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Spectral sequences via examples

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Abstract

These are lecture notes for a short course about spectral sequences that was held at Málaga, October 18–20 (2016), during the "Fifth Young Spanish Topologists Meeting". The approach is to illustrate the basic notions via fully computed examples arising from Algebraic Topology and Group Theory. We give particular attention to extension and lifting problems in spectral sequences, which is an aspect often occulted in introductory texts.

MSC2010. Primary 55T;Secondary 18G40,55R20,20J06

1 Foreword

These notes give a simplified and hands-on introduction to spectral sequences based on examples. The aim is to introduce the basic notions involved when working with spectral sequences with the minimum necessary background. We go along the way bringing in notions and illustrating them by concrete computations. The target reader is someone not acquainted with this topic. For full details the reader should consult [17], [21] or [16].

The following list describes the contents of the different sections of these notes:

(2) **Spectral sequences and filtrations.** We introduce the intimate relation between spectral sequences, filtrations, quotients and (co)homology.

(3) **Differentials and convergence.** We give definitions of what a (homological type) spectral sequence is and what is convergence in this setup. We also talk about extension problems.

(4) **Cohomological type spectral sequences.** We introduce cohomological type spectral sequences and say a few more words about the extension problem.

(5) **Additional structure: algebra.** We equip everything with products, yielding graded algebras and bigraded algebras. We start discussing the lifting problem.

(6) **Lifting problem.** We go in detail about the lifting problem and discuss several particular cases.

(7) **Edge morphisms.** After introducing the relevant notions, a couple of examples where the edge morphisms play a central role are fully described.

(8) **Different spectral sequences with same target.** This section is included to make the reader aware of this fact.

(9) **A glimpse into the black box.** We sketch the technicalities that have been avoided in the rest of the sections. In particular, we provide some detail on the construction of a spectral sequence from a filtration of a chain complex.

(10) **Appendix I.** Here, precise statements for the following spectral sequences are given: Serre spectral sequence of a fibration, Lyndon-Hochschild-Serre spectral sequence of a short exact sequence of groups, Atiyah-Hirzebruch spectral sequence of a fibration. All examples in these notes are based on these spectral sequences. Also, sketch of proofs of the two first spectral sequences are included and the filtrations employed are described.

(11) **Appendix II.** We list some important results in Topology whose proofs involve spectral sequences and we briefly comment on the role they play in the proofs.

(12) **Solutions.** Answers to the proposed exercises. Some of them are just a reference to where to find a detailed solution.

We denote by R the base ring (commutative with unit) over which we work.

Acknowledgements: I am grateful to Oihana Garaialde, David Méndez and Jose Manuel Moreno for reading a preliminary version of these notes. Also, thanks to Urtzi Buijs for its nice version of the bridge-river-tree picture using the TikZ package. I thank the referee and the editors for their careful reading and their many helpful suggestions and recommendations.

2 Spectral sequences and filtrations

A spectral sequence is a tool to compute the homology of a chain complex. It arises from a filtration of the chain complex and it provides an alternative way to determine the homology of the chain complex. A spectral sequence consists of a sequence of intermediate chain complexes called "pages", $E^0, E^1, E^2, E^3, \ldots$, with differentials denoted by $d_0, d_1, d_2, d_3, \ldots$, such that E^{r+1} is the homology of E^r. The various pages have accessible homology groups which form a finer and finer approximation of the homology H we wish to find out. This limit process may converge, in which case the "limit page" is denoted by E^∞, see Section 3 for more details. Even if there is convergence to E^∞, reconstruction is still needed to obtain H from E^∞. This reconstruction procedure involves solving the extension and lifting problems, see Sections 3, 4, 5 and 6.

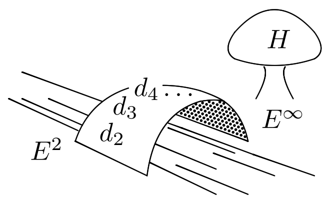
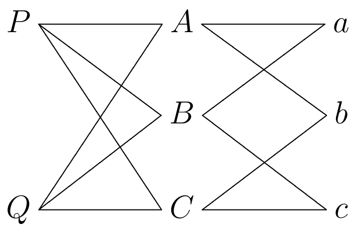


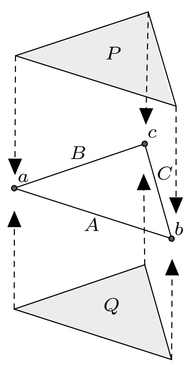
Fig. 1: You want to cross the bridge and then to climb the tree. Don't fall into the river of madness!

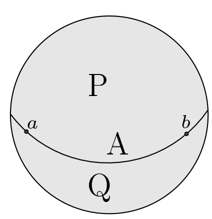
Although the differentials d_r's cannot always be all computed, the existence of the spectral sequence often reveals deep facts about the chain complex. The spectral sequence and its internal mechanisms can still lead to very useful and deep applications, see Appendix II 11.

