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Simplicial Complexes and the Evasiveness Conjecture

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Abstract

We review the topological approach to evasiveness in [5], study the sizes of automorphism groups of graphs, and use this to estimate the Euler characteristic of the simplicial complex associated to a nontrivial graph property $\mathcal{P}$ on 6 vertices. Based on these estimates, we give an alternative proof of the evasiveness conjecture in the 6 vertices case, different from the given in [5]. The condition of being nonevasive implies that the Euler characteristic of these simplicial complexes is 1, and our estimate of $\chi(\mathcal{P})$ obligates $\mathcal{P}$ to contain some classes of special graphs. The use of Oliver groups (as in [5]) is essential to deal with the different cases that appear.

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1 Introduction

In this paper we consider only simple graphs on the fixed set of n vertices $V = \{1, 2, \ldots, n\}$. A graph $G = (V, E)$ is thus determined by its edge set $E \subseteq \binom{V}{2}$, which allows us to identify G with E. Two graphs $G = (V, E)$ and $G' = (V, E')$ on vertices V are *isomorphic* if there is a permutation σ of V, i.e. $\sigma \in S_n$, such that $\{i, j\} \in E$ if and only if $\{\sigma(i), \sigma(j)\} \in E'$. A *graph property* $\mathcal{P}$ is a collection of graphs, or a family of subsets of $\binom{V}{2}$, which is closed under isomorphism of graphs: namely, a graph G is in $\mathcal{P}$ if and only if any graph G' isomorphic to G is also in $\mathcal{P}$.

Now consider the following game in which there are two players X and Y, there is a graph property $\mathcal{P}$ that both X and Y know, and a graph G that only X knows. The goal of Y is to determine whether the graph G is in $\mathcal{P}$; player Y is allowed to ask X questions of the form "*is the edge* $\{i, j\}$ *in* G*?*", which X answers truthfully *yes* or *no*. The game ends when Y has determined if G is in $\mathcal{P}$ or not. A *strategy* for Y is an algorithm that, depending on the answer player X gives at each stage of the game, assigns an edge for asking the next question, or if possible, gives one of the answers, "G is in $\mathcal{P}$" or "G is not in $\mathcal{P}$" (ending the game). The minimal number k for which there is a strategy for player Y such that regardless of the graph G and the answers of player X, player Y can always end the game by asking at most k questions, is the *complexity* $c(\mathcal{P})$ of the graph property $\mathcal{P}$. This says that there is a strategy for Y such that Y can always reach his goal by asking at most $c(\mathcal{P})$ questions, but there is some *extreme* case (the "worst case") in which exactly $c(\mathcal{P})$ questions are required for ending up the game. We have that $c(\mathcal{P}) \leq \binom{n}{2}$. In the extreme case that $c(\mathcal{P}) = \binom{n}{2}$, we say that the property $\mathcal{P}$ is *evasive*, otherwise we say $\mathcal{P}$ is *nonevasive*.

For example, let us fix $V = \{1, 2, 3\}$ so that $e_1 = \{1, 2\}, e_2 = \{1, 3\}$ and $e_3 = \{2, 3\}$ are all possible edges graphs on 3 vertices can have. Let $\mathcal{P}$ be the property "having exactly one edge". This is a graph property consisting of exactly three graphs (one isomorphism class). The first question in our strategy can be "is e_1 in G?", and suppose the answer to this question is yes. At this point we know that the (unknown) graph G has at least one edge, but it is impossible for us to know if any other edge is in G or not. We are forced to ask, say, "is e_2 in G?". If the answer to this question is yes, then the game ends. But let us think of the worst case, that is, the answer to that question is no. Because we do not know if e_3 is in G, and there is no way of knowing but asking for it, we are forced to ask the third question. In fact, for any strategy, there is some "worst case" in which all 3 questions have to be asked in order to finish the game. Thus, this property $\mathcal{P}$ is evasive.

Few graph properties are known to be nonevasive. For example, the property $\mathcal{P}$ consisting of all graphs on six vertices isomorphic to one of the graphs shown in figure 1 is nonevasive (see [6] for more examples). A famous example is the property of being a *scorpion graph* (see [1],[6]), defined for $n \geq 5$ and that has complexity $\leq 6n - 13$, so that for $n \geq 11$ it is nonevasive.

We say a graph property $\mathcal{P}$ is *monotone* if it is closed under removal of edges. The property $\mathcal{P}$ is called *trivial* if it is either empty or is the family of all subsets of $\binom{V}{2}$, otherwise $\mathcal{P}$ is called *nontrivial*. The *evasiveness* conjecture or Karp's conjecture asserts that every nontrivial monotone graph property is evasive.

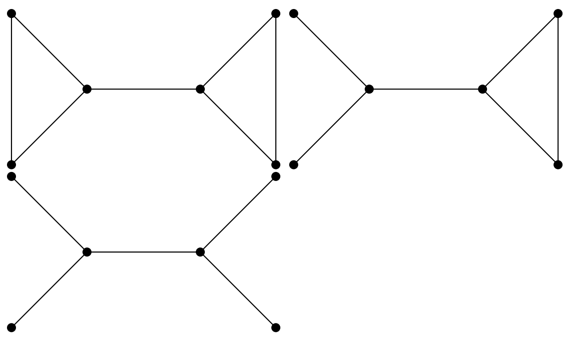


Fig. 1: A non-evasive graph property.

It was conjectured in [11] that there is a constant $\epsilon > 0$ such that any nontrivial monotone *digraph* property on n vertices has complexity at least ϵn^2. This statement, known as the Aanderaa-Rosenberg conjecture, was proven in [10]. Karp's conjecture is a very strong version of this conjecture (see [11]). Karp's conjecture is the starting point for the work of Kahn, Saks and Sturtevant [5], along with the fact that the conjecture was known to be true for many specific graph properties. In [5], an ingenious connection was established between topology and this complexity problem, which enabled Kahn, Saks and Sturtevant to give a proof of Karp's conjecture for the prime power case. They also prove the six vertices case by using this topological approach. Karp's conjecture is proven in [15] by A. C.-C. Yao for *bipartite graph properties*. Yao's proof is based on this topological approach.

