^1}{5^2}c_{14}\Big]$$

$$\hat{A}_8 = \frac{1}{2^{15}\cdot 3^7\cdot 5^3\cdot 7^1\cdot 17^1}\Big[\frac{11^1\cdot 1249^1}{2^8\cdot 3^1\cdot 7^1\cdot 13^1}c_4^4 - \frac{275593^1}{2^5\cdot 3^1\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}c_4^2c_8 - \frac{31^1}{5^1\cdot 11^1}c_4c_6^2 + \frac{11299^1}{2^1\cdot 3^1\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}c_4c_{12}$$
$$+ \frac{11^1\cdot 181^1}{2^2\cdot 3^1\cdot 5^1\cdot 7^1\cdot 13^1}c_6c_{10} + \frac{73^1\cdot 199^1}{2^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}c_8^2 - \frac{3617^1}{2^2\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}c_{16}\Big]$$

Note that, in this case, the first Chern class is necessarily trivial, so indeed $c_2 = 0$ defines a complex String structure. Also, similarly to the previous case, the denominators in the above expressions are exactly those showing up in the corresponding term in the expansion with a p_1 or String structure, i.e., Expansion 19.

4.3 The L-genus

The strategy and implementation for the L-genus as well as for Todd genus below follow the same pattern as that of the Â-genus, since most of the Mathematica functions are shared. Here, we are presenting the results of the L-genus using the characteristic power series $\frac{\sqrt{x_j}}{\tanh\sqrt{x_j}}$, as defined in Section 4.1.

Expansion 27. *L-genus in terms of Pontrjagin classes.*

Expansion 27:L-genus in terms of Pontrjagin classes

$$L_0 = 1$$

$$L_1 = \frac{1}{3^1}\Big[p_1\Big]$$

$$L_2 = \frac{1}{3^2\cdot 5^1}\Big[-1^1 p_1^2 + 7^1 p_2\Big]$$

$$L_3 = \frac{1}{3^3\cdot 5^1\cdot 7^1}\Big[+2^1 p_1^3 - 13^1 p_1 p_2 + 2^1\cdot 31^1 p_3\Big]$$

$$L_4 = \frac{1}{3^3\cdot 5^2\cdot 7^1}\Big[-1^1 p_1^4 + \frac{2^1\cdot 11^1}{3^1} p_1^2 p_2 - \frac{71^1}{3^1} p_1 p_3 - \frac{19^1}{3^1} p_2^2 + 127^1 p_4\Big]$$

$$L_5 = \frac{1}{3^4\cdot 5^1\cdot 11^1}\Big[+\frac{2^1}{3^1\cdot 7^1} p_1^5 - \frac{83^1}{3^1\cdot 5^1\cdot 7^1} p_1^3 p_2 + \frac{79^1}{5^1\cdot 7^1} p_1^2 p_3 + \frac{127^1}{3^1\cdot 5^1\cdot 7^1} p_1 p_2^2 - \frac{919^1}{3^1\cdot 5^1\cdot 7^1} p_1 p_4 - \frac{2^4}{5^1} p_2 p_3 + \frac{2^1\cdot 73^1}{3^1} p_5\Big]$$

$$L_6 = \frac{1}{3^5\cdot 5^2\cdot 7^2\cdot 11^1\cdot 13^1}\Big[-\frac{2^1\cdot 691^1}{3^1\cdot 5^1} p_1^6 + \frac{2^1\cdot 6421^1}{3^1\cdot 5^1} p_1^4 p_2 - \frac{33863^1}{3^1\cdot 5^1} p_1^3 p_3 - \frac{5527^1}{3^1} p_1^2 p_2^2 + \frac{40841^1}{5^1} p_1^2 p_4 + \frac{83^1\cdot 349^1}{5^1} p_1 p_2 p_3$$
$$- \frac{2^5\cdot 29^1\cdot 181^1}{5^1} p_1 p_5 + \frac{2^1\cdot 1453^1}{5^1} p_2^3 - \frac{159287^1}{3^1\cdot 5^1} p_2 p_4 - \frac{167^1\cdot 241^1}{3^1\cdot 5^1} p_3^2 + \frac{2^1\cdot 23^1\cdot 89^1\cdot 691^1}{3^1\cdot 5^1} p_6\Big]$$

$$L_7 = \frac{1}{3^2\cdot 5^1\cdot 7^1\cdot 13^1}\Big[+\frac{2^2}{3^4\cdot 5^1\cdot 11^1} p_1^7 - \frac{2^1\cdot 2161^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} p_1^5 p_2 + \frac{2^2}{5^2\cdot 7^1} p_1^4 p_3 + \frac{2^3}{3^2\cdot 5^1\cdot 7^1} p_1^3 p_2^2 - \frac{2^1\cdot 113^1}{3^4\cdot 5^1\cdot 7^1} p_1^3 p_4 - \frac{39341^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} p_1^2 p_2 p_3$$
$$+ \frac{2^4\cdot 277^1}{3^4\cdot 5^2\cdot 7^1} p_1^2 p_5 - \frac{2^1\cdot 3989^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} p_1 p_2^3 + \frac{1399^1}{3^3\cdot 5^2\cdot 11^1} p_1 p_2 p_4 + \frac{22027^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} p_1 p_3^2 - \frac{2^1\cdot 305633^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} p_1 p_6 + \frac{2^3\cdot 2087^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} p_2^2 p_3$$
$$- \frac{2^1\cdot 23^2}{3^5\cdot 11^1} p_2 p_5 - \frac{2^1\cdot 97^1\cdot 107^1}{3^4\cdot 5^2\cdot 7^1\cdot 11^1} p_3 p_4 + \frac{2^2\cdot 8191^1}{3^4\cdot 5^1\cdot 11^1} p_7\Big]$$

$$L_8 = \frac{1}{3^3\cdot 5^3\cdot 7^1\cdot 17^1}\Big[-\frac{3617^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1^8 + \frac{2^2\cdot 41^1\cdot 83^1}{3^3\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1^6 p_2 - \frac{2^1\cdot 143483^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1^5 p_3 - \frac{2^1\cdot 43^1\cdot 431^1}{3^5\cdot 5^1\cdot 7^1\cdot 13^1} p_1^4 p_2^2 + \frac{2^1\cdot 162011^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1^4 p_4$$
$$+ \frac{2^6\cdot 29^1\cdot 739^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1^3 p_2 p_3 - \frac{97^1\cdot 12889^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1^3 p_5 + \frac{2^7\cdot 661^1}{3^5\cdot 7^1\cdot 11^1\cdot 13^1} p_1^2 p_2^3 - \frac{97^1\cdot 2273^1}{3^4\cdot 7^1\cdot 11^1\cdot 13^1} p_1^2 p_2 p_4 - \frac{107^1\cdot 857^1}{3^3\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1^2 p_3^2$$
$$+ \frac{1091^1\cdot 13789^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1^2 p_6 - \frac{2^1\cdot 9619^1}{3^5\cdot 7^1\cdot 11^1} p_1 p_2^2 p_3 + \frac{2^1\cdot 617147^1}{3^5\cdot 5^1\cdot 11^1\cdot 13^1} p_1 p_2 p_5 + \frac{29^1\cdot 98057^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_1 p_3 p_4 - \frac{37117^1}{5^1\cdot 11^1\cdot 13^1} p_1 p_7$$
$$- \frac{13687^1}{3^5\cdot 7^1\cdot 11^1\cdot 13^1} p_2^4 + \frac{2^1\cdot 101^1\cdot 6491^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_2^2 p_4 + \frac{2^3\cdot 311^1}{3^4\cdot 5^1\cdot 11^1} p_2 p_3^2 - \frac{191^1\cdot 14143^1}{3^5\cdot 5^1\cdot 11^1\cdot 13^1} p_2 p_6 - \frac{29^1\cdot 31^1\cdot 79^1}{3^5\cdot 5^1\cdot 13^1} p_3 p_5 - \frac{167^1\cdot 2663^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} p_4^2$$
$$+ \frac{31^1\cdot 151^1\cdot 3617^1}{3^4\cdot 5^1\cdot 11^1\cdot 13^1} p_8\Big]$$

$$L_9 = \frac{1}{3^7\cdot 5^3\cdot 7^1\cdot 19^1}\Big[+\frac{2^1\cdot 43867^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^9 - \frac{41^1\cdot 53^1\cdot 827^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^7 p_2 + \frac{1337617^1}{5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^6 p_3 + \frac{541^1\cdot 761^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1} p_1^5 p_2^2$$
$$- \frac{29^1\cdot 79^1\cdot 3467^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^5 p_4 - \frac{3931^1\cdot 17981^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^4 p_2 p_3 + \frac{61^1\cdot 491^1\cdot 1013^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^4 p_5 - \frac{2^1\cdot 31^1\cdot 475957^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^3 p_2^3$$
$$+ \frac{2^2\cdot 1697^1\cdot 25951^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^3 p_2 p_4 + \frac{2^1\cdot 70003^1}{7^1\cdot 11^1\cdot 13^1\cdot 17^1} p_1^3 p_3^2 - \frac{29^1\cdot 53^1\cdot 181^1\cdot 433^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^3 p_6 + \frac{2^1\cdot 93133^1}{3^2\cdot 5^1\cdot 13^1\cdot 17^1} p_1^2 p_2^2 p_3$$
$$- \frac{2^1\cdot 83892287^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^2 p_2 p_5 - \frac{743^1\cdot 73597^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1^2 p_3 p_4 + \frac{887^1\cdot 210719^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 17^1} p_1^2 p_7 + \frac{13^1\cdot 7297^1}{5^1\cdot 7^2\cdot 11^1\cdot 17^1} p_1 p_2^4$$
$$- \frac{2^1\cdot 8359009^1}{5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1 p_2^2 p_4 - \frac{2^2\cdot 83^1\cdot 15667^1}{3^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1 p_2 p_3^2 + \frac{47^1\cdot 1291^1\cdot 22709^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1 p_2 p_6 + \frac{641^1\cdot 121169^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1 p_3 p_5$$
$$+ \frac{67^1\cdot 101^1\cdot 13883^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1 p_4^2 - \frac{43^1\cdot 237398563^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_1 p_8 - \frac{2^4\cdot 31^1\cdot 139^1\cdot 293^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_2^3 p_3 + \frac{2^2\cdot 31^1\cdot 314173^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_2^2 p_5$$
$$+ \frac{2^2\cdot 89^1\cdot 11071^1}{3^1\cdot 5^1\cdot 7^2\cdot 13^1\cdot 17^1} p_2 p_3 p_4 - \frac{2^1\cdot 23^1\cdot 1933^1\cdot 6857^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_2 p_7 + \frac{2^3\cdot 89^1\cdot 17929^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_3^3 - \frac{2^1\cdot 367^1\cdot 1150249^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_3 p_6$$
$$- \frac{2^2\cdot 13569497^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_4 p_5 + \frac{2^1\cdot 43867^1\cdot 131071^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} p_9\Big]$$

L-genus with simplifications

As in the case of Â-genus one can get simplifications in the presence of the higher structures. These expressions were obtained by using the two genus simplification formulas stated in the Â-genus section above.