Our first two examples illustrate how the E_r's and the d_r's arise from a filtration. To read the precise definition of what a filtration is and how exactly it gives rise to a spectral sequence, see Section 9.

Example 2.1. Consider a (semi-simplicial) model of the 2-sphere S^2 with vertices $\{a, b, c\}$, edges $\{A, B, C\}$ and solid triangles $\{P, Q\}$ and with inclusions as follows:







The homology $H = H_*(C, d)$ of the following chain complex (C, d) computes the integral homology of the sphere S^2:

$$0 \longrightarrow \mathbb{Z}\{P, Q\} \xrightarrow{\ d\ } \mathbb{Z}\{A, B, C\} \xrightarrow{d} \mathbb{Z}\{a, b, c\} \to 0.$$

$$\begin{array}{ll} d(P) = C - B + A & d(A) = b - a \\ d(Q) = C - B + A & d(B) = c - a \\ & d(C) = c - b \end{array}$$

Instead of directly calculating H, we instead consider the following submodules on each dimension:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}\{P, Q\} & \to & \mathbb{Z}\{A, B, C\} & \to & \mathbb{Z}\{a, b, c\} & \to & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{A, B\} & \longrightarrow & \mathbb{Z}\{a, b, c\} & \to & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{A\} & \longrightarrow & \mathbb{Z}\{a, b\} & \to & 0. \end{array}$$

As you can see, the differential restricts to each row, making each row into a chain complex, and each row is contained in the row above. A family of such subcomplexes is known as a *filtration* of (C, d). We may then consider the quotient of successive rows. The object obtained is known as the *associated graded object* with respect to the given filtration. We call it E^0 and it inherits a differential d_0 from d:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}\{P, Q\} & \to & \mathbb{Z}\{C\} & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{B\} & \to & \mathbb{Z}\{c\} & \to & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{A\} & \to & \mathbb{Z}\{a, b\} & \to & 0 \end{array}$$

$$d_0(P) = C \ , \ d_0(Q) = C \ , \ d_0(B) = c \ , \ d_0(A) = b - a.$$

If we compute the homology with respect to d_0 we obtain E^1:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}\{P - Q\} & \to & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \to & \mathbb{Z}\{\bar{a}\} & \to & 0. \end{array}$$

Clearly, the differential d_1, which is also induced by d, is zero. This implies that $E^2 = E^1$. For the same reason, $d_n = 0$ for all $n \geq 2$ and $E^\infty = \ldots = E^3 = E^2 = E^1$. The two classes $P - Q$ and $\bar{a}$ correspond respectively to generators of the homology groups $H_2(S^2; \mathbb{Z}) = \mathbb{Z}$ and $H_0(S^2; \mathbb{Z}) = \mathbb{Z}$.

Of course in Example 2.1, $H = H_*(C, d)$ is easy to compute directly and the use of a spectral sequence is not needed. However and in practice, $H_*(C, d)$ is very hard to obtain directly and passing to a spectral sequence is necessary.

Exercise 2.2. Cook up a different filtration to that in Example 2.1 and compute E^0 and E^1. Obtain again the integral homology of S^2.

We can learn much from this example. The first fact is that, in general,

> **E^0 is the associated graded object for the filtration on C,**

and d_0 is induced by d. Then E^1 is the homology of (E^0, d_0) and d_1 is induced by d again. In fact, this is the general rule to pass from E^n to E^{n+1}:

> **E^{n+1} is the homology of (E^n, d_n) and d_{n+1} is induced from d.**

Although there exists a closed expression for d_{n+1} in terms of d and C, it is so complicated that, in practice, you are only able to fully describe (E^0, d_0), (E^1, d_1) and E^2. Then E^2 is what you write in the statement of your theorem about your spectral sequence. You are seldom capable of figuring out what $d_2, d_3, \ldots$ are.

This gives you some hints about the bridge in Figure 2. The next example provides information on how the tree in that figure looks like.

Example 2.3. Recall the filtration of chain complexes we gave in Example 2.1:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P,Q\} & \to & \mathbb{Z}\{A,B,C\} & \to & \mathbb{Z}\{a,b,c\} & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{A,B\} & \to & \mathbb{Z}\{a,b,c\} & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{A\} & \to & \mathbb{Z}\{a,b\} & \to & 0.
\end{array}$$

Now, instead of taking quotient followed by homology, we do it the other way around, i.e., we first take homology,

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P-Q\} & \to & 0 & \to & \mathbb{Z}\{\bar{a}\} & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & \mathbb{Z}\{\bar{a}\} & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & \mathbb{Z}\{\bar{a}\} & \to & 0,
\end{array}$$

obtaining a filtration of H, and then we take quotient by successive rows,

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P-Q\} & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & \mathbb{Z}\{\bar{a}\} & \to & 0.
\end{array}$$

So we obtain the associated graded object with respect to the induced filtration of H. We call it E^∞. As you can observe, in this particular example we have $E^\infty = E^1$ of Example 2.1. In general, E^∞ is the "limit" of the pages $E^0, E^1, E^2, E^3, \ldots$.