Besides the results in [5] and [15], there are plenty of families of nontrivial monotone graph properties which are known to be evasive (see for example [4]). The techniques used are based mainly on the topological approach, in particular on *discrete Morse theory* (see [2], [4]).

2 Review of the Topological Approach

Let V be a finite set. An *(abstract) simplicial complex* on V is a collection K of non-empty subsets of V such that

(i) $\{v\} \in K$ for all $v \in V$ and

(ii) $A \in K$ and $B \subseteq A$ implies $B \in K$.

If $A \in K$ we say that A is a *face* or a *simplex* of K, and $|A| - 1$ is the *dimension* of A ($dimA$). If the whole set of vertices V is a face of K we say that K is a *simplex*, that is, K consists of all subsets of V. The *automorphism group* of K, $Aut(K)$, is the collection of all permutations of V which leave K invariant. There is a topological space associated to K, $|K|$, called the *geometric realization* of K; if $V = \{v_1, v_2, \ldots, v_n\}$, identifying v_i with the standard basis vector $e_i \in \mathbb{R}^n$, $|K|$ is the subspace of $\mathbb{R}^n$ obtained as the union of all convex hulls $\langle A\rangle = conv\{e_i : \ v_i \in A\}$ for $A \in K$. If K has f_i faces of dimension i, then the Euler characteristic of K, $\chi(K)$, is defined as

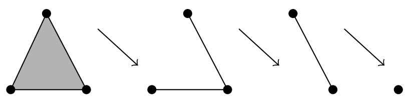


Fig. 2: Collapsing a simplicial complex to a vertex.

$$\chi(K) = \sum_{i\geq 0}(-1)^i f_i.$$

If Γ is a subgroup of $Aut(K)$, then Γ acts on $|K|$ by extending linearly the action on vertices, and we write $|K|^\Gamma$ for the fixed points of this action. The space $|K|^\Gamma$ can be described in an abstract way as follows: define K^Γ to be a simplicial complex such that

(i) the vertices of K^Γ are the orbits of the action of Γ on V that are also faces of K and

(ii) if $A_1, A_2, \ldots, A_r$ are vertices of K^Γ then

$$\{A_1, A_2, \ldots, A_r\}$$

is a face of K^Γ if $A_1 \cup A_2 \cup \cdots \cup A_r$ is a face of K.

If we identify each vertex A_i of K_Γ with the barycenter of $|A_i|$ in $|K|$, then the geometric realization of K^Γ is just $|K|^\Gamma$.

A *free face* of K is a nonempty face A of K such that it is not maximal under inclusion in K, but it is contained in exactly one inclusion maximal face B of K, where we require that $dimB = dimA + 1$. An *elementary collapse* of K consists of the removal of a free face along with the maximal face containing it. We say that K *collapses* to a complex K', and denote this by $K \searrow K'$, if K' can be obtained from K by a sequence of elementary collapses and say that K is *collapsible* if it collapses to a complex consisting of a single vertex.

We assume as known some concepts from algebraic topology such as singular and simplicial homology, contractibility of spaces, etc. We also make use of the following definition: K is called Γ-acyclic if its reduced homology with coefficients in Γ is trivial. We have the following sequence of implications: K is collapsible $\Longrightarrow$ K is contractible $\Longrightarrow$ K is $\mathbb{Z}$-acyclic $\Longrightarrow$ K is $\mathbb{Z}/p$-acyclic for every prime p.

For a vertex v we define the *link* of v, $lk_K(v)$ as the simplicial complex on vertices $V \setminus \{v\}$ given by $lk_K(v) = \{A \subseteq V \setminus \{v\} : \ A \cup \{v\} \in K\}$. We also define the *deletion* of v, $del_K(v)$ as the simplicial complex on $V \setminus \{v\}$ given by $del_K(v) = \{A \subseteq V \setminus \{v\} : \ A \in K\}$. Finally we define the *(Alexander) dual* of K, K^*, as the complex on V given by $K^* = \{A \subseteq V : \ V \setminus A \notin K\}$.

The concept of evasiveness can be defined for simplicial complexes as follows: just as before there is a game, two players X and Y, a simplicial complex K which is known to both X and Y, and a subset A of V

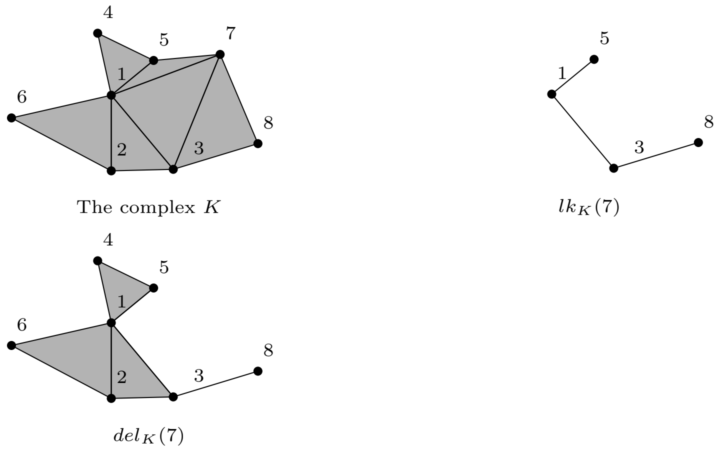


Fig. 3: Link and deletion of a vertex.

known only to X. Player Y has to determine if A is in K by asking questions of the form "*is the vertex v in A?*" As before we can define the complexity of K, and call K evasive, if its complexity equals the size of V, otherwise call K nonevasive. The dual of a nonevasive complex is again nonevasive, for determining whether A is in K is equivalent to determining whether $V \setminus A$ is not in K^*. We can regard a nonempty monotone graph property $\mathcal{P}$ on n vertices as an abstract simplicial complex $\Delta\mathcal{P}$ as follows: the set of vertices of $\Delta\mathcal{P}$ is the set of two-element subsets of $V = \{1, 2, \dots, n\}$ (that is, the set of all possible edges $\{i, j\}$, $1 \le i < j \le n$), and the simplices of $\Delta\mathcal{P}$ are the collections of such edges that correspond to graphs belonging to $\mathcal{P}$. Thus, simplices of $\Delta\mathcal{P}$ of dimension k correspond to graphs in $\mathcal{P}$ having $k+1$ edges. By abuse of notation we denote both the monotone graph property and the associated simplicial complex by $\mathcal{P}$. Note that the two concepts of evasiveness defined for properties of graphs $\mathcal{P}$ coincide.