Expansion 28. *L-genus for a String manifolds or p_1-structure ($p_1 = 0$).*

Expansion 28: L-genus with String pr p_1 structure

$$L_0 = 1$$
$$L_1 = 0$$
$$L_2 = \frac{1}{3^2 \cdot 5^1}\Big[+7^1 p_2\Big]$$
$$L_3 = \frac{1}{3^3 \cdot 5^1 \cdot 7^1}\Big[+2^1 \cdot 31^1 p_3\Big]$$
$$L_4 = \frac{1}{3^3 \cdot 5^2 \cdot 7^1}\Big[-\frac{19^1}{3^1} p_2^2 + 127^1 p_4\Big]$$
$$L_5 = \frac{1}{3^4 \cdot 5^1 \cdot 11^1}\Big[-\frac{2^4}{5^1} p_2 p_3 + \frac{2^1 \cdot 73^1}{3^1} p_5\Big]$$
$$L_6 = \frac{1}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[+2^1 \cdot 1453^1 p_2^3 - \frac{159287^1}{3^1} p_2 p_4 - \frac{167^1 \cdot 241^1}{3^1} p_3^2 + \frac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1} p_6\Big]$$
$$L_7 = \frac{1}{3^6 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1}\Big[+\frac{2^3 \cdot 2087^1}{3^1 \cdot 5^2 \cdot 7^1} p_2^2 p_3 - \frac{2^1 \cdot 23^2}{3^1} p_2 p_5 - \frac{2^1 \cdot 97^1 \cdot 107^1}{5^2 \cdot 7^1} p_3 p_4 + \frac{2^2 \cdot 8191^1}{5^1} p_7\Big]$$
$$L_8 = \frac{1}{3^7 \cdot 5^3 \cdot 7^1 \cdot 17^1}\Big[-\frac{13687^1}{3^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} p_2^4 + \frac{2^1 \cdot 101^1 \cdot 6491^1}{3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} p_2^2 p_4 + \frac{2^3 \cdot 311^1}{5^1 \cdot 11^1} p_2 p_3^2 - \frac{191^1 \cdot 14143^1}{3^1 \cdot 5^1 \cdot 11^1 \cdot 13^1} p_2 p_6 - \frac{29^1 \cdot 31^1 \cdot 79^1}{3^1 \cdot 5^1 \cdot 13^1} p_3 p_5$$
$$- \frac{167^1 \cdot 2663^1}{5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} p_4^2 + \frac{31^1 \cdot 151^1 \cdot 3617^1}{5^1 \cdot 11^1 \cdot 13^1} p_8\Big]$$
$$L_9 = \frac{1}{3^8 \cdot 5^3 \cdot 7^3 \cdot 13^1 \cdot 17^1 \cdot 19^1}\Big[-\frac{2^4 \cdot 31^1 \cdot 139^1 \cdot 293^1}{3^1 \cdot 5^1 \cdot 11^1} p_2^3 p_3 + \frac{2^2 \cdot 31^1 \cdot 314173^1}{3^1 \cdot 11^1} p_2^2 p_5 + \frac{2^2 \cdot 89^1 \cdot 11071^1}{5^1} p_2 p_3 p_4 - \frac{2^1 \cdot 23^1 \cdot 1933^1 \cdot 6857^1}{3^1 \cdot 11^1} p_2 p_7$$
$$+ \frac{2^3 \cdot 89^1 \cdot 17929^1}{3^1 \cdot 5^1 \cdot 11^1} p_3^3 - \frac{2^1 \cdot 367^1 \cdot 1150249^1}{3^1 \cdot 5^1 \cdot 11^1} p_3 p_6 - \frac{2^2 \cdot 13569497^1}{3^1 \cdot 11^1} p_4 p_5 + \frac{2^1 \cdot 43867^1 \cdot 131071^1}{3^1 \cdot 11^1} p_9\Big]$$

Expansion 29. *L-genus with p_2-structure manifolds ($p_2 = 0$).*

Expansion 29: L-genus with p_2-structure

$$L_0 = 1$$
$$L_1 = \frac{1}{3^1}\Big[p_1\Big]$$
$$L_2 = \frac{1}{3^2 \cdot 5^1}\Big[-1^1 p_1^2\Big]$$
$$L_3 = \frac{1}{3^3 \cdot 5^1 \cdot 7^1}\Big[+2^1 p_1^3 + 2^1 \cdot 31^1 p_3\Big]$$
$$L_4 = \frac{1}{3^3 \cdot 5^2 \cdot 7^1}\Big[-1^1 p_1^4 - \frac{71^1}{3^1} p_1 p_3 + 127^1 p_4\Big]$$
$$L_5 = \frac{1}{3^4 \cdot 5^1 \cdot 11^1}\Big[+\frac{2^1}{3^1 \cdot 7^1} p_1^5 + \frac{79^1}{5^1 \cdot 7^1} p_1^2 p_3 - \frac{919^1}{3^1 \cdot 5^1 \cdot 7^1} p_1 p_4 + \frac{2^1 \cdot 73^1}{3^1} p_5\Big]$$
$$L_6 = \frac{1}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[-\frac{2^1 \cdot 691^1}{3^1} p_1^6 - \frac{33863^1}{3^1} p_1^3 p_3 + 40841^1 p_1^2 p_4 - 2^5 \cdot 29^1 \cdot 181^1 p_1 p_5 - \frac{167^1 \cdot 241^1}{3^1} p_3^2 + \frac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1} p_6\Big]$$
$$L_7 = \frac{1}{3^2 \cdot 5^2 \cdot 7^1 \cdot 13^1}\Big[+\frac{2^2}{3^4 \cdot 11^1} p_1^7 + \frac{2^2}{5^1 \cdot 7^1} p_1^4 p_3 - \frac{2^1 \cdot 113^1}{3^4 \cdot 7^1} p_1^3 p_4 + \frac{2^4 \cdot 277^1}{3^4 \cdot 5^1 \cdot 7^1} p_1^2 p_5 + \frac{22027^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1 p_3^2 - \frac{2^1 \cdot 305633^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1 p_6$$
$$- \frac{2^1 \cdot 97^1 \cdot 107^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_3 p_4 + \frac{2^2 \cdot 8191^1}{3^4 \cdot 11^1} p_7\Big]$$
$$L_8 = \frac{1}{3^3 \cdot 5^4 \cdot 7^1 \cdot 13^1 \cdot 17^1}\Big[-\frac{3617^1}{3^4 \cdot 7^1 \cdot 11^1} p_1^8 - \frac{2^1 \cdot 143483^1}{3^5 \cdot 7^1 \cdot 11^1} p_1^5 p_3 + \frac{2^1 \cdot 162011^1}{3^4 \cdot 7^1 \cdot 11^1} p_1^4 p_4 - \frac{97^1 \cdot 12889^1}{3^4 \cdot 7^1 \cdot 11^1} p_1^3 p_5 - \frac{107^1 \cdot 857^1}{3^3 \cdot 7^1 \cdot 11^1} p_1^2 p_3^2$$
$$+ \frac{1091^1 \cdot 13789^1}{3^5 \cdot 7^1 \cdot 11^1} p_1^2 p_6 + \frac{29^1 \cdot 98057^1}{3^5 \cdot 7^1 \cdot 11^1} p_1 p_3 p_4 - \frac{37117^1}{11^1} p_1 p_7 - \frac{29^1 \cdot 31^1 \cdot 79^1}{3^5} p_3 p_5 - \frac{167^1 \cdot 2663^1}{3^4 \cdot 7^1 \cdot 11^1} p_4^2 + \frac{31^1 \cdot 151^1 \cdot 3617^1}{3^4 \cdot 11^1} p_8\Big]$$
$$L_9 = \frac{1}{3^7 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 17^1 \cdot 19^1}\Big[+\frac{2^1 \cdot 43867^1}{3^2 \cdot 7^1 \cdot 13^1} p_1^9 + \frac{1337617^1}{5^1 \cdot 7^1 \cdot 13^1} p_1^6 p_3 - \frac{29^1 \cdot 79^1 \cdot 3467^1}{3^2 \cdot 7^1 \cdot 13^1} p_1^5 p_4 + \frac{61^1 \cdot 491^1 \cdot 1013^1}{3^2 \cdot 7^1 \cdot 13^1} p_1^4 p_5 + \frac{2^1 \cdot 70003^1}{13^1} p_1^3 p_3^2$$
$$- \frac{29^1 \cdot 53^1 \cdot 181^1 \cdot 433^1}{3^2 \cdot 7^1 \cdot 13^1} p_1^3 p_6 - \frac{743^1 \cdot 73597^1}{3^1 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_1^2 p_3 p_4 + \frac{887^1 \cdot 210719^1}{3^2 \cdot 5^1 \cdot 7^1} p_1^2 p_7 + \frac{641^1 \cdot 121169^1}{3^2 \cdot 7^1 \cdot 13^1} p_1 p_3 p_5 + \frac{67^1 \cdot 101^1 \cdot 13883^1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_1 p_4^2$$
$$- \frac{43^1 \cdot 237398563^1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_1 p_8 + \frac{2^3 \cdot 89^1 \cdot 17929^1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_3^3 - \frac{2^1 \cdot 367^1 \cdot 1150249^1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_3 p_6 - \frac{2^2 \cdot 13569497^1}{3^2 \cdot 7^1 \cdot 13^1} p_4 p_5 + \frac{2^1 \cdot 43867^1 \cdot 131071^1}{3^2 \cdot 7^1 \cdot 13^1} p_9\Big]$$