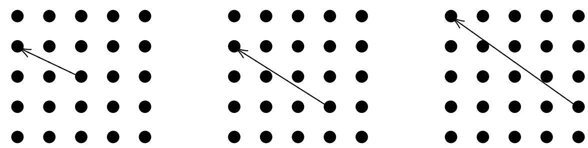
What happened in the above example is not a coincidence and, in general,

> **E^∞ is the associated graded object for the induced filtration on H.**

All notions discussed above will now be made precise.

3 Differentials and convergence

A (homological type) spectral sequence consists of a sequence of pages and differentials $\{E^r, d_r\}_{r\geq 2}$, such that E^{r+1} is the homology of (E^r, d_r). Each page is a differential bigraded module over the base ring R. This means that at each position (p,q), there is an R-module $E^r_{p,q}$. Moreover, the differential d_r squares to zero and has bidegree $(-r, r-1)$. This last bit means that it maps $d_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$, as in these pictures:



(E^2, d_2) (E^3, d_3) (E^4, d_4)

One does not need to know neither (E^0, d_0) nor (E^1, d_1) to start computing with a spectral sequence. You can proceed even if you do not know C or the filtration that gave rise to the spectral sequence. If you know H and you are told that the spectral sequence *converges to* H, written as $E^2_{p,q} \Rightarrow H_{p+q}$, this roughly means that:

(a) the value at each position (p,q) stabilizes after a finite number of steps, and the stable value is denoted by $E^\infty_{p,q}$, and that

(b) you can recover H from E^∞.

Before unfolding details about point (b), we work out an example.

Example 3.1. The sphere of dimension S^n is a Moore space $M(n,\mathbb{Z})$, i.e., its homology is given by $H_i(S^n;\mathbb{Z}) = \mathbb{Z}$ if $i = n, 0$ and 0 otherwise. Moreover, the spheres of dimensions 1, 3 and 2 nicely fit into the Hopf fibration:

$$S^1 \longrightarrow S^3 \longrightarrow S^2$$

The Hopf map $S^3 \to S^2$, we recall, takes $(z_1, z_2) \in S^3 \subseteq \mathbb{C}\times\mathbb{C}$ to $\frac{z_1}{z_2} \in \mathbb{C}\cup\{\infty\} = S^2$. Associated to a fibration over a CW-complex, there is the Serre spectral sequence which converges to the homology of the total space. The filtration on the chain complex for E is obtained by taking preimages of the skeleta of the base space. See Theorems 10.1 and 10.2 in Appendix I 10 for more details. As explained there, the second page of the spectral sequence consist of the homology of the base space with (local) coefficients in the homology of the fiber. In this case, $\pi_1(S^2) = 1$ and the second page consists of the homology of S^2 with coefficients in the homology of S^1, and it *converges* to the homology of S^3:

$$E^2_{p,q} = H_p(S^2; H_q(S^1;\mathbb{Z})) \Rightarrow H_{p+q}(S^3;\mathbb{Z}).$$

So we have:

$$E^2_{p,q} = H_p(S^2; H_q(S^1;\mathbb{Z})) = \begin{cases} \mathbb{Z}, & \text{if } p = 0,2 \text{ and } q = 0,1, \\ 0, & \text{otherwise,} \end{cases}$$

and E^2 looks like this:

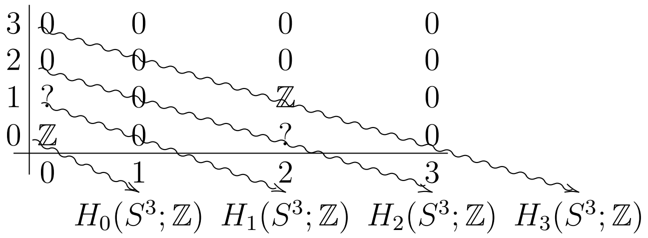
$$\begin{array}{c|cccc} 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\ \hline & 0 & 1 & 2 & 3 \end{array}$$

The only possible non-trivial differential d_2 in E^2 is $d_2\colon E^2_{2,0} = \mathbb{Z} \to E^2_{0,1} = \mathbb{Z}$. All higher differentials $d_3, d_4, \ldots$ must be zero because of the location of the $\mathbb{Z}$'s in the diagram. This ensures that point (a) above holds. As $d_2\colon \mathbb{Z} \to \mathbb{Z}$ is a $\mathbb{Z}$-homomorphism, it is determined by $d_2(1)$. Depending on this value, we get the following page $E^3 = E^\infty$:

$$\begin{array}{c|cccc} 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\ \hline & 0 & 1 & 2 & 3 \end{array} \qquad \begin{array}{c|cccc} 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & 0 & 0 \\ \hline & 0 & 1 & 2 & 3 \end{array} \qquad \begin{array}{c|cccc} 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 1 & \mathbb{Z}_n & 0 & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & 0 & 0 \\ \hline & 0 & 1 & 2 & 3 \end{array}$$