The result that relates the complexity problem to topology is the following.

Theorem 2.1 (Kahn-Saks-Sturtevant, [5])**.** *A nonevasive complex is collapsible.*

Proof. Let K be a nonevasive complex on a set of n vertices V. Then there is a strategy for player Y in which not all n vertices have to be asked in order to determine if a (unknown subset) $A \subseteq V$ is in K or not. This strategy starts with some vertex v_0. Now, determining whether A is in K is equivalent to determining whether one of the following holds: $A \setminus \{v_0\}$ is in $lk_K(v_0)$ or $A \setminus \{v_0\}$ is in $del_K(v_0)$. Thus, we have that either of these two queries on $A \setminus \{v_0\}$ can be determined by asking less than $n-1$ questions, that is, both $lk_K(v_0)$ and $del_K(v_0)$ are nonevasive complexes. An induction argument applies. Assuming that the theorem is true for all nonevasive complexes on $n-1$ vertices, then $lk_K(v_0)$ and $del_K(v_0)$ are collapsible. If a sequence of free faces $A_1, A_2, \dots, A_m$ permits collapsing $lk_K(v_0)$ to a vertex x, then the sequence $A_1 \cup \{v_0\}, A_2 \cup \{v_0\}, \dots, A_m \cup \{v_0\}$ permits collapsing K to $del_K(v_0)$, and since $del_K(v_0)$ is collapsible, then so is K. □

Remark 2.1. In the proof of theorem 2.1 we showed that if K is nonevasive, then there exists some vertex $v_0 \in V$ such that both $lk_K(v_0)$ and $del_K(v_0)$ are nonevasive complexes. This can be used to give an equivalent inductive definition of a nonevasive complex. Define a simplicial complex K as nonevasive if it consists of a single vertex or if there exists some vertex $v_0 \in V$ such that both $lk_K(v_0)$ and $del_K(v_0)$ are nonevasive. Theorem 2.1 is true if we adopt this inductive definition of a nonevasive complex.

Corollary 2.2. *A nonevasive complex is $\mathbb{Z}$-acyclic.*

The next ingredient for giving the proof of Karp's conjecture in the prime power case is the action of the group $Aut(K)$. We use the following result of R. Oliver (see [9]):

Theorem 2.3 (Oliver, [9])**.** *Let K be a simplicial complex, Γ be a finite subgroup of $Aut(K)$ and p be a fixed prime. Assume that*
i) $|K|$ is $\mathbb{Z}/p$-acyclic and
ii) Γ has a normal p-subgroup Γ_1 such that the quotient Γ/Γ_1 is cyclic.
Then $\chi(|K|^\Gamma) = 1$. In particular, $|K|^\Gamma$ is nonempty.

We call a group Γ satisfying condition ii) in theorem 2.3 an *Oliver group*. For example, all finite p-groups are Oliver groups.

Lemma 2.4. *Let K be a nonempty $\mathbb{Z}$-acyclic complex on V and Γ be a vertex-transitive subgroup of $Aut(K)$. If Γ is an Oliver group, then K is a simplex.*

Proof. We claim that there is a face A of K which is invariant under Γ. A face A is invariant under Γ if and only if, regarding Γ acting on $|K|$, there is some point x in the relative interior of $|A|$ fixed by Γ. As Γ is an Oliver group, we know from Theorem 2.3 that there is a point x in $|K|$ fixed by Γ. Take A to be the unique face of K such that x is in the relative interior of $|A|$. Since Γ acts transitively on V, $A = V$ and so K is a simplex. □

Theorem 2.5 (Kahn-Saks-Sturtevant, [5])**.** *The evasiveness conjecture is true if $|V| = p^r$ is a power of a prime, that is, every nontrivial and monotone property of graphs on p^r vertices, is evasive.*

Proof. Let $|V| = p^r$ be a prime power and $\mathcal{P}$ be a nonempty monotone and nonevasive graph property on vertices V. Identify V with the finite field $GF(p^r)$ and consider the group Γ of all affine transformations $\phi_{a,b} : GF(p^r) \to GF(p^r)$, $x \mapsto ax + b$, for fixed a, b $GF(p^r)$, $a \neq 0$. This group Γ acts doubly transitively on V, that is, Γ acts transitively on the set of ordered pairs of elements of V and so it acts transitively on the set of unordered pairs. We are just saying that Γ is a vertex-transitive subgroup of $Aut(\mathcal{P})$. The group Γ has a normal p-subgroup $\Gamma_1 = \{\phi_{1,b} : b \in GF(p^r)\}$ with quotient $\Gamma/\Gamma_1 \cong GF(p^r)^*$ (the nonzero elements of $GF(p^r)$), which is known to be cyclic. Thus Γ is an Oliver group. Lemma 2.4 ends the proof, because $\mathcal{P}$ has to be the simplex on V, that is, $\mathcal{P}$ is trivial. □

3 Euler Characteristic and Automorphisms of Graphs

If $\mathcal{P}$ is a nonempty monotone and nonevasive graph property, then $\mathcal{P}$, considered as a simplicial complex is collapsible and this implies that $\chi(\mathcal{P}) = 1$. Therefore, if $\chi(\mathcal{P}) \neq 1$, then $\mathcal{P}$, being monotone and nontrivial, is evasive. This suggests the idea of examining the Euler characteristic of monotone graph properties of graphs on n vertices.