Expansion 30. *L-genus with p_3-structure manifolds ($p_3 = 0$).*

Expansion 30: L-genus with p_3-structure

$$L_1 = \frac{1}{3^1}\Big[p_1\Big]$$
$$L_2 = \frac{1}{3^2 \cdot 5^1}\Big[-1^1 p_1^2 + 7^1 p_2\Big]$$
$$L_3 = \frac{1}{3^3 \cdot 5^1 \cdot 7^1}\Big[+2^1 p_1^3 - 13^1 p_1 p_2\Big]$$
$$L_4 = \frac{1}{3^3 \cdot 5^2 \cdot 7^1}\Big[-1^1 p_1^4 + \frac{2^1 \cdot 11^1}{3^1} p_1^2 p_2 - \frac{19^1}{3^1} p_2^2 + 127^1 p_4\Big]$$
$$L_5 = \frac{1}{3^5 \cdot 5^1 \cdot 11^1}\Big[+\frac{2^1}{7^1} p_1^5 - \frac{83^1}{5^1 \cdot 7^1} p_1^3 p_2 + \frac{127^1}{5^1 \cdot 7^1} p_1 p_2^2 - \frac{919^1}{5^1 \cdot 7^1} p_1 p_4 + 2^1 \cdot 73^1 p_5\Big]$$
$$L_6 = \frac{1}{3^5 \cdot 5^2 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[-\frac{2^1 \cdot 691^1}{3^1 \cdot 5^1} p_1^6 + \frac{2^1 \cdot 6421^1}{3^1 \cdot 5^1} p_1^4 p_2 - \frac{5527^1}{3^1} p_1^2 p_2^2 + \frac{40841^1}{5^1} p_1^2 p_4 - \frac{2^5 \cdot 29^1 \cdot 181^1}{5^1} p_1 p_5 + \frac{2^1 \cdot 1453^1}{5^1} p_2^3$$
$$- \frac{159287^1}{3^1 \cdot 5^1} p_2 p_4 + \frac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1 \cdot 5^1} p_6\Big]$$
$$L_7 = \frac{1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 13^1}\Big[+\frac{2^2}{3^2 \cdot 5^1 \cdot 11^1} p_1^7 - \frac{2^1 \cdot 2161^1}{3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_1^5 p_2 + \frac{2^3}{5^1 \cdot 7^1} p_1^3 p_2^2 - \frac{2^1 \cdot 113^1}{3^2 \cdot 5^1 \cdot 7^1} p_1^3 p_4 + \frac{2^4 \cdot 277^1}{3^2 \cdot 5^2 \cdot 7^1} p_1^2 p_5 - \frac{2^1 \cdot 3989^1}{3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_1 p_2^3$$
$$+ \frac{1399^1}{3^1 \cdot 5^2 \cdot 11^1} p_1 p_2 p_4 - \frac{2^1 \cdot 305633^1}{3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1} p_1 p_6 - \frac{2^1 \cdot 23^2}{3^3 \cdot 11^1} p_2 p_5 + \frac{2^2 \cdot 8191^1}{3^2 \cdot 5^1 \cdot 11^1} p_7\Big]$$
$$L_8 = \frac{1}{3^3 \cdot 5^3 \cdot 7^1 \cdot 13^1 \cdot 17^1}\Big[-\frac{3617^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1^8 + \frac{2^2 \cdot 41^1 \cdot 83^1}{3^3 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1^6 p_2 - \frac{2^1 \cdot 43^1 \cdot 431^1}{3^5 \cdot 5^1 \cdot 7^1} p_1^4 p_2^2 + \frac{2^1 \cdot 162011^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1^4 p_4 - \frac{97^1 \cdot 12889^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1^3 p_5$$
$$+ \frac{2^7 \cdot 661^1}{3^5 \cdot 7^1 \cdot 11^1} p_1^2 p_2^3 - \frac{97^1 \cdot 2273^1}{3^4 \cdot 7^1 \cdot 11^1} p_1^2 p_2 p_4 + \frac{1091^1 \cdot 13789^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1^2 p_6 + \frac{2^1 \cdot 617147^1}{3^5 \cdot 5^1 \cdot 11^1} p_1 p_2 p_5 - \frac{37117^1}{5^1 \cdot 11^1} p_1 p_7$$
$$- \frac{13687^1}{3^5 \cdot 7^1 \cdot 11^1} p_2^4 + \frac{2^1 \cdot 101^1 \cdot 6491^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_2^2 p_4 - \frac{191^1 \cdot 14143^1}{3^5 \cdot 5^1 \cdot 11^1} p_2 p_6 - \frac{167^1 \cdot 2663^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_4^2 + \frac{31^1 \cdot 151^1 \cdot 3617^1}{3^4 \cdot 5^1 \cdot 11^1} p_8\Big]$$
$$L_9 = \frac{1}{3^7 \cdot 5^3 \cdot 7^3 \cdot 11^1 \cdot 19^1}\Big[+\frac{2^1 \cdot 43867^1}{3^2 \cdot 13^1 \cdot 17^1} p_1^9 - \frac{41^1 \cdot 53^1 \cdot 827^1}{3^1 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1^7 p_2 + \frac{541^1 \cdot 761^1}{3^1 \cdot 5^1 \cdot 13^1} p_1^5 p_2^2 - \frac{29^1 \cdot 79^1 \cdot 3467^1}{3^2 \cdot 13^1 \cdot 17^1} p_1^5 p_4 + \frac{61^1 \cdot 491^1 \cdot 1013^1}{3^2 \cdot 13^1 \cdot 17^1} p_1^4 p_5$$
$$- \frac{2^1 \cdot 31^1 \cdot 475957^1}{3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1^3 p_2^3 + \frac{2^2 \cdot 1697^1 \cdot 25951^1}{3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1^3 p_2 p_4 - \frac{29^1 \cdot 53^1 \cdot 181^1 \cdot 433^1}{3^2 \cdot 13^1 \cdot 17^1} p_1^3 p_6 - \frac{2^1 \cdot 83892287^1}{3^1 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1^2 p_2 p_5 + \frac{887^1 \cdot 210719^1}{3^2 \cdot 5^1 \cdot 17^1} p_1^2 p_7$$
$$+ \frac{13^1 \cdot 7297^1}{5^1 \cdot 17^1} p_1 p_2^4 - \frac{2^1 \cdot 8359009^1}{5^1 \cdot 13^1 \cdot 17^1} p_1 p_2^2 p_4 + \frac{47^1 \cdot 1291^1 \cdot 22709^1}{3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1 p_2 p_6 + \frac{67^1 \cdot 101^1 \cdot 13883^1}{3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1 p_4^2 - \frac{43^1 \cdot 237398563^1}{3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} p_1 p_8$$
$$+ \frac{2^2 \cdot 31^1 \cdot 314173^1}{3^2 \cdot 13^1 \cdot 17^1} p_2^2 p_5 - \frac{2^1 \cdot 23^1 \cdot 1933^1 \cdot 6857^1}{3^2 \cdot 13^1 \cdot 17^1} p_2 p_7 - \frac{2^2 \cdot 13569497^1}{3^2 \cdot 13^1 \cdot 17^1} p_4 p_5 + \frac{2^1 \cdot 43867^1 \cdot 131071^1}{3^2 \cdot 13^1 \cdot 17^1} p_9\Big]$$

Expansion 31. *L-genus for Fivebrane manifolds ($p_1 = p_2 = 0$).*

Expansion 31: L-genus with Fivebrane structure

$$L_3 = \frac{1}{3^3 \cdot 5^1 \cdot 7^1}\Big[+2^1 \cdot 31^1 p_3\Big]$$
$$L_4 = \frac{1}{3^3 \cdot 5^2 \cdot 7^1}\Big[+127^1 p_4\Big]$$
$$L_5 = \frac{1}{3^5 \cdot 5^1 \cdot 11^1}\Big[+2^1 \cdot 73^1 p_5\Big]$$
$$L_6 = \frac{1}{3^6 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[-167^1 \cdot 241^1 p_3^2 + 2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1 p_6\Big]$$
$$L_7 = \frac{1}{3^6 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1}\Big[-\frac{2^1 \cdot 97^1 \cdot 107^1}{5^1 \cdot 7^1} p_3 p_4 + 2^2 \cdot 8191^1 p_7\Big]$$
$$L_8 = \frac{1}{3^7 \cdot 5^4 \cdot 7^1 \cdot 13^1 \cdot 17^1}\Big[-\frac{29^1 \cdot 31^1 \cdot 79^1}{3^1} p_3 p_5 - \frac{167^1 \cdot 2663^1}{7^1 \cdot 11^1} p_4^2 + \frac{31^1 \cdot 151^1 \cdot 3617^1}{11^1} p_8\Big]$$
$$L_9 = \frac{1}{3^9 \cdot 5^3 \cdot 7^3 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1}\Big[+\frac{2^3 \cdot 89^1 \cdot 17929^1}{5^1} p_3^3 - \frac{2^1 \cdot 367^1 \cdot 1150249^1}{5^1} p_3 p_6 - 2^2 \cdot 13569497^1 p_4 p_5 + 2^1 \cdot 43867^1 \cdot 131071^1 p_9\Big]$$