$d_2(1) = 0$ $\qquad$ $d_2(1) = \pm 1$ $\qquad$ $d_2(1) = n$, $n \neq 0, \pm 1$

Now, point (b) above means that each $\mathbb{Z}$-module $H_n(S^3;\mathbb{Z})$ can be recovered from the "diagonal" $\mathbb{Z}$-modules $\{E^\infty_{p,q}\}_{p+q=n}$:



In the three different cases discussed above, $d_2(1) = 0$, $d_2(1) = \pm 1$, $d_2(1) = n, n \neq 0, \pm 1$, we would get the following values:

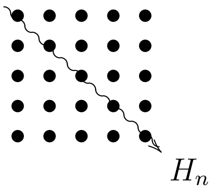
$$\begin{array}{lll} H_0(S^3;\mathbb{Z}) = \mathbb{Z} & H_0(S^3;\mathbb{Z}) = \mathbb{Z} & H_0(S^3;\mathbb{Z}) = \mathbb{Z} \\ H_1(S^3;\mathbb{Z}) = \mathbb{Z} & H_1(S^3;\mathbb{Z}) = 0 & H_1(S^3;\mathbb{Z}) = \mathbb{Z}_n \\ H_2(S^3;\mathbb{Z}) = \mathbb{Z} & H_2(S^3;\mathbb{Z}) = 0 & H_2(S^3;\mathbb{Z}) = 0 \\ H_3(S^3;\mathbb{Z}) = \mathbb{Z} & H_3(S^3;\mathbb{Z}) = \mathbb{Z} & H_3(S^3;\mathbb{Z}) = \mathbb{Z} \\ (d_2(1) = 0) & (d_2(1) = \pm 1) & (d_2(1) = n \\ & & \quad n \neq 0, \pm 1). \end{array}$$

As we know the homology of S^3, we deduce that we must have $d_2(1) = \pm 1$ and that E^∞ contains exactly two entries different from zero. This brings up another aspect: in order to compute some key differentials in the Serre spectral sequence, it is necessary to have a deep insight into the geometry of the fibration. In the case above, d_2 is determined by the holonomy of the fibration and it is possible to argue geometrically why $d_2(1) = \pm 1$. This then recovers our knowledge of $H_*(S^3;\mathbb{Z})$.

Remark 3.2. Note that the double indexing in examples 2.1 and 3.1 are different. The one used in the latter case is customary.

Exercise 3.3. Deduce using the same arguments that if $S^l \to S^m \to S^n$ is a fibration then $n = l + 1$ and $m = 2n - 1$.

Next we unravel what point (b) above means with more detail: You can recover the R-module H_n from the "diagonal" R-modules $\{E^\infty_{p,q}\}_{p+q=n}$ via a finite number of extensions.



More precisely, there exist numbers $s \leq r$ and a finite *increasing* filtration of H_n by R-modules,

$$0 = A_{s-1} \subseteq A_s \subseteq A_{s+1} \subseteq \ldots \subseteq A_{r-1} \subseteq A_r = H_n,$$

together with short exacts sequences of R-modules:

$$\begin{aligned} &A_{r-1} \to H_n \to E^\infty_{r,n-r} \qquad (3.1)\\ &A_{r-2} \to A_{r-1} \to E^\infty_{r-1,n-r+1} \\ &\ldots \\ &A_s \to A_{s+1} \to E^\infty_{s+1,n-s-1} \\ &0 \to A_s \to E^\infty_{s,n-s}. \end{aligned}$$

So, starting from the bottom, $A_s = E^\infty_{s,n-s}$, one expects to find out from these extensions what the R-modules $A_{s+1}, \ldots, A_{r-1}, A_r = H_n$ are. This is in general not possible without further information. For instance, in Example 3.1 we could deduce the homology groups H_* because there was only a non-zero entry on each diagonal of E^∞. In the particular case of R being a field, all extensions are trivial being extensions of free modules. In particular, H_n is the direct sum $H_n \cong \oplus_{l=s}^r E^\infty_{l,n-l}$. The next example exhibits the kind of extension problems one can find when R is not a field.

Remark 3.4. If H is graded and $a \in H_p$ is homogeneous then $|a| = p$ is called its degree or its total degree. If E is bigraded and $a \in E_{p,q}$ then $|a| = (p,q)$ is called its bidegree and $|a| = p+q$ is called its total degree. So Equation (3.1) says that elements from $E^\infty_{*,*}$ of a given total degree contribute to H_* on the same total degree.

Remark 3.5. Already in the previous examples 2.1 and 3.1, plenty of the slang used by "spectral sequencers" becomes useful. For instance, regarding Example 2.1, one says that "B dies killing c" and that "c dies killed by B", being the reason that some differential (d_0) applied to B is exactly c. Also, as $d_n = 0$ for all $n \geq 1$, one says that the spectral sequence "collapses" at E^1. The terminology refers to the fact that $E^n = E^1$ for all $n \geq 2$. In Example 3.1, the $\mathbb{Z}$ in $E^2_{2,0}$ dies killing the $\mathbb{Z}$ in $E^2_{0,1}$ and the spectral sequence collapses at E^3.