For a given graph G on n vertices let $[G]$ denote its isomorphism class and for any pair of graphs G, H on n vertices let us write $[G] \leq [H]$ if and only if G is isomorphic to some subgraph of H. We can write the Euler characteristic $\chi(\mathcal{P})$ as

$$\chi(\mathcal{P}) = \sum_{[G]\subseteq\mathcal{P}} (-1)^{m_G-1}|[G]|,$$

where $\mathcal{P}$ is supposed to be nonempty, m_G represents the number of edges of the graph G (so that G corresponds to a face of dimension $m_G - 1$ of the simplicial complex $\mathcal{P}$), $|[G]|$ is the size of the isomorphism class $[G]$ and the sum is taken over all isomorphism classes of graphs contained in $\mathcal{P}$. The idea is that in many cases there is a common divisor $d > 1$ of all the sizes $|[G]|$ for $G \in \mathcal{P}$ and, as a result, d divides $\chi(\mathcal{P})$; consequently $\chi(\mathcal{P}) \neq 1$ and we conclude that $\mathcal{P}$ is evasive.

In order to study the divisors of $|[G]|$, we observe that $|[G]| = \frac{n!}{|Aut(G)|}$, where $Aut(G)$ is the subgroup of S_n that leaves G invariant (the *automorphism group* of G). So we investigate the divisors of $|Aut(G)|$.
For each graph G on n vertices we have its complement $\overline{G} = \binom{V}{2} \setminus G$ and given a monotone graph property $\mathcal{P}$ we have its dual $\mathcal{P}^* = \{\overline{G} : \ G \notin \mathcal{P}\} = \{G : \ \overline{G} \notin \mathcal{P}\}$. We also have that $Aut(G) = Aut(\overline{G})$.

A *permutation group* Γ of *degree* m is a subgroup of S_m. For permutation groups Γ_1 of degree m_1 acting on the set $V_1 = \{1, 2, \ldots, m_1\}$ and Γ_2 of degree m_2 acting on $V_2 = \{1, 2, \ldots, m_2\}$, we can describe the *wreath product* $\Gamma_1 \wr \Gamma_2$ as the permutation group with elements represented by $(\sigma_1, \ldots, \sigma_{m_2}, \sigma)$ for $\sigma_i \in \Gamma_1$ and $\sigma \in \Gamma_2$, acting on $V_1 \times V_2$ by the rule $(\sigma_1, \ldots, \sigma_{m_2}, \sigma) \cdot (i, j) = (\sigma(i), \sigma_i(j))$. The size of $\Gamma_1 \wr \Gamma_2$ is $|\Gamma_2| \cdot |\Gamma_1|^{m_2}$. For the following basic result see [3], chapter 14.

Lemma 3.1. *Let G be a graph on n vertices, then the group $Aut(G)$ decomposes as*

$$Aut(G) \cong (Aut(G_1) \wr S_{n_1}) \times \cdots \times (Aut(G_s) \wr S_{n_s})$$

where the G_i's are the distinct connected components of G and n_i is the number of components of G isomorphic to G_i. Let m_i be the number of vertices of G_i so that $n = n_1m_1 + n_2m_2 + \cdots + n_sm_s$ and $Aut(G_i)$ is isomorphic to a subgroup of S_{m_i}. Then $|Aut(G)|$ divides $\prod_i n_i! \cdot (m_i!)^{n_i}$.

Let $2K_3$ be the graph on 6 vertices consisting of the union of two disjoint copies of the graph $K_3 = C_3$ (a 3-cycle graph). In this case $s = 1, m_1 = 3, n_1 = 2$, then $Aut(2K_3) \cong Aut(C_3) \wr S_2 \cong S_3 \wr S_2$. For the graph $3K_2$ we have that $Aut(3K_2) \cong Aut(K_2) \wr S_3 \cong S_2 \wr S_3$, in particular $|Aut(3K_2)| = 3! \cdot 2^3$.

Lemma 3.2. *Suppose that a graph G on n vertices has exactly k vertices of a given degree r, where $0 \leq k \leq n$. Then $Aut(G)$ is isomorphic to a subgroup of $S_k \times S_{n-k}$. In particular $\binom{n}{k}$ divides $|[G]|$.*

This is just because every element of $Aut(G)$ preserves the set of vertices of degree r and also preserves the set of vertices that are not of degree r. Then $|Aut(G)|$ divides $k!(n-k)!$ and $\frac{n!}{k!(n-k)!}$ divides $|[G]|$.
For example, for a graph G on 6 vertices that has exactly k vertices of some degree r we have that $\binom{6}{k}$ divides $|[G]|$. If we require that 3 does not divide $|[G]|$ then 3 does not divide $\binom{6}{k}$, but the only values of k for which this is true are 0, 3 and 6.

A graph G is called *regular* if all of its vertices have the same degree. If such degree is r, we say that G is r-regular. When studying the divisors of automorphism groups of regular graphs, the following result of N. Wormald, is very useful.

Theorem 3.3 (Wormald, [14])**.** *Let G be a connected r-regular graph on n vertices, where $r > 0$. Then $|Aut(G)|$ divides*

$$rn \prod_p p^\beta$$

where the product is taken over all prime numbers $p \leq r - 1$, and β is given by

$$\sum_{p^\alpha \leq r-1} \left\lfloor \frac{n-2}{p^\alpha} \right\rfloor$$

Corollary 3.4. *In the hypothesis of theorem 3.3, if $r < 3$, then $|Aut(G)|$ divides rn.*

Let us show how with these ideas we can give a proof of Karp's conjecture in the case of 5 vertices. We look for those graphs G on 5 vertices such that 5 does not divide $|[G]|$ or, equivalently, that 5 divides the size of $Aut(G)$. This is to say that $Aut(G)$ contains a 5-cycle, say (12345); the orbits of this 5-cycle acting on the two-element subsets of $\{1, 2, 3, 4, 5\}$ are

$$A = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\}$$

and $B = \{\{1, 3\}, \{2, 4\}, \{3, 5\}, \{1, 4\}, \{2, 5\}\}$ (see figure 4). If G contains some edge in A, then G contains A; similarly, if G contains some edge in B, then G contains B. Thus, there are just 4 options: $G = \overline{K}_5, G = A, G = B, G = A \cup B = K_5$. Note that $A \cong B$.
What we obtain is that if 5 divides $|Aut(G)|$ then G is isomorphic to one of the three graphs: $\overline{K}_5$ (the graph without edges), $G \cong C_5$ (a 5-cycle graph), $G =$

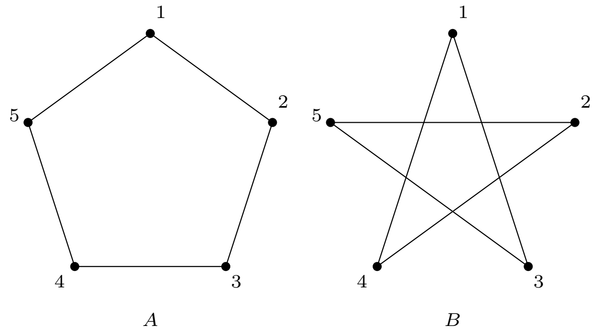


Fig. 4: The 2 5-cycles fixed under the action of (12345).