Expansion 32. *L-genus with Ninebrane manifolds ($p_1 = p_2 = p_3 = 0$).*

Expansion 32: L-genus with Ninebrane structure

$$L_4 = \frac{1}{3^3 \cdot 5^2 \cdot 7^1}\Big[+127^1 p_4\Big]$$
$$L_5 = \frac{1}{3^5 \cdot 5^1 \cdot 11^1}\Big[+2^1 \cdot 73^1 p_5\Big]$$
$$L_6 = \frac{1}{3^6 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[+2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1 p_6\Big]$$
$$L_7 = \frac{1}{3^6 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1}\Big[+2^2 \cdot 8191^1 p_7\Big]$$
$$L_8 = \frac{1}{3^7 \cdot 5^4 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1}\Big[-\frac{167^1 \cdot 2663^1}{7^1} p_4^2 + 31^1 \cdot 151^1 \cdot 3617^1 p_8\Big]$$
$$L_9 = \frac{1}{3^9 \cdot 5^3 \cdot 7^3 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1}\Big[-2^2 \cdot 13569497^1 p_4 p_5 + 2^1 \cdot 43867^1 \cdot 131071^1 p_9\Big]$$

There is indeed no limit to how far one could go with the vanishing classes, as long as the machine is able to process the computations. Observe that, as with the Â-genus, if all odd p_i's were set to zero, then all the odd L-genus expansions would vanish too. In the following, all L_{2i+1} vanish.

Expansion 33. *L-genus simplification - vanishing of all odd Pontrjagin classes ($p_{2i+1} = 0$).*

Expansion 33: L-genus with only even Pontrjagin classes

$$L_2 = \frac{1}{3^2 \cdot 5^1} \Big[+ 7^1 p_2 \Big]$$

$$L_4 = \frac{1}{3^3 \cdot 5^2 \cdot 7^1} \Big[- \frac{19^1}{3^1} p_2^2 + 127^1 p_4 \Big]$$

$$L_6 = \frac{1}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1} \Big[+ 2^1 \cdot 1453^1 p_2^3 - \frac{159287^1}{3^1} p_2 p_4 + \frac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1} p_6 \Big]$$

$$L_8 = \frac{1}{3^7 \cdot 5^3 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1} \Big[- \frac{13687^1}{3^1 \cdot 7^1} p_2^4 + \frac{2^1 \cdot 101^1 \cdot 6491^1}{3^1 \cdot 5^1 \cdot 7^1} p_2^2 p_4 - \frac{191^1 \cdot 14143^1}{3^1 \cdot 5^1} p_2 p_6 - \frac{167^1 \cdot 2663^1}{5^1 \cdot 7^1} p_4^2 + \frac{31^1 \cdot 151^1 \cdot 3617^1}{5^1} p_8 \Big]$$

L-genus and complexification

The L-genus is expressed in terms of the Chern classes similarly to way as it was described for the Â-genus.

Expansion 34. *L-genus in terms of the Chern classes of complexification.*

Expansion 34: L-genus in terms of the Chern classes of complexification

$$L_1 = \frac{1}{3^1} \Big[- 1^1 c_2 \Big]$$

$$L_2 = \frac{1}{3^2 \cdot 5^1} \Big[- 1^1 c_2^2 + 7^1 c_4 \Big]$$

$$L_3 = \frac{1}{3^3 \cdot 5^1 \cdot 7^1} \Big[- 2^1 c_2^3 + 13^1 c_2 c_4 - 2^1 \cdot 31^1 c_6 \Big]$$

$$L_4 = \frac{1}{3^3 \cdot 5^2 \cdot 7^1} \Big[- 1^1 c_2^4 + \frac{2^1 \cdot 11^1}{3^1} c_2^2 c_4 - \frac{71^1}{3^1} c_2 c_6 - \frac{19^1}{3^1} c_4^2 + 127^1 c_8 \Big]$$

$$L_5 = \frac{1}{3^4 \cdot 5^1 \cdot 11^1} \Big[- \frac{2^1}{3^1 \cdot 7^1} c_2^5 + \frac{83^1}{3^1 \cdot 5^1 \cdot 7^1} c_2^3 c_4 - \frac{79^1}{5^1 \cdot 7^1} c_2^2 c_6 - \frac{127^1}{3^1 \cdot 5^1 \cdot 7^1} c_2 c_4^2 + \frac{919^1}{3^1 \cdot 5^1 \cdot 7^1} c_2 c_8 + \frac{2^4}{5^1} c_4 c_6 - \frac{2^1 \cdot 73^1}{3^1} c_{10} \Big]$$

$$L_6 = \frac{1}{3^5 \cdot 5^2 \cdot 7^2 \cdot 11^1 \cdot 13^1} \Big[- \frac{2^1 \cdot 691^1}{3^1 \cdot 5^1} c_2^6 + \frac{2^1 \cdot 6421^1}{3^1 \cdot 5^1} c_2^4 c_4 - \frac{33863^1}{3^1 \cdot 5^1} c_2^3 c_6 - \frac{5527^1}{3^1} c_2^2 c_4^2 + \frac{40841^1}{5^1} c_2^2 c_8 + \frac{83^1 \cdot 349^1}{5^1} c_2 c_4 c_6$$
$$- \frac{2^5 \cdot 29^1 \cdot 181^1}{5^1} c_2 c_{10} + \frac{2^1 \cdot 1453^1}{5^1} c_4^3 - \frac{159287^1}{3^1 \cdot 5^1} c_4 c_8 - \frac{167^1 \cdot 241^1}{3^1 \cdot 5^1} c_6^2 + \frac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1 \cdot 5^1} c_{12} \Big]$$

$$L_7 = \frac{1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} \Big[- \frac{2^2}{3^4 \cdot 5^1 \cdot 11^1} c_2^7 + \frac{2^1 \cdot 2161^1}{3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1} c_2^5 c_4 - \frac{2^2}{5^2 \cdot 7^1} c_2^4 c_6 - \frac{2^3}{3^2 \cdot 5^1 \cdot 7^1} c_2^3 c_4^2 + \frac{2^1 \cdot 113^1}{3^4 \cdot 5^1 \cdot 7^1} c_2^3 c_8 + \frac{39341^1}{3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1} c_2^2 c_4 c_6$$
$$- \frac{2^4 \cdot 277^1}{3^4 \cdot 5^2 \cdot 7^1} c_2^2 c_{10} + \frac{2^1 \cdot 3989^1}{3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1} c_2 c_4^3 - \frac{1399^1}{3^3 \cdot 5^2 \cdot 11^1} c_2 c_4 c_8 - \frac{22027^1}{3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1} c_2 c_6^2 + \frac{2^1 \cdot 305633^1}{3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1} c_2 c_{12} - \frac{2^3 \cdot 2087^1}{3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1} c_4^2 c_6$$
$$+ \frac{2^1 \cdot 23^2}{3^5 \cdot 11^1} c_4 c_{10} + \frac{2^1 \cdot 97^1 \cdot 107^1}{3^4 \cdot 5^2 \cdot 7^1 \cdot 11^1} c_6 c_8 - \frac{2^2 \cdot 8191^1}{3^4 \cdot 5^1 \cdot 11^1} c_{14} \Big]$$

$$L_8 = \frac{1}{3^3 \cdot 5^3 \cdot 7^1 \cdot 17^1} \Big[- \frac{3617^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^8 + \frac{2^2 \cdot 41^1 \cdot 83^1}{3^3 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^6 c_4 - \frac{2^1 \cdot 143483^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^5 c_6 - \frac{2^1 \cdot 43^1 \cdot 431^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 13^1} c_2^4 c_4^2 + \frac{2^1 \cdot 162011^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^4 c_8$$
$$+ \frac{2^6 \cdot 29^1 \cdot 739^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^3 c_4 c_6 - \frac{97^1 \cdot 12889^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^3 c_{10} + \frac{2^7 \cdot 661^1}{3^5 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^2 c_4^3 - \frac{97^1 \cdot 2273^1}{3^4 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^2 c_4 c_8 - \frac{107^1 \cdot 857^1}{3^3 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^2 c_6^2$$
$$+ \frac{1091^1 \cdot 13789^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2^2 c_{12} - \frac{2^1 \cdot 9619^1}{3^5 \cdot 7^1 \cdot 11^1} c_2 c_4^2 c_6 + \frac{2^1 \cdot 617147^1}{3^5 \cdot 5^1 \cdot 11^1 \cdot 13^1} c_2 c_4 c_{10} + \frac{29^1 \cdot 98057^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_2 c_6 c_8 - \frac{37117^1}{5^1 \cdot 11^1 \cdot 13^1} c_2 c_{14}$$
$$- \frac{13687^1}{3^5 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_4^4 + \frac{2^1 \cdot 101^1 \cdot 6491^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_4^2 c_8 + \frac{2^3 \cdot 311^1}{3^4 \cdot 5^1 \cdot 11^1} c_4 c_6^2 - \frac{191^1 \cdot 14143^1}{3^5 \cdot 5^1 \cdot 11^1 \cdot 13^1} c_4 c_{12} - \frac{29^1 \cdot 31^1 \cdot 79^1}{3^5 \cdot 5^1 \cdot 13^1} c_6 c_{10}$$
$$- \frac{167^1 \cdot 2663^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_8^2 + \frac{31^1 \cdot 151^1 \cdot 3617^1}{3^4 \cdot 5^1 \cdot 11^1 \cdot 13^1} c_{16} \Big]$$