Example 3.6. The groups C_2 and C_4 fit into the short exact sequence

$$C_2 \to C_4 \to C_2.$$

Associated to a short exact sequence of groups there is the Lyndon-Hochschild-Serre spectral sequence 10.3, and we are going to study some extension problems arising in this situation. The second page of the spectral sequence consists of the homology of C_2 with coefficients in the homology of C_2, and it *converges* to the homology of C_4:

$$E^2_{p,q} = H_p(C_2; H_q(C_2; \mathbb{Z})) \Rightarrow H_{p+q}(C_4; \mathbb{Z}).$$

Because $H_*(C_2; \mathbb{Z}) = H_*(\mathbb{R}P^\infty; \mathbb{Z})$, where $\mathbb{R}P^\infty$ is the infinite-dimensional projective space, it is easy to see that E^2 looks like this:

$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	0	0	0	0	0
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$

To deduce the differentials we would need some extra information. For instance, a multiplicative structure would suffice as in Example 5.2. Although it seems a little bit artificial, we are going to use the fact that $H_n(C_4; \mathbb{Z}) = 0$ for $n > 0$ even to deduce the differentials. Then we will study the extension problems for $H_n(C_4; \mathbb{Z})$ with n odd. The full homology of C_4 can be computed with (periodic) resolutions, see for instance [5, II.3].

So, assume that we know that $H_n(C_4; \mathbb{Z}) = 0$ for $n > 0$ even. Then the $\mathbb{Z}_2$'s in $E^2_{1,1}$, $E^2_{3,1}$, $E^2_{5,1}$, etc, must die and their only chance is being killed by d_2 from the $\mathbb{Z}_2$'s in $E^2_{3,0}$, $E^2_{5,0}$, $E^2_{7,0}$, etc:

$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	0	0	0	0	0
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$

There are no other possible non-trivial differentials d_2 and hence E^3 is as follows:

$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	0	0	0	0	0	0
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	0	0	0	0	0	0
$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	0	0	0

Again there are $\mathbb{Z}_2$'s in diagonals contributing to $H_n(C_4; \mathbb{Z})$ with $n > 0$ even. A careful analysis shows that all must die killed by d_3:

$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	0	0	0	0	0	0
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	0	0	0	0	0	0
$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	0	0	0

Hence, E^4 looks as follows.

$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0
0	0	0	0	0	0	0
$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0
0	0	0	0	0	0	0
$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0
$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	0	0	0

It is clear from the positions of the non-zero entries in E^4 that the rest of the differentials $d_4, d_5, d_6, \ldots$ must be zero. So the spectral sequence collapses at $E^4 = E^\infty$. For n odd, we have an extension of $\mathbb{Z}$-modules:

$$\mathbb{Z}_2 \to H_n(C_4; \mathbb{Z}) \to \mathbb{Z}_2.$$

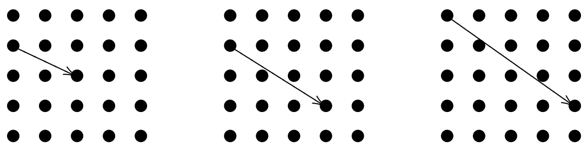
There are two solutions, either $H_n(C_4; \mathbb{Z}) = \mathbb{Z}_4$ or $H_n(C_4; \mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2$, and one would need extra information to decide which one is the right one. For instance, bringing in some topology, we quickly deduce that $H_1(C_4; \mathbb{Z}) = H_1(K(C_4, 1); \mathbb{Z}) = \pi_1(K(C_4, 1))_{ab} = C_4$, where $K(C_4, 1)$ is an Eilenberg-MacLane space (see also Example 5.7). In fact, $H_n(C_4; \mathbb{Z}) = C_4$ for all n odd, and to extend this fact from $n = 1$ to $n \neq 1$, one could use results about periodic (co)homology as [5, VI.9]. In any case and as aforementioned, it is easier to compute $H_*(C_4; \mathbb{Z})$ via resolutions.

Exercise 3.7. From the pathspace fibration $\Omega S^n \to PS^n \to S^n$, where $PS^n \simeq *$, and the Serre spectral sequence 10.1, deduce the integral homology of the loops on the sphere ΩS^n. Solution in 12.1.

4 Cohomological type spectral sequences

There are homological type and cohomological type spectral sequences. In Section 3, we described homological type spectral sequences. In this section, we work with cohomological type spectral sequences.