K_5 (the complete graph on 5 vertices). For any other graph G on 5 vertices we have that 5 divides $|[G]|$. Now let $\mathcal{P}$ be a nontrivial monotone and nonevasive graph property, so that $\chi(\mathcal{P}) = 1$; then $\mathcal{P}$ has to contain some of these three graphs for which 5 does not divide the size of their isomorphism classes. As $\overline{K}_5$ represents the empty simplex, it does not contribute to $\chi(\mathcal{P})$. As $\mathcal{P}$ is nontrivial, it does not contain the complete graph K_5. Then $\mathcal{P}$ must contain $[C_5]$. We know that $Aut(C_5) \cong D_5$ and $|[C_5]| = 5!/10 = 12$. Finally, since C_5 represents a face of dimension 4, $\chi(\mathcal{P})$ has the form $5m + 12$; but this can never equal 1. This shows that nontrivial monotone and nonevasive graph properties on 5 vertices do not exist.

Remark 3.1. Note that we have proved indeed that if $\mathcal{P}$ is any nontrivial monotone graph property on 5 vertices that contains the 5-cycle C_5, then $\chi(\mathcal{P})$ has the form $5m + 2$. In particular, $\mathcal{P}$ is not $\mathbb{Z}$-acyclic. This is also true for any other nontrivial monotone graph property $\mathcal{P}$ on 5 vertices, for then $\chi(\mathcal{P})$ is divisible by 5, in particular $\chi(\mathcal{P}) \neq 1$.

4 Evasiveness of Graph Properties on Six Vertices

In this section a proof of the evasiveness conjecture for properties of graphs on six vertices is shown in detail. The idea of estimating the Euler characteristic of the simplicial complex associated to a graph property yields that a nontrivial monotone and nonevasive graph property on 6 vertices contains exactly one of the two graphs $2K_3, K_{3,3}$. This is also shown in [5], by using the action of several Oliver groups. We start with a nontrivial monotone and nonevasive graph property $\mathcal{P}$ on 6 vertices, equivalently, $\mathcal{P}$ can be regarded as a nonevasive simplicial complex on the 15 elements which are the two-element subsets of $\{1, \dots, 6\}$. We want to show that such a $\mathcal{P}$ cannot exist. If a group Γ acts on the set $\{1, 2, 3, 4, 5, 6\}$ then it acts on the 2-element subsets of $\{1, 2, 3, 4, 5, 6\}$ and the orbits $X_1, \dots, X_r$ represent graphs on 6 vertices. The number 3 is going to play the role of the common divisor. The first step in the proof is to find those graphs G on 6 vertices for which 3 is not a divisor of $|[G]|$ (equivalently, 9 is a divisor of $|Aut(G)|$). Then, as for the 5 vertices case, $\mathcal{P}$ has to contain some

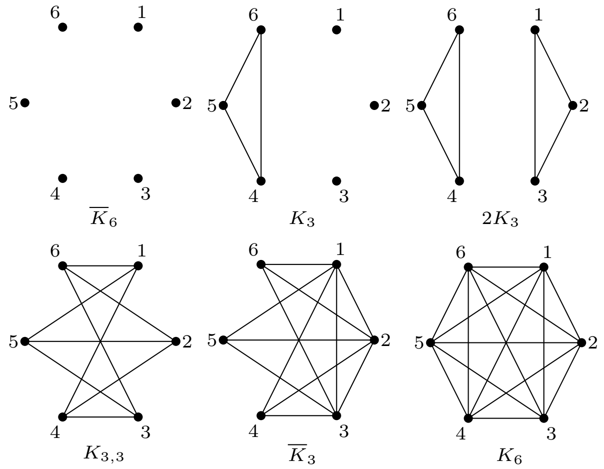


Fig. 5: Graphs G on 6 vertices such that $3 \nmid |[G]|$.

of these graphs in order to satisfy $\chi(\mathcal{P}) = 1$, and this yields various cases for $\mathcal{P}$, depending on which of those graphs $\mathcal{P}$ contains. Finally, by considering the action of several Oliver groups, we show that none of those cases can happen, reaching a contradiction and finishing the proof. The first step, the classification of those graphs G on 6 vertices for which 3 does not divide $|[G]|$ is the content of the following lemma.

Lemma 4.1. *Let G be a graph on 6 vertices and suppose that 3 is not a divisor of $|[G]|$, then G is isomorphic to one of the following 6 graphs: $\overline{K}_6$, K_3, $2K_3$, $K_{3,3}$, $\overline{K}_3$, K_6 (See figure 5).*

Proof. We start with the observation that 3 does not divide $|[G]|$ if and only if 9 divides $|Aut(G)|$. If $G = \overline{K}_6$, then $|Aut(G)| = 6!$ and it follows that 3 does not divide $|[G]|$. Suppose that G has at least one edge. To use lemma 3.2, suppose there are $k > 0$ vertices of some fixed degree $r > 0$ in G, then $\binom{6}{k}$ divides $|[G]|$, but 3 divides $\binom{6}{k}$ except for $k = 3, 6$.