$$L_9 = \frac{1}{3^7 \cdot 5^3 \cdot 7^1 \cdot 19^1} \Big[- \frac{2^1 \cdot 43867^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^9 + \frac{41^1 \cdot 53^1 \cdot 827^1}{3^1 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^7 c_4 - \frac{1337617^1}{5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^6 c_6 - \frac{541^1 \cdot 761^1}{3^1 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1} c_2^5 c_4^2$$
$$+ \frac{29^1 \cdot 79^1 \cdot 3467^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^5 c_8 + \frac{3931^1 \cdot 17981^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^4 c_4 c_6 - \frac{61^1 \cdot 491^1 \cdot 1013^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^4 c_{10} + \frac{2^1 \cdot 31^1 \cdot 475957^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^3 c_4^3$$
$$- \frac{2^2 \cdot 1697^1 \cdot 25951^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^3 c_4 c_8 - \frac{2^1 \cdot 70003^1}{7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^3 c_6^2 + \frac{29^1 \cdot 53^1 \cdot 181^1 \cdot 433^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^3 c_{12} - \frac{2^1 \cdot 93133^1}{3^2 \cdot 5^1 \cdot 13^1 \cdot 17^1} c_2^2 c_4^2 c_6$$
$$+ \frac{2^1 \cdot 83892287^1}{3^1 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^2 c_4 c_{10} + \frac{743^1 \cdot 73597^1}{3^1 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2^2 c_6 c_8 - \frac{887^1 \cdot 210719^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 17^1} c_2^2 c_{14} - \frac{13^1 \cdot 7297^1}{5^1 \cdot 7^2 \cdot 11^1 \cdot 17^1} c_2 c_4^4$$
$$+ \frac{2^1 \cdot 8359009^1}{5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2 c_4^2 c_8 + \frac{2^2 \cdot 83^1 \cdot 15667^1}{3^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2 c_4 c_6^2 - \frac{47^1 \cdot 1291^1 \cdot 22709^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2 c_4 c_{12} - \frac{641^1 \cdot 121169^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2 c_6 c_{10}$$
$$- \frac{67^1 \cdot 101^1 \cdot 13883^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2 c_8^2 + \frac{43^1 \cdot 237398563^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_2 c_{16} + \frac{2^4 \cdot 31^1 \cdot 139^1 \cdot 293^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_4^3 c_6 - \frac{2^2 \cdot 31^1 \cdot 314173^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_4^2 c_{10}$$
$$- \frac{2^2 \cdot 89^1 \cdot 11071^1}{3^1 \cdot 5^1 \cdot 7^2 \cdot 13^1 \cdot 17^1} c_4 c_6 c_8 + \frac{2^1 \cdot 23^1 \cdot 1933^1 \cdot 6857^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_4 c_{14} - \frac{2^3 \cdot 89^1 \cdot 17929^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_6^3 + \frac{2^1 \cdot 367^1 \cdot 1150249^1}{3^2 \cdot 5^1 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_6 c_{12}$$
$$+ \frac{2^2 \cdot 13569497^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_8 c_{10} - \frac{2^1 \cdot 43867^1 \cdot 131071^1}{3^2 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1} c_{18} \Big]$$

We now consider the higher structures in the complex case as well.

Expansion 35. *L-genus for complex String manifolds ($c_1 = c_2 = 0$) in terms of Chern classes.*

Expansion 35: L-genus with complex String structure

$$L_2 = \tfrac{1}{3^2.5^1}\Big[+7^1 c_4\Big]$$
$$L_3 = \tfrac{1}{3^3.5^1.7^1}\Big[-2^1\cdot 31^1 c_6\Big]$$
$$L_4 = \tfrac{1}{3^3.5^2.7^1}\Big[-\tfrac{19^1}{3^1}c_4^2 + 127^1 c_8\Big]$$
$$L_5 = \tfrac{1}{3^4.5^1.11^1}\Big[+\tfrac{2^4}{5^1}c_4c_6 - \tfrac{2^1\cdot 73^1}{3^1}c_{10}\Big]$$
$$L_6 = \tfrac{1}{3^5.5^3.7^2.11^1.13^1}\Big[+2^1\cdot 1453^1 c_4^3 - \tfrac{159287^1}{3^1}c_4c_8 - \tfrac{167^1\cdot 241^1}{3^1}c_6^2 + \tfrac{2^1\cdot 23^1\cdot 89^1\cdot 691^1}{3^1}c_{12}\Big]$$
$$L_7 = \tfrac{1}{3^6.5^1.7^1.11^1.13^1}\Big[-\tfrac{2^3\cdot 2087^1}{3^1.5^2.7^1}c_4^2c_6 + \tfrac{2^1\cdot 23^2}{3^1}c_4c_{10} + \tfrac{2^1\cdot 97^1\cdot 107^1}{5^2.7^1}c_6c_8 - \tfrac{2^2\cdot 8191^1}{5^1}c_{14}\Big]$$
$$L_8 = \tfrac{1}{3^7.5^3.7^1.17^1}\Big[-\tfrac{13687^1}{3^1.7^1.11^1.13^1}c_4^4 + \tfrac{2^1\cdot 101^1\cdot 6491^1}{3^1.5^1.7^1.11^1.13^1}c_4^2c_8 + \tfrac{2^3\cdot 311^1}{5^1.11^1}c_4c_6^2 - \tfrac{191^1\cdot 14143^1}{3^1.5^1.11^1.13^1}c_4c_{12} - \tfrac{29^1\cdot 31^1\cdot 79^1}{3^1.5^1.13^1}c_6c_{10}$$
$$- \tfrac{167^1\cdot 2663^1}{5^1.7^1.11^1.13^1}c_8^2 + \tfrac{31^1\cdot 151^1\cdot 3617^1}{5^1.11^1.13^1}c_{16}\Big]$$
$$L_9 = \tfrac{1}{3^8.5^3.7^3.13^1.17^1.19^1}\Big[+\tfrac{2^4\cdot 31^1\cdot 139^1\cdot 293^1}{3^1.5^1.11^1}c_4^3c_6 - \tfrac{2^2\cdot 31^1\cdot 314173^1}{3^1.11^1}c_4^2c_{10} - \tfrac{2^2\cdot 89^1\cdot 11071^1}{5^1}c_4c_6c_8 + \tfrac{2^1\cdot 23^1\cdot 1933^1\cdot 6857^1}{3^1.11^1}c_4c_{14}$$
$$- \tfrac{2^3\cdot 89^1\cdot 17929^1}{3^1.5^1.11^1}c_6^3 + \tfrac{2^1\cdot 367^1\cdot 1150249^1}{3^1.5^1.11^1}c_6c_{12} + \tfrac{2^2\cdot 13569497^1}{3^1.11^1}c_8c_{10} - \tfrac{2^1\cdot 43867^1\cdot 131071^1}{3^1.11^1}c_{18}\Big]$$

4.4 The Todd genus

The Todd genus (see Section 4) has characteristic power series $x_j/(1-e^{-x_j})$. As with the Â-genus and L-genus, the characteristic power series were plugged into the corresponding function to produce the expansion below. However, unlike the two genera considered earlier, the Todd genus is expressed in terms of the Chern classes.