A cohomological type spectral sequence consists of a sequence of bigraded differential modules $\{E_r, d_r\}_{r \geq 2}$ such that E_{r+1} is the cohomology of (E_r, d_r) and such that d_r has bidegree $(r, 1-r)$, $d_r \colon E^r_{p,q} \to E^r_{p+r,q+1-r}$, as in these pictures:



(E_2, d_2) (E_3, d_3) (E_4, d_4)

In this situation, $H = H^*$ is the cohomology of some *cochain* complex, C^*, and convergence is written $E_2^{p,q} \Rightarrow H^{p+q}$. As for homological type, convergence means that the pages eventually stabilize at each particular position (p, q) and that you can recover the R-module H^n from the "diagonal" R-modules $\{E_\infty^{p,q}\}_{p+q=n}$ via a finite number of extensions. More precisely, there exist numbers $s \leq r$ and a finite *decreasing* filtration of H^n by R-modules,

$$0 = A^{r+1} \subseteq A^r \subseteq A^{r-1} \subseteq \ldots \subseteq A^{s+1} \subseteq A^s = H^n,$$

together with short exact sequences of R-modules:

$$\begin{aligned} &A^{s+1} \to H^n \to E_\infty^{s,n-s} \qquad (4.1)\\ &A^{s+2} \to A^{s+1} \to E_\infty^{s+1,n-s-1}\\ &\ldots\\ &A^r \to A^{r-1} \to E_\infty^{r-1,n-r+1}\\ &0 \to A^r \to E_\infty^{r,n-r}. \end{aligned}$$

Remark 4.1. Pay attention to the difference between (3.1) and (4.1): the indexing is upside down.

Example 4.2. Topological complex K-theory is a generalized cohomology theory. For any compact topological space X, $K^0(X)$ is the Grothendieck group of isomorphism classes of complex vector bundles over X, and $K^n(X) \cong K^{n+2}(X)$ for all $n \in \mathbb{Z}$. For a point, we have $K^0(*) \cong \mathbb{Z}$ and $K^1(*) = 0$. Let $\mathbb{C}P^n$ be the complex projective space of dimension n and consider the fibration

$$* \to \mathbb{C}P^n \xrightarrow{id} \mathbb{C}P^n.$$

For a fibration we have the Atiyah-Hirzebruch spectral sequence 10.5. In this case, the second page is given by the cohomology of $\mathbb{C}P^n$ with coefficients in the K-theory of a point, and it converges to the K-theory of $\mathbb{C}P^n$:

$$E_2^{p,q} \cong H^p(\mathbb{C}P^n; K^q(*)) \Rightarrow K^{p+q}(\mathbb{C}P^n).$$

The integral cohomology groups of $\mathbb{C}P^n$ are given by $H^p(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$ for $0 \leq p \leq n$, p even, and 0 otherwise. So $E_2^{p,q} \cong \mathbb{Z}$ if p and q are even and $0 \leq p \leq 2n$, and it is 0 otherwise. This is the picture for $\mathbb{C}P^3$:

$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0	0	0	0	0	0	0
$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0	0	0	0	0	0	0
$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0	0	0	0	0	0	0

Because for every r, either r or $r-1$ is odd, it follows that $d_r = 0$ for all $r \geq 2$. Hence the spectral sequence collapses at $E_2 = E_\infty$. Because $\mathbb{Z}$ is a free abelian group, all extension problems have a unique solution up to isomorphism, and we deduce that:

$$K^k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}^{n+1}, \text{ if } k \text{ is even,} \\ 0, \text{ otherwise.} \end{cases}$$

5 Additional structure: algebra

In some cases, the cohomology groups we want to compute, H, have some additional structure. For instance, there may be a product

$$H \otimes H \to H$$

that makes H into an algebra. A *spectral sequences of algebras* is a spectral sequence $\{E_r, d_r\}_{r \geq 2}$ such that E_r is a differential bigraded algebra, i.e., there is a product,

$$E_r \otimes E_r \to E_r,$$

and the product in E_{r+1} is that induced by the product in E_r after taking cohomology with respect to d_r. A spectral sequence of algebras *converges as an algebra* to the algebra H if it converges as a spectral sequence and there is a decreasing filtration of H^*,

$$\ldots \subseteq F^{n+1} \subseteq F^n \subseteq F^{n-1} \subseteq \ldots \subseteq H^*,$$

which is compatible with the product in H,

$$F^n \cdot F^m \subseteq F^{n+m},$$

and which satisfies the conditions below. When one restricts this filtration to each dimension, one gets the filtration explained in (4.1). More precisely, if we define $F^p H^{p+q} = F^p \cap H^{p+q}$, we have short exact sequences

$$F^{p+1} H^{p+q} \to F^p H^{p+q} \to E_\infty^{p,q}.$$

Then there are two products in the page E_∞:

(c) The one induced by E_∞ being the limit of the algebras $E_2, E_3, \ldots$,

(d) The one induced by the filtration as follows: For elements $\overline{a}$ and $\overline{b}$ belonging to

$$E_\infty^{p,q} = F^p H^{p+q} / F^{p+1} H^{p+q}$$

and $E_\infty^{p',q'} = F^{p'} H^{p'+q'} / F^{p'+1} H^{p'+q'}$ respectively, we set:

$$\overline{a} \cdot \overline{b} = \overline{a \cdot b}$$

in $E_\infty^{p+p',q+q'} = F^{p+p'} H^{p+p'+q+q'} / F^{p+p'+1} H^{p+p'+q+q'}$.