Consider first the case $k = 3$ and suppose that the vertices of degree r are $1, 2, 3$. By an application of lemma 3.2 again, the remaining 3 vertices, 4,5,6, have degree s for some $s \neq r$. Let G_1 be the subgraph of G consisting of those edges of G connecting the vertices 1,2,3; G_2 be the subgraph of G consisting of the edges connecting 4,5,6 and H be the graph that has vertices 1,2,3,4,5,6 and whose edges are the edges of G connecting some vertex in $\{1, 2, 3\}$ to some vertex in $\{4, 5, 6\}$ (see figure 6). We know that $Aut(G)$ is isomorphic to a subgroup of $S_{\{1,2,3\}} \times S_{\{4,5,6\}}$ and that $Aut(G)$ contains a subgroup of order 9, but a subgroup of order 9 of $S_{\{1,2,3\}} \times S_{\{4,5,6\}}$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ where the first copy of $\mathbb{Z}/3$ is generated for some element α of order 3 that permutes transitively 1,2,3 and fixes $4, 5, 6$; and the second copy is generated by an element β of order 3 that transitively permutes 4,5,6 and fixes $1, 2, 3$. Now, if H has at least one edge, say $\{1, 4\}$, by combining the action of α and β we find that H contains the edges $\{1, 4\}$, $\{1, 5\}$, $\{1, 6\}$, $\{2, 4\}$, $\{2, 5\}$, $\{1, 6\}$, $\{3, 4\}$, $\{3, 5\}$, $\{3, 6\}$. So, H has no edges or $H = K_{3,3}$. As G_1

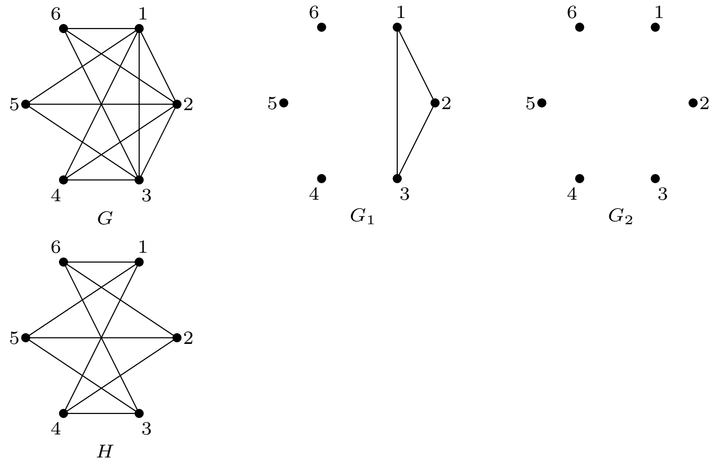


Fig. 6: The graphs G, G_1, G_2 and H.

has α as an automorphism, if G_1 has at least one edge, then it is equal to K_3. The same is true for G_2. So the only possibilities for $G = G_1 \cup G_2 \cup H$ are K_3 and $\overline{K}_3$ (remember that in this case $r \neq s$).

Now the case $k = 6$. If $r = 5$, then $G = K_6$, $|Aut(G)| = 6!$ and 3 does not divide $|[G]|$. Suppose $0 < r < 5$ and consider $\overline{G}$ which is a regular graph of degree $5 - r > 0$. Note that one of the two, r or $5 - r$ is less than 3. If both G and $\overline{G}$ are connected, choose the one that has degree less than 3, without loss of generality suppose it is G. Apply Wormald's theorem or corollary 3.4 to conclude that $|Aut(G)|$ divides $6r$, so that 9 does not divide $|Aut(G)|$. If some of the two, G or $\overline{G}$ is disconnected, say G, then apply lemma 3.1 to conclude that $|Aut(G)|$ divides $\prod_i n_i! \cdot (m_i!)^{n_i}$ where $6 = n_1m_1 + n_2m_2 + \cdots + n_sm_s$. As there are no isolated vertices in G we have that $m_i \geq 2$ for all i. We look for the *partitions* of 6 such that 9 divides $\prod_i n_i! \cdot (m_i!)^{n_i}$. If some $n_i \geq 3$, then $6 \geq n_im_i \geq 2n_i \geq 6$ which implies $s = 1$, $n_1 = 3$, $m_1 = 2$ and G is the union of 3 (disjoint) copies of the graph K_2; in this case $|Aut(G)| = 3!2^3$ is not divisible by 9. If $n_i < 3$ for all i, the only cases in which 9 divides $\prod_i n_i! \cdot (m_i!)^{n_i}$ are $m_i = 6$ for some i, or $m_i = 3$ and $m_j = 3$ for $i \neq j$, or $m_i = 3$ and $n_i = 2$ for some i. In the first case G is a regular connected graph (but we are assuming G is disconnected), in the second case G is the union of two nonisomorphic regular connected graphs on 3 vertices (which is impossible because there is only one regular connected graph on three vertices: K_3) and in the third case G is the union of two copies of a regular connected graph on 3 vertices (that is, $G = 2K_3$).
In case G is connected but $\overline{G}$ is not, then by the above argument $\overline{G} = 2K_3$ and $G = K_{3,3}$. This ends the proof. □

By being nontrival, $\mathcal{P}$ does not contain K_6, and the graph $\overline{K}_6$ represents the empty simplex which does not contribute to $\chi(\mathcal{P})$. If none of the graphs K_3, $2K_3$, $K_{3,3}$, $\overline{K}_3$ belongs to $\mathcal{P}$, then for each $G \in \mathcal{P}$, 3 is a divisor of $|[G]|$ and then $\mathcal{P}$ cannot be nonevasive. Then $\mathcal{P}$ must contain some of these 4 graphs. The following table shows the automorphism group, the size of the isomorphism classes of these 4 graphs and the dimension of the faces that they represent:

G	$Aut(G)$	$\|[G]\|$	$dimG$
K_3	$S_3 \times S_3$	20	2
$2K_3$	$S_3 \wr S_2$	10	5
$K_{3,3}$	$S_3 \wr S_2$	10	8
$\overline{K}_3$	$S_3 \times S_3$	20	11

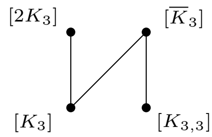
Lemma 4.2. *If $\mathcal{P}$ is a nontrivial monotone and nonevasive graph property on 6 vertices, then $\mathcal{P}$ satisfies exactly one of the following:*
1) *From the 4 graphs $K_3, 2K_3, K_{3,3}, \overline{K}_3$, $\mathcal{P}$ just contains $K_{3,3}$.*
2) *From the 4 graphs $K_3, 2K_3, K_{3,3}, \overline{K}_3$, $\mathcal{P}$ just contains K_3 and $2K_3$.*
3) *From the 4 graphs $K_3, 2K_3, K_{3,3}, \overline{K}_3$, $\mathcal{P}$ just contains $K_3, K_{3,3}$ and $\overline{K}_3$.*

We will say that $\mathcal{P}$ is of *type* 1, 2 or 3 if $\mathcal{P}$ satisfies 1), 2) or 3) in lemma 4.2, respectively. Note that if $\mathcal{P}$ is of type i, then $\mathcal{P}^*$ is of type $3 - i$.