Expansion 36. *Todd genus in terms of Chern classes.*

Expansion 36: Todd genus in terms of Chern classes

$$T_1 = \tfrac{1}{2^1}\Big[c_1\Big]$$
$$T_2 = \tfrac{1}{2^2.3^1}\Big[c_1^2 + c_2\Big]$$
$$T_3 = \tfrac{1}{2^3.3^1}\Big[c_1c_2\Big]$$
$$T_4 = \tfrac{1}{2^2.3^1.5^1}\Big[-\tfrac{1^1}{2^2.3^1}c_1^4 + \tfrac{1^1}{3^1}c_1^2c_2 + \tfrac{1^1}{2^2.3^1}c_1c_3 + \tfrac{1^1}{2^2}c_2^2 - \tfrac{1^1}{2^2.3^1}c_4\Big]$$
$$T_5 = \tfrac{1}{2^5.3^1.5^1}\Big[-\tfrac{1^1}{3^1}c_1^3c_2 + \tfrac{1^1}{3^1}c_1^2c_3 + c_1c_2^2 - \tfrac{1^1}{3^1}c_1c_4\Big]$$
$$T_6 = \tfrac{1}{2^4.3^1.7^1}\Big[+\tfrac{1^1}{2^1.3^2.5^1}c_1^6 - \tfrac{1^1}{3^1.5^1}c_1^4c_2 + \tfrac{1^1}{2^2.3^2}c_1^3c_3 + \tfrac{11^1}{2^2.3^2.5^1}c_1^2c_2^2 - \tfrac{1^1}{2^2.3^2}c_1^2c_4 + \tfrac{11^1}{2^2.3^2.5^1}c_1c_2c_3$$
$$- \tfrac{1^1}{2^1.3^2.5^1}c_1c_5 + \tfrac{1^1}{2^1.3^2}c_2^3 - \tfrac{1^1}{2^2.5^1}c_2c_4 - \tfrac{1^1}{2^2.3^2.5^1}c_3^2 + \tfrac{1^1}{2^1.3^2.5^1}c_6\Big]$$
$$T_7 = \tfrac{1}{2^6.3^1.7^1}\Big[+\tfrac{1^1}{3^2.5^1}c_1^5c_2 - \tfrac{1^1}{3^2.5^1}c_1^4c_3 - \tfrac{1^1}{3^2}c_1^3c_2^2 + \tfrac{1^1}{3^2.5^1}c_1^3c_4 + \tfrac{11^1}{2^1.3^2.5^1}c_1^2c_2c_3 - \tfrac{1^1}{3^2.5^1}c_1^2c_5 + \tfrac{1^1}{3^2}c_1c_2^3$$
$$- \tfrac{1^1}{2^1.5^1}c_1c_2c_4 - \tfrac{1^1}{2^1.3^2.5^1}c_1c_3^2 + \tfrac{1^1}{3^2.5^1}c_1c_6\Big]$$
$$T_8 = \tfrac{1}{2^4.3^3}\Big[-\tfrac{1^1}{2^4.5^2.7^1}c_1^8 + \tfrac{1^1}{2^1.5^2.7^1}c_1^6c_2 - \tfrac{1^1}{2^3.3^1.5^2}c_1^5c_3 - \tfrac{1^1}{2^3.3^1.7^1}c_1^4c_2^2 + \tfrac{1^1}{2^3.3^1.5^2}c_1^4c_4 + \tfrac{13^1}{2^3.3^1.5^2.7^1}c_1^3c_2c_3$$
$$- \tfrac{1^1}{2^4.3^1.5^2}c_1^3c_5 + \tfrac{1^1}{2^1.3^1.5^2.7^1}c_1^2c_2^3 - \tfrac{19^1}{2^4.3^1.5^2.7^1}c_1^2c_2c_4 + \tfrac{1^1}{2^4.5^2.7^1}c_1^2c_3^2 + \tfrac{1^1}{2^4.3^1.5^2}c_1^2c_6 + \tfrac{1^1}{2^3.3^1.7^1}c_1c_2^2c_3$$
$$- \tfrac{1^1}{3^1.5^2.7^1}c_1c_2c_5 - \tfrac{13^1}{2^4.3^1.5^2.7^1}c_1c_3c_4 + \tfrac{1^1}{2^4.5^2.7^1}c_1c_7 + \tfrac{1^1}{2^4.5^2}c_2^4 - \tfrac{17^1}{2^3.3^1.5^2.7^1}c_2^2c_4 - \tfrac{1^1}{2^1.3^1.5^2.7^1}c_2c_3^2$$
$$+ \tfrac{13^1}{2^4.3^1.5^2.7^1}c_2c_6 + \tfrac{1^1}{2^4.5^2.7^1}c_3c_5 + \tfrac{1^1}{2^4.3^1.5^1.7^1}c_4^2 - \tfrac{1^1}{2^4.5^2.7^1}c_8\Big]$$
$$T_9 = \tfrac{1}{2^5.3^3}\Big[-\tfrac{1^1}{2^4.5^2.7^1}c_1^7c_2 + \tfrac{1^1}{2^4.5^2.7^1}c_1^6c_3 + \tfrac{1^1}{2^4.5^2}c_1^5c_2^2 - \tfrac{1^1}{2^4.5^2.7^1}c_1^5c_4 - \tfrac{29^1}{2^4.3^1.5^2.7^1}c_1^4c_2c_3 + \tfrac{1^1}{2^4.5^2.7^1}c_1^4c_5$$
$$- \tfrac{1^1}{2^3.5^2}c_1^3c_2^3 + \tfrac{13^1}{2^3.3^1.5^2.7^1}c_1^3c_2c_4 + \tfrac{1^1}{2^1.3^1.5^2.7^1}c_1^3c_3^2 - \tfrac{1^1}{2^4.5^2.7^1}c_1^3c_6 + \tfrac{1^1}{2^3.3^1.7^1}c_1^2c_2^2c_3 - \tfrac{1^1}{3^1.5^2.7^1}c_1^2c_2c_5$$
$$- \tfrac{13^1}{2^4.3^1.5^2.7^1}c_1^2c_3c_4 + \tfrac{1^1}{2^4.5^2.7^1}c_1^2c_7 + \tfrac{1^1}{2^4.5^2}c_1c_2^4 - \tfrac{17^1}{2^3.3^1.5^2.7^1}c_1c_2^2c_4 - \tfrac{1^1}{2^1.3^1.5^2.7^1}c_1c_2c_3^2$$
$$+ \tfrac{13^1}{2^4.3^1.5^2.7^1}c_1c_2c_6 + \tfrac{1^1}{2^4.5^2.7^1}c_1c_3c_5 + \tfrac{1^1}{2^4.3^1.5^1.7^1}c_1c_4^2 - \tfrac{1^1}{2^4.5^2.7^1}c_1c_8\Big]$$

4.4.1 Todd genus with simplifications

Here we present expansions of the Todd genus with vanishing one or several characteristic classes. In particular, complex spaces with vanishing first Chern class are very prominent in algebraic geometry, complex differential geometry and mathematical physics (see [15][16][11]). Observe the significance of c_1 characteristic class for the Todd genus - by setting $c_1 = 0$ all odd Todd genus expansions vanished as well at least up to the degree presented.

Expansion 37. *Todd genus for Calabi-Yau manifolds ($c_1 = 0$).*

Expansion 37: Todd genus for Calabi-Yau spaces

$$T_2 = \frac{1}{2^2.3^1}\left[c_2\right]$$

$$T_4 = \frac{1}{2^4.3^1.5^1}\left[c_2^2 - \frac{1^1}{3^1}c_4\right]$$

$$T_6 = \frac{1}{2^5.3^1.7^1}\left[+\frac{1^1}{3^2}c_2^3 - \frac{1^1}{2^1.5^1}c_2c_4 - \frac{1^1}{2^1.3^2.5^1}c_3^2 + \frac{1^1}{3^2.5^1}c_6\right]$$

$$T_8 = \frac{1}{2^5.3^3.5^1}\left[+\frac{1^1}{2^3.5^1}c_2^4 - \frac{17^1}{2^2.3^1.5^1.7^1}c_2^2c_4 - \frac{1^1}{3^1.5^1.7^1}c_2c_3^2 + \frac{13^1}{2^3.3^1.5^1.7^1}c_2c_6 + \frac{1^1}{2^3.5^1.7^1}c_3c_5 + \frac{1^1}{2^3.3^1.7^1}c_4^2 - \frac{1^1}{2^3.5^1.7^1}c_8\right]$$

Sometimes a complex manifold might have vanishing higher Chern classes without the lower ones necessarily being zero. These can be viewed as complex analogues of the p_1-structures encountered in the case of Â-genus and L-genus. These can be done similarly and we will not list them.

The next expansions present Todd genus simplifications with more than one vanishing Chern class. Observe how drastically the following expansions simplify from their initial form in *Expansion 36* or even from the above expansions involving vanishing of a single Chern class.

Expansion 38. *Todd genus for complex String manifolds ($c_1 = c_2 = 0$).*

Expansion 38: Todd genus with complex String structure

$$T_4 = \frac{1}{2^4.3^2.5^1}\left[-1^1c_4\right]$$

$$T_6 = \frac{1}{2^5.3^3.5^1.7^1}\left[-\frac{1^1}{2^1}c_3^2 + c_6\right]$$

$$T_8 = \frac{1}{2^8.3^3.5^1.7^1}\left[+\frac{1^1}{5^1}c_3c_5 + \frac{1^1}{3^1}c_4^2 - \frac{1^1}{5^1}c_8\right]$$

$$T_{10} = \frac{1}{2^5.3^4.5^1.11^1}\left[+\frac{1^1}{2^5.5^1}c_3^2c_4 - \frac{1^1}{2^4.3^1.7^1}c_3c_7 - \frac{1^1}{3^1.5^1.7^1}c_4c_6 - \frac{1^1}{2^5.3^1.7^1}c_5^2 + \frac{1^1}{2^4.3^1.7^1}c_{10}\right]$$

Expansion 39. *Todd genus with complex Fivebrane structure ($c_1 = c_2 = c_3 = c_4 = 0$).*

Expansion 39: Todd genus complex Fivebrane structure

$$T_6 = \frac{1}{2^5.3^3.5^1.7^1}\left[c_6\right]$$

$$T_8 = \frac{1}{2^8.3^3.5^2.7^1}\left[-1^1c_8\right]$$

$$T_{10} = \frac{1}{2^9.3^5.5^1.7^1.11^1}\left[-\frac{1^1}{2^1}c_5^2 + c_{10}\right]$$

Expansion 40. *Todd-genus simplification - vanishing of the odd Chern classes ($c_{2i+1} = 0$).*

Expansion 40: Todd genus with even Chern classes

$$T_2 = \frac{1}{2^2.3^1}\left[c_2\right]$$

$$T_4 = \frac{1}{2^4.3^1.5^1}\left[c_2^2 - \frac{1^1}{3^1}c_4\right]$$

$$T_6 = \frac{1}{2^5.3^1.7^1}\left[+\frac{1^1}{3^2}c_2^3 - \frac{1^1}{2^1.5^1}c_2c_4 + \frac{1^1}{3^2.5^1}c_6\right]$$

$$T_8 = \frac{1}{2^7.3^3.5^1}\left[+\frac{1^1}{2^1.5^1}c_2^4 - \frac{17^1}{3^1.5^1.7^1}c_2^2c_4 + \frac{13^1}{2^1.3^1.5^1.7^1}c_2c_6 + \frac{1^1}{2^1.3^1.7^1}c_4^2 - \frac{1^1}{2^1.5^1.7^1}c_8\right]$$

$$T_{10} = \frac{1}{2^5.3^3.5^1.7^1.11^1}\left[+\frac{1^1}{2^4}c_2^5 - \frac{73^1}{2^5.3^1.5^1}c_2^3c_4 + \frac{109^1}{2^5.3^2.5^1}c_2^2c_6 + \frac{29^1}{2^5.3^1.5^1}c_2c_4^2 - \frac{43^1}{2^5.3^2.5^1}c_2c_8 - \frac{1^1}{3^2.5^1}c_4c_6 - \frac{1^1}{2^5.3^2}c_5^2 + \frac{1^1}{2^4.3^2}c_{10}\right]$$

4.4.2 Todd genus of realification

The Todd genus may be written in terms of the Pontrjagin classes as well, due to the relation between Chern and Pontrjagin classes shown in *Equation* (3.2). The expansion below straightforwardly implements that relation by first imposing vanishing all odd degrees of the Todd genus and then converting the Chern classes to Pontrjagin classes in even degrees.