We say that the spectral sequence *converges as an algebra* if these two products are equal.

Remark 5.1. Note that, by definition, the graded and bigraded products satisfy $H^n \cdot H^m \subseteq H^{n+m}$ and $E_r^{p,q} \cdot E_r^{p',q'} \subseteq E_r^{p+p',q+q'}$ for $r \in \mathbb{N} \cup \{\infty\}$. All algebras we consider are graded (or bigraded) commutative, i.e.,

$$ab = (-1)^{|a||b|} ba, \tag{5.1}$$

where $|a|$ and $|b|$ are the degrees (total degrees) of the homogeneous elements a and b respectively. Moreover, every differential d is a derivation, i.e.,

$$d(ab) = d(a)b + (-1)^{|a|} a d(b).$$

Finally, if $C^{*,*}$ is a bigraded module or a double cochain complex, by $\mathrm{Total}(C)$ we denote the graded algebra or the cochain complex such that

$$\mathrm{Total}(C)^n = \bigoplus_{p+q=n} C^{p,q}.$$

Example 5.2. Consider the short exact sequence of cyclic groups:

$$C_2 \to C_4 \to C_2.$$

The cohomology ring of C_2 with coefficients in the field of two elements is given by $H^*(C_2; \mathbb{F}_2) = \mathbb{F}_2[x]$ with $|x| = 1$. For the extension above there is the Lyndon-Hochschild-Serre spectral sequence of algebras converging as an algebra 10.4,

$$E_2^{p,q} = H^p(C_2, H^q(C_2; \mathbb{F}_2)) \Rightarrow H^{p+q}(C_4; \mathbb{F}_2).$$

As the extension is central, we have that

$$\begin{aligned} H^p(C_2, H^q(C_2; \mathbb{F}_2)) &\cong H^p(C_2; \mathbb{F}_2) \otimes H^q(C_2; \mathbb{F}_2) \\ &\cong \mathbb{F}_2[x] \otimes \mathbb{F}_2[y] \end{aligned}$$

(see the comments after the statement of Theorem 10.4). Hence, the corner of the page $E_2 \cong \mathbb{F}_2[x, y]$ has the following generators:

y^3	y^3x	y^3x^2	y^3x^3
y^2	y^2x	y^2x^2	y^2x^3
y	yx	yx^2	yx^3
1	x	x^2	x^3

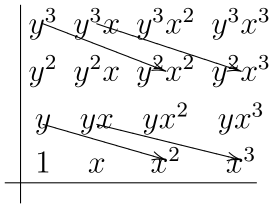
Because the extension is non-split, $d_2(y) = x^2$. Another way of deducing this is to use the fact

$$H^1(C_4; \mathbb{F}_2) \cong \mathrm{Hom}(C_4, \mathbb{F}_2) \cong \mathbb{F}_2.$$

Then, as the terms $E_\infty^{1,0} = \langle x \rangle$ and $E_\infty^{0,1}$ contribute to $H^1(C_4; \mathbb{F}_2)$, the y must die, and $d_2(y) = x^2$ is its only chance. From here, we can deduce the rest of the differentials as d_2 is a derivation, for instance:

$$d_2(yx) = d_2(y)x + y d_2(x) = x^3, \text{ as } d_2(x) = 0,$$
$$d_2(y^2) = d_2(y)y + y d_2(y) = 2 d_2(y) y = 0 \text{ , over } \mathbb{F}_2.$$

Analogously, one deduces that $d_2(y^{2i+1}x^j) = y^{2i}x^{j+2}$ and that $d_2(y^{2i}x^j) = 0$. Hence d_2 kills elements as follows,



and E_3 is

0	0	0	0
y^2	y^2x	0	0
0	0	0	0
1	x	0	0

By degree reasons, there cannot be any other nontrivial differentials and hence the spectral sequence collapses at $E_3 = E_\infty$. Moreover, the bigraded algebra E_∞ is given by

$$E_\infty = \mathbb{F}_2[x, x']/(x^2) = \Lambda(x) \otimes \mathbb{F}_2[x'],$$

where $x' = y^2$, $|x| = (1, 0)$ and $|x'| = (0, 2)$. It turns out that $H^*(C_4; \mathbb{F}_2) = \Lambda(z) \otimes \mathbb{F}_2[z']$ with $|z| = 1$, $|z'| = 2$. We cannot deduce this directly from Theorem 5.5 below as the relation $x^2 = 0$ in E_∞ is not one of the "free" relations 5.1 for the field $\mathbb{F}_2$. But it is easy to deduce it from Theorem 6.2 below in a similar way to that of Example 6.4.

Exercise 5.3. Compare last example and Example 3.6.

Exercise 5.4. Do the analogous computation for the extension $C_3 \to C_9 \to C_3$ given that $H^*(C_3; \mathbb{F}_3) = \Lambda(y) \otimes \mathbb{F}_3[x]$ with $|y| = 1$, $|x| = 2$. Solution in 12.2.

In general we cannot recover the graded algebra structure in H^* from the bigraded algebra structure of E_∞. This problem is known as the *lifting problem.* The next theorem gives some special conditions under which we can reconstruct H^* from $E_\infty^{*,*}$. They apply to Exercise 5.4.

Theorem 5.5 ([17, Example 1.K, p. 25])**.** *If E_∞ is a free, graded-commutative, bigraded algebra, then H^* is a free, graded commutative algebra isomorphic to* $\mathrm{Total}(E_\infty)$.

A free graded (or bigraded) commutative algebra is the quotient of the free algebra on some graded (bigraded) symbols modulo the relations (5.1). Theorem 5.5 translates as that, if $x_1, \ldots, x_r$ are the free generators of E_∞ with bidegrees $(p_1, q_1), \ldots, (p_r, q_r)$, then H^* is a free graded commutative algebra on generators $z_1, \ldots, z_r$ of degrees $p_1 + q_1, \ldots, p_r + q_r$. The next example is the integral version of the earlier Example 5.2 and here the lifting problem becomes apparent.

Example 5.6. . Consider the short exact sequence of cyclic groups:

$$C_2 \to C_4 \to C_2.$$

The integral cohomology ring of the cyclic group C_2 is given by $H^*(C_2;\mathbb{Z}) = \mathbb{Z}[x]/(2x)$, with $|x| = 2$. The Lyndon-Hochschild-Serre spectral 10.4 is

$$E_2^{p,q} = H^p(C_2, H^q(C_2;\mathbb{Z})) \Rightarrow H^{p+q}(C_4;\mathbb{Z}).$$

As the extension is central, the corner of the page $E_2 = \mathbb{F}_2[x,y]$ has the following generators:

y^2	0	y^2x	0	y^2x^2	0	y^2x^3
0	0	0	0	0	0	0
y	0	yx	0	yx^2	0	yx^3
0	0	0	0	0	0	0
1	0	x	0	x^2	0	x^3

Recall that $|x| = (2,0)$ and $|y| = (0,2)$. As in Example 4.2, parity implies that the spectral sequence collapses at $E_2 = E_\infty$ and hence $E_\infty = \mathbb{Z}[x,y]/(2x,2y)$. Nevertheless, $H^*(C_4;\mathbb{Z}) = \mathbb{Z}[z]/(4z)$ with $|z| = 2$.

In the next section, we deepen into the lifting problem to pin down the relation between the bigraded algebra E_∞ and the graded algebra H^*. We finish this section with a topological example.

Example 5.7. We are set to determine the ring $H^*(K(\mathbb{Z},n);\mathbb{Q})$, where $K(\mathbb{Z},n)$ is by definition the *Eilenberg-MacLane space* whose homotopy groups satisfy

$$\pi_i(K(\mathbb{Z},n)) = \begin{cases} \mathbb{Z}, \text{ if } i = n, \\ 0, \text{ otherwise.} \end{cases}$$

So, we know that $K(\mathbb{Z},1) = S^1$ and hence

$$H^*(K(\mathbb{Z},1);\mathbb{Q}) = \Lambda(z) \text{ with } |z| = 1.$$

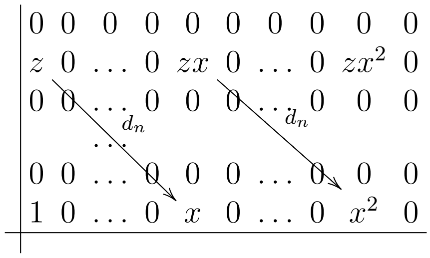
We prove by induction that

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \mathbb{Q}[z], \text{ if } n \text{ is even,} \\ \Lambda(z), \text{ if } n \text{ is odd,} \end{cases}$$

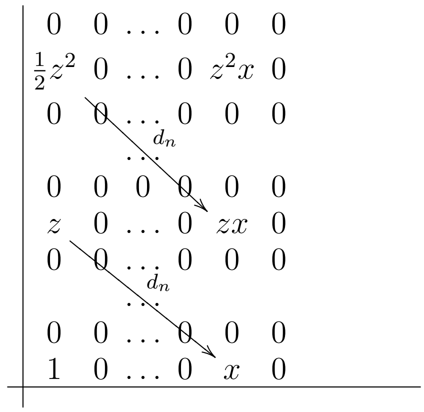
where $|z| = n$. We consider the pathspace fibration

$$\Omega K(\mathbb{Z},n) \to PK(\mathbb{Z},n) \to K(\mathbb{Z},n),$$

where $\Omega K(\mathbb{Z},n) \simeq K(\mathbb{Z},n-1)$ and $PK(\mathbb{Z},n) \simeq *$, and its Serre spectral sequence 10.2. Notice that $K(\mathbb{Z},n)$ is simply connected as $n \geq 2$. Then $E_2^{p,q} = H