Proof. We have the following relations:

$$[K_3] \leq [2K_3], [K_3] \leq [\overline{K}_3], \quad [K_{3,3}] \leq [\overline{K}_3],$$

that can be represented by the following poset:



As we said before, by satisfying $\chi(\mathcal{P}) = 1$, $\mathcal{P}$ has to contain some of these 4 graphs. $\mathcal{P}$ could just contain any of the 15 nonempty subsets of $\{K_3, 2K_3, K_{3,3}, \overline{K}_3\}$, but we note that $\mathcal{P}$ cannot just contain $2K_3$, for instance, for if $2K_3 \in \mathcal{P}$, then $K_3 \in \mathcal{P}$, for $\mathcal{P}$ is monotone and for the relation $[K_3] \leq [2K_3]$. $\mathcal{P}$ can just contain K_3, for example; if this is the case, then we estimate $\chi(\mathcal{P})$. Because for all graphs G in $\mathcal{P}$ that are not isomorphic to K_3, 3 is a divisor of $|[G]|$, $\chi(\mathcal{P})$ has the form $3m + 20$ (K_3 is a 2-dimensional face of $\mathcal{P}$, and $|[K_3]| = 20$). The following table shows all the possibilities of which nonempty subsets of $\{K_3, 2K_3, K_{3,3}, \overline{K}_3\}$ can be contained in $\mathcal{P}$ and the form that $\chi(\mathcal{P})$ takes in each case:

$\mathcal{P}$ contains just	$\chi(\mathcal{P})$
K_3	$3m + 20$
$K_{3,3}$	$3m + 10$
$K_3, 2K_3$	$3m + 10$
$K_3, K_{3,3}$	$3m + 30$
$K_3, 2K_3, K_{3,3}$	$3m + 20$
$K_3, K_{3,3}, \overline{K}_3$	$3m + 10$
$K_3, 2K_3, K_{3,3}, \overline{K}_3$	$3m$

The only cases in which $\chi(\mathcal{P}) = 1$ can happen are the cases in which $\chi(\mathcal{P})$ has the form $3m + 10$, and this ends the proof. □

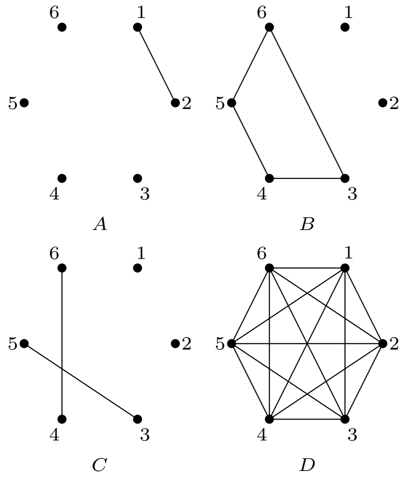


Fig. 7: Orbits under the action of $\langle(12),(3456),(35)\rangle$

Up to this point, we have reduced the possibilities for $\mathcal{P}$ to be of type 1, 2 or 3. Now, considering some Oliver groups acting on $\mathcal{P}$, and using theorem 2.3, we will prove that none of the three types can really happen. The ideas that are going to be used are already found in [5], but there are some results we can obtain directly from lemma 4.2. For instance, notice that, in any case, exactly one of the two graphs $2K_3, K_{3,3}$ belongs to $\mathcal{P}$.

First, we want to show that $\mathcal{P}$ cannot be of type 1 nor 3. Assume that $\mathcal{P}$ is of type 1. Thus, $\mathcal{P}$ contains $K_{3,3}$, but it does not contain K_3. Following [5], consider the group $\Gamma = \langle(12),(3456),(35)\rangle$ which is a 2-group (all its elements have order equal to a power of 2) and therefore it is an Oliver group. This group acts on the vertices of the simplicial complex $\mathcal{P}$, and the orbits A, B, C, D of this action are shown in figure 7. As all of the graphs $A, B, C, A \cup B$ and $A \cup C$ are isomorphic to subgraphs of $K_{3,3}$, they belong to $\mathcal{P}$. We can check that $A \cup D$, $B \cup D$, $C \cup D$ and $B \cup C$, contain a copy of K_3 as a subgraph, so they cannot be in $\mathcal{P}$ (as $\mathcal{P}$ is monotone and does not contain K_3). If $D \in \mathcal{P}$, then $\mathcal{P}^\Gamma$ is a simplicial complex whose faces are $\varnothing, A, B, C, D, \{A, B\}, \{A, C\}$; but this implies that $\chi(\mathcal{P}^\Gamma) = 2$ which contradicts theorem 2.3. Hence, $\mathcal{P}_\Gamma$ is a simplicial complex with faces $\varnothing, A, B, C, \{A, B\}, \{A, C\}$.

As $\mathcal{P}$ is nonevasive, there is some vertex X of $\mathcal{P}$ such that $lk_\mathcal{P}(X)$ and $del_\mathcal{P}(X)$ are nonevasive. The transitivity of $Aut(\mathcal{P})$ (remember that being a property of graphs means that the group S_6, acting in the natural way on the vertices of the simplicial complex $\mathcal{P}$, is a subgroup of $Aut(\mathcal{P})$) permits us to take $X = A$ (that is, the game can be started with the question *is A an edge of your graph?*). The vertex A of $\mathcal{P}$ is fixed by Γ, so Γ acts on $lk_\mathcal{P}(A)$, and theorem 2.3 can be applied to conclude that $\chi((lk_\mathcal{P}(A))^\Gamma) = 1$. By the definition of the link, we find that $(lk_\mathcal{P}(A))^\Gamma = lk_{\mathcal{P}^\Gamma}(A)$, which is a complex with just two vertices (see figure 8) and no edges, then its Euler characteristic is 2, a contradiction. This proves that $\mathcal{P}$ cannot be of type 1.