Expansion 41. *Todd genus in terms of Pontrjagin classes of realification.*

Expansion 41: Todd genus in terms of Pontrjagin classes

$$T_2 = \frac{1}{2^2.3^1}\Big[-1^1 p_1\Big]$$
$$T_4 = \frac{1}{2^4.3^1.5^1}\Big[p_1^2 - \frac{1^1}{3^1}p_2\Big]$$
$$T_6 = \frac{1}{2^5.3^1.7^1}\Big[-\frac{1^1}{3^2}p_1^3 + \frac{1^1}{2^1.5^1}p_1p_2 - \frac{1^1}{3^2.5^1}p_3\Big]$$
$$T_8 = \frac{1}{2^7.3^3.5^1}\Big[+\frac{1^1}{2^1.5^1}p_1^4 - \frac{17^1}{3^1.5^1.7^1}p_1^2p_2 + \frac{13^1}{2^1.3^1.5^1.7^1}p_1p_3 + \frac{1^1}{2^1.3^1.7^1}p_2^2 - \frac{1^1}{2^1.5^1.7^1}p_4\Big]$$
$$T_{10} = \frac{1}{2^5.3^3.5^1.7^1.11^1}\Big[-\frac{1^1}{2^4}p_1^5 + \frac{73^1}{2^5.3^1.5^1}p_1^3p_2 - \frac{109^1}{2^5.3^2.5^1}p_1^2p_3 - \frac{29^1}{2^5.3^1.5^1}p_1p_2^2 + \frac{43^1}{2^5.3^2.5^1}p_1p_4 + \frac{1^1}{3^2.5^1}p_2p_3 - \frac{1^1}{2^4.3^2}p_5\Big]$$

Expansion 42. *String or p_1-structure - Todd genus in terms of Pontrjagin classes with $p_1 = 0$.*

Expansion 42: Todd genus with String or p_1-structure

$$T_4 = \frac{1}{2^4.3^2.5^1}\Big[-1^1 p_2\Big]$$
$$T_6 = \frac{1}{2^5.3^3.5^1.7^1}\Big[-1^1 p_3\Big]$$
$$T_8 = \frac{1}{2^8.3^3.5^1.7^1}\Big[+\frac{1^1}{3^1}p_2^2 - \frac{1^1}{5^1}p_4\Big]$$
$$T_{10} = \frac{1}{2^5.3^5.5^1.7^1.11^1}\Big[+\frac{1^1}{5^1}p_2p_3 - \frac{1^1}{2^4}p_5\Big]$$

Expansion 43. *Todd genus in terms of Pontrjagin classes with Fivebrane structure ($p_1 = 0 = p_2$).*

Expansion 43: Todd genus with Fivebrane structure

$$T_6 = -\frac{1}{2^5.3^3.5\cdot 7}p_3$$
$$T_8 = -\frac{1}{2^8.3^3.5^2.7}p_4$$
$$T_{10} = -\frac{1}{2^9.3^5.5\cdot 7\cdot 11}p_5$$
$$T_{12} = \frac{1}{2^{12}.3^6.5^3.7^2.11\cdot 13}[+3^1\cdot 7^1\cdot 101^1 p_3^2 - 2^1\cdot 691^1 p_6]$$

Other higher cases can be done similarly, as the expressions for the Todd genus are generally much simpler than those of other genera.

5 Relations among genera

Having computed the genera expansions for various manifolds, we can now relate the genera with each other for the corresponding structures, since all of them can be expressed in both the Pontrjagin and Chern classes. Some such relations can also be predicted theoretically; see Proposition 4 for an illustration of one case.

Relations among the genera are important, as they provide information about which manifolds are comparable and in what ways are their structures are similar. In fact, relations among the genera lead to nontrivial effects in geometry, topology, and physics. In particular, in mathematical approaches to quantum field theory and string theory they lead to what is called *cancellation of anomalies*, thereby rendering the theories properly defined (see [2], [29], and [31]). The relations presented here are intended as ready-made results and references for people working on the relevant topics.

It is explicitly indicated in each case, which expansions are being related. The method of computing was substitution - we expressed one genus in terms of its characteristic classes degree by degree (e.g. from *Expansions 28* we have $p_2 = \frac{45}{7}L_2$, $p_3 = \frac{945}{62}L_3$, ...), then inserted these expressions into the other genus for the corresponding structure (e.g., L-genus for string manifolds corresponds to the Â-genus for String manifolds). The complexity of the resulting equation was the main reason for curtailing the number of equations at a certain degree in each case.

5.1 Â and L genera

Since both the Â-genus and L-genus are expressed in terms of the Pontrjagin classes, then the relations can be computed without any further conversions. In the first case below, we relate Â-genus and L-genus of String manifolds (see *Expansion 19* and *28*). Observe how the $\hat{A}_2$ and L_2 as well as $\hat{A}_3$ and L_3 have a linear relationship.

Expansion 44. *Relation between Â-genus and L-genus for String manifolds* ($p_1 = 0$).

Expansion 44: Relation between Â-genus and L-genus for String manifolds

$$\hat{A}_2 = \frac{1}{2^5 \cdot 7^1}\Big[-1L_2\Big]$$

$$\hat{A}_3 = \frac{1}{2^7 \cdot 31^1}\Big[-1L_3\Big]$$

$$\hat{A}_4 = \frac{1}{2^{11} \cdot 7^2 \cdot 127^1}\Big[+3^1 \cdot 5^1 \cdot 17^1 L_2^2 - 2^2 \cdot 7^2 L_4\Big]$$

$$\hat{A}_5 = \frac{1}{2^{12} \cdot 7^1 \cdot 31^1 \cdot 73^1}\Big[+3^3 \cdot 5^1 L_2 L_3 - 2^1 \cdot 31^1 L_5\Big]$$

$$\hat{A}_6 = \frac{1}{2^{16} \cdot 7^3 \cdot 23^1 \cdot 31^2 \cdot 89^1 \cdot 127^1}\Big[-3^2 \cdot 5^2 \cdot 11^1 \cdot 31^2 \cdot 163^1 L_2^3 + 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 17^1 \cdot 31^2 L_2 L_4 + 2^1 \cdot 7^3 \cdot 127^1 \left(3^4 \cdot 7^2 L_3^2 - 2^2 \cdot 31^2 L_6\right)\Big]$$

Next we relate *Expansions 20* and *29*. Again notice the linear relationship in the first two equations.

Expansion 45. *Relation between Â-genus and L-genus of* p_2*-structure manifolds* ($p_2 = 0$).

Expansion 45: Relation between Â-genus and L-genus with p_2-structure

$$\hat{A}_1 = \frac{1}{2^3}\Big[-1L_1\Big]$$

$$\hat{A}_2 = \frac{1}{2^7}\Big[-7L_2\Big]$$

$$\hat{A}_3 = \frac{1}{2^{10} \cdot 31^1}\Big[-3^3 L_1^3 - 2^3 L_3\Big]$$

$$\hat{A}_4 = \frac{1}{2^{15} \cdot 3^1 \cdot 5^1 \cdot 31^1 \cdot 127^1}\Big[+37^1 \cdot 41^1 L_1^4 + 2^5 \cdot 3^3 \cdot 5^1 \cdot 7^1 L_1 L_3 - 2^6 \cdot 3^1 \cdot 5^1 \cdot 31^1 L_4\Big]$$

$$\hat{A}_5 = \frac{1}{2^{18} \cdot 5^1 \cdot 7^1 \cdot 31^1 \cdot 73^1 \cdot 127^1}\Big[-3^3 \cdot 17^1 \cdot 26863^1 L_1^5 - 2^4 \cdot 3^2 \cdot 5^2 \cdot 17^1 \cdot 251^1 L_1^2 L_3 + 2^6 \cdot 3^2 \cdot 5^2 \cdot 17^1 \cdot 31^1 L_1 L_4 - 2^7 \cdot 5^1 \cdot 31^1 \cdot 127^1 L_5\Big]$$

The equations below were written based on *Expansions 21* and *30*.

Expansion 46. *Relation between Â-genus and L-genus of* p_3*-structure manifolds* ($p_3 = 0$).

Expansion 46: Relation between Â-genus and L-genus with p_3-structure

$$\hat{A}_1 = \frac{1}{2^3}\Big[-1L_1\Big]$$

$$\hat{A}_2 = \frac{1}{2^7 \cdot 7^1}\Big[+3^2 L_1^2 - 2^2 L_2\Big]$$

$$\hat{A}_3 = \frac{1}{2^{10} \cdot 13^1}\Big[-3^2 L_1^3 - 2^2 \cdot 11^1 L_3\Big]$$

$$\hat{A}_4 = \frac{1}{2^{15} \cdot 5^1 \cdot 7^2 \cdot 127^1}\Big[+3^2 \cdot 13^1 \cdot 389^1 L_1^4 - 2^3 \cdot 3^2 \cdot 5^2 \cdot 59^1 L_1^2 L_2 + 2^4 \cdot 5^1 \cdot \left(3^2 \cdot 5^2 L_2^2 - 2^2 \cdot 7^2 L_4\right)\Big]$$

For the next result we used *Expansions 22* and *31*. Observe the linear relationship between in degrees 3 to 5.

Expansion 47. *Relation between Â-genus and L-genus of Fivebrane manifolds* ($p_1 = p_2 = 0$).

Expansion 47: Relation between Â-genus and L-genus with Fivebrane structure

$$\hat{A}_3 = \frac{1}{2^7 \cdot 31^1}\Big[-1L_3\Big]$$

$$\hat{A}_4 = \frac{1}{2^9 \cdot 127^1}\Big[-1L_4\Big]$$

$$\hat{A}_5 = \frac{1}{2^{11} \cdot 7^1 \cdot 73^1}\Big[-1L_5\Big]$$

$$\hat{A}_6 = \frac{1}{2^{15} \cdot 3^1 \cdot 5^1 \cdot 23^1 \cdot 31^2 \cdot 89^1}\Big[+3^5 \cdot 5^1 \cdot 7^2 L_3^2 - 2^2 \cdot 17^1 \cdot 31^2 L_6\Big]$$

$$\hat{A}_7 = \frac{1}{2^{16} \cdot 31^1 \cdot 127^1 \cdot 8191^1}\Big[+3^3 \cdot 5^1 \cdot 7^1 \cdot 17^1 L_3 L_4 - 2^1 \cdot 31^1 \cdot 127^1 L_7\Big]$$

In the next result we relate *Expansions 23* and *32*. It is indeed an interesting observation that from degree 4 to 7 the genera are related linearly. It appears that there is a clear trade-off between vanishing of the characteristic classes and the simplicity of the resulting relations among the two genera.

Expansion 48. *Relation between Â-genus and L-genus of Ninebrane manifolds ($p_1 = p_2 = p_3 = 0$).*

Expansion 48: Relation between Â-genus and L-genus with Ninebrane structure

$$\hat{A}_4 = \frac{1}{2^9 \cdot 127^1}\Big[-1L_4\Big]$$
$$\hat{A}_5 = \frac{1}{2^{11} \cdot 7^1 \cdot 73^1}\Big[-1L_5\Big]$$
$$\hat{A}_6 = \frac{1}{2^{13} \cdot 23^1 \cdot 89^1}\Big[-1L_6\Big]$$
$$\hat{A}_7 = \frac{1}{2^{15} \cdot 8191^1}\Big[-1L_7\Big]$$
$$\hat{A}_8 = \frac{1}{2^{19} \cdot 7^1 \cdot 31^1 \cdot 127^2 \cdot 151^1}\Big[+3^2 \cdot 5^2 \cdot 17^2 L_4^2 + 2^2 \cdot 127^2 L_8\Big]$$
$$\hat{A}_9 = \frac{1}{2^{20} \cdot 7^1 \cdot 73^1 \cdot 127^1 \cdot 131071^1}\Big[+3^2 \cdot 5^1 \cdot 11^1 \cdot 17^1 \cdot 31^1 L_4 L_5 - 2^1 \cdot 7^1 \cdot 73^1 \cdot 127^1 L_9\Big]$$

The result below was computed from *Expansions 24* and *33*.

Expansion 49. *Relation between Â-genus and L-genus of manifolds with vanishing odd Pontrjagin classes ($p_{2i+1} = 0$).*

In this case the odd classes on both sides are trivial and we are left with only even degree classes, i.e. those of dimension multiple of 8.

Expansion 49: Relation between Â-genus and L-genus with even Pontrjagin classes

$$\hat{A}_2 = \frac{1}{2^5 \cdot 7^1}\Big[-1L_2\Big]$$
$$\hat{A}_4 = \frac{1}{2^{11} \cdot 7^2 \cdot 127^1}\Big[+3^2 \cdot 5^2 L_2^2 - 2^2 \cdot 7^2 L_4\Big]$$
$$\hat{A}_6 = \frac{1}{2^{16} \cdot 7^3 \cdot 23^1 \cdot 89^1 \cdot 127^1}\Big[-3^2 \cdot 5^2 \cdot 11^1 \cdot 163^1 L_2^3 + 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 17^1 L_2 L_4 - 2^3 \cdot 7^3 \cdot 127^1 L_6\Big]$$

5.2 Todd and L genera

In order to compare either L-genus or Â-genus to Todd genus one needs to ensure that both have either complex bundles or real bundles. Therefore, we use the Todd genus of complexification in terms of Pontrjagin classes (see Expansion 41) and relate it to the standard L-genus and Â-genus expansions (see *Expansion 27* and *18*, respectively).

Expansion 50. *L-genus in terms of Todd genus of complexification.*

Expansion 50: L-genus in terms of Todd genus of complexification (or realification)

$$T_1 = \frac{1}{2^2}\Big[-1L_1\Big]$$
$$T_2 = \frac{1}{2^4 \cdot 7^1}\Big[+2^2 L_1^2 - 1^1 L_2\Big]$$
$$T_3 = \frac{1}{2^6 \cdot 7^1 \cdot 31^1}\Big[-2^1 \cdot 3^3 L_1^3 + 2^1 \cdot 19^1 L_1 L_2 - 7^1 L_3\Big]$$
$$T_4 = \frac{1}{2^8 \cdot 7^2 \cdot 31^1 \cdot 127^1}\Big[+2^1 \cdot 3^3 \cdot 331^1 L_1^4 - 2^1 \cdot 3^4 \cdot 5^1 \cdot 29^1 L_1^2 L_2 + 2^1 \cdot 7^2 \cdot 79^1 L_1 L_3 + 31^1\big(2^3 \cdot 11^1 L_2^2 - 7^2 L_4\big)\Big]$$

5.3 Todd and Â genera

For completeness let us restate that the equations below were obtained from *Expansions 41* and *18*. We illustrate a theoretical relationship as follows.

Proposition 4. **(i)** *For any complex vector bundle E, its realification $E_{\mathbb{R}}$ satisfies* $\mathrm{Td}(E_{\mathbb{R}} \otimes \mathbb{C}) = [\hat{A}(E_{\mathbb{R}})]^2$.

(ii) *For any oriented real vector bundle E we have the relation* $\mathrm{Td}(E \otimes \mathbb{C}) = [\hat{A}(E)]^2$.

Proof. **(i)** Start with a complex vector bundle of rank n, the (total) Todd class $\mathrm{Td}(E)$ is formally given by the series

$$\mathrm{Td}(E) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \ ,$$

where $x_1, \cdots, x_n$ are the Chern roots of E. We have the isomorphism $E_{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus \overline{E}$ as complex vector bundles. Hence, we obtain

$$\begin{aligned} \mathrm{Td}(E_{\mathbb{R}} \otimes \mathbb{C}) = \mathrm{Td}(E \oplus \overline{E}) = \mathrm{Td}(E) \cdot \mathrm{Td}(\overline{E}) &= \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \cdot \prod_{i=1}^{n} \frac{-x_i}{1 - e^{x_i}} \\ &= \prod_{i=1}^{n} \left[\frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \right]^2 \\ &= \prod_{i=1}^{n} \left[\frac{x_i/2}{\frac{1}{2}(e^{x_i/2} - e^{-x_i/2})} \right]^2 \\ &= \prod_{i=1}^{n} \left[\frac{x_i/2}{\sinh(x_i/2)} \right]^2 \\ &= \left[\hat{A}(E_{\mathbb{R}}) \right]^2 . \end{aligned}$$

(ii) If one starts from a real vector bundle E of rank n, we can form its complexification $E \otimes \mathbb{C}$. The total Todd class of the latter will be of the form

$$\mathrm{Td}(E \otimes \mathbb{C}) = \prod_{i=1}^{[\frac{n}{2}]} \frac{y_i}{1 - e^{-y_i}} \cdot \frac{-y_i}{1 - e^{y_i}},$$

because the Chern roots of $E \otimes \mathbb{C}$ are formally given by $\{\pm y_1, \cdots, \pm y_{[\frac{n}{2}]}\}$ if n is even and by $\{\pm y_1, \cdots, \pm y_{[\frac{n}{2}]}, 0\}$ if n is odd. Then the same computation as in part **(i)** yields the indicated formula. □

Expansion 51. *Â-genus in terms of Todd genus of complexification.*

Observation 1. *Note the relations in term of components*

$$\begin{aligned} T_{2n} &= \hat{A}_n^2 + 2(\hat{A}_1 \hat{A}_{2n-1} + \hat{A}_2 \hat{A}_{2n-2} + \ldots + \hat{A}_{2n} \hat{A}_0) \ , \\ T_{2n+1} &= 2(\hat{A}_{2n+1} \hat{A}_0 + \hat{A}_{2n} \hat{A}_1 + \hat{A}_{2n-1} \hat{A}_2 + \ldots + \hat{A}_n \hat{A}_{n+1}) \ . \end{aligned}$$

Considering how complex the expressions for the Â-genus and the Todd genus of complexification are, particularly in higher degrees, it is perhaps surprising how the two genera can be related through simple formulas with a clear pattern. In fact, the pattern evident in the formulas above is an expanded form of the relation between Todd and Â- genus demonstrated theoretically above.

Expansion 51: Â-genus in terms of Todd genus of complexification (or realification)

$T_1 = \frac{1}{1^1}\left[2\hat{A}_1\right]$

$T_2 = \frac{1}{1^1}\left[\hat{A}_1^2 + 2^1 \hat{A}_2\right]$

$T_3 = 2^1\left[\hat{A}_1 \hat{A}_2 + \hat{A}_3\right]$

$T_4 = \frac{1}{1^1}\left[\hat{A}_2^2 + 2^1(\hat{A}_1 \hat{A}_3 + \hat{A}_4)\right]$

$T_5 = 2^1\left[\hat{A}_2 \hat{A}_3 + \hat{A}_1 \hat{A}_4 + \hat{A}_5\right]$

$T_6 = \hat{A}_3^2 + 2^1\left[\hat{A}_2 \hat{A}_4 + \hat{A}_1 \hat{A}_5 + \hat{A}_6\right]$

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