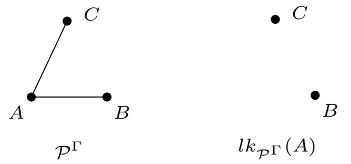


Fig. 8:

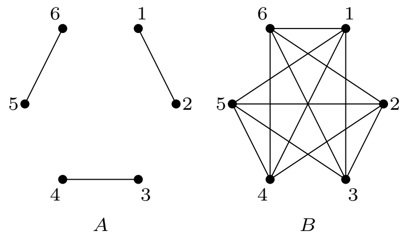


Fig. 9: Orbits under the action of $\langle(12),(34),(56),(135)(246)\rangle$

If $\mathcal{P}$ is of type 3, then $\mathcal{P}^*$ is of type 1; but the above argument applied to $\mathcal{P}^*$ gives as result that $\mathcal{P}^*$ cannot be of type 1 and as consequence, $\mathcal{P}$ cannot be of type 3.

Finally suppose that $\mathcal{P}$ is of type 2. In this case, $\mathcal{P}^*$ is also of type 2. Both $\mathcal{P}$ and $\mathcal{P}^*$ contain K_3 and $2K_3$, but they do not contain $K_{3,3}$ nor $\overline{K}_3$. The following lemma holds for $\mathcal{P}$ without the assumption of being of type 2.

Lemma 4.3. *If $\mathcal{P}$ is a nontrivial monotone and nonevasive graph property on 6 vertices, then all perfect matchings belong to $\mathcal{P}$.*

Proof. Consider the group Γ generated by the permutations $(12), (34), (56)$ and $(135)(246)$. This Γ is an Oliver group, for Γ_1 can be taken as the subgroup of Γ generated by $(12), (34)$ and (56), which has order 8 and the quotient Γ/Γ_1 is cyclic and isomorphic to $\langle(135)(246)\rangle$. Thus, theorem 2.3 implies that $|\mathcal{P}|^\Gamma \neq \varnothing$.

The group Γ acts on the set of 2-element subsets of $\{1, 2, 3, 4, 5, 6\}$ and the two orbits of this action, A and B, are shown in figure 9. In the abstract version of $\mathcal{P}^\Gamma$, theorem 2.3 says that at least one of the graphs in figure 9 belongs to $\mathcal{P}$. Note that A is a perfect matching. We claim that $\mathcal{P}$ does not contain B. On the contrary, suppose B is in $\mathcal{P}$ and note that B contains a perfect matching as a subgraph; thus $\mathcal{P}$ contains both A and B. This two, A and B are all possible vertices of $\mathcal{P}^\Gamma$ and theorem 2.3 says that $\chi(\mathcal{P}^\Gamma) = 1$; this obligates $\mathcal{P}^\Gamma$ to contain the face $\{A, B\}$, which means that $A \cup B = K_6$ belongs to $\mathcal{P}$, a contradiction. Therefore, $\mathcal{P}$ contains A (and so $\mathcal{P}$ contains all perfect matchings). Note that we have proved that from the two graphs, A and B, $\mathcal{P}$ contains exactly A. □

Now we prove that $\mathcal{P}$ cannot be of type 2. Assume $\mathcal{P}$ is of type 2. Consider the group $\Gamma = \langle(153624)\rangle$. The orbits A, B, C of Γ are in figure 10. This Γ is an Oliver group as it is cyclic of order 6, so Γ_1 can be taken as any subgroup of order 3 (as Γ_1 has index 2,

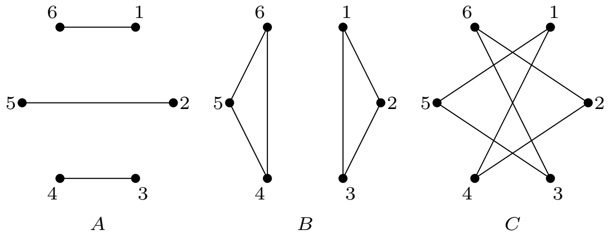


Fig. 10: Orbits under the action of $\langle(153624)\rangle$

it is normal) whose quotient is cyclic of order 2. By lemma 4.3, $A \in \mathcal{P}$. As $\mathcal{P}$ is of type 2, $B \in \mathcal{P}$. We claim that C is also in $\mathcal{P}$. If not, $\mathcal{P}^{\Gamma}$ has just two vertices A, B and theorem 2.3 implies that $A \cup B \in \mathcal{P}$, but C is isomorphic to a subgraph of $A \cup B$, so C is in $\mathcal{P}$. This argument also applies to $\mathcal{P}^*$ (as it is also of type 2), so we have that $A, B, C \in \mathcal{P}^*$. By the definition of $\mathcal{P}^*$, $\overline{A} = B \cup C, \overline{B} = A \cup C, \overline{C} = A \cup B$ are not in $\mathcal{P}$, so $\mathcal{P}^{\Gamma}$ consists only of three vertices and this contradicts theorem 2.3. This finishes the proof that every nontrivial monotone graph property on 6 vertices has to be evasive.

Remark 4.1. The estimation of the Euler characteristic by means of divisors of the sizes of isomorphism classes of graphs on n vertices is not generally enough to give a proof of evasiveness. The use of another tool is necessary like theorem 2.3, for instance. In fact, there are nontrivial monotone graph properties $\mathcal{P}$ satisfying $\chi(\mathcal{P}) = 1$; see [4], pp. 124, or also [7] for an example on 6 vertices. In [4] there is an example of a monotone graph property on 6 vertices which is $\mathbb{Q}$-acyclic, but it is not $\mathbb{Z}$-acyclic.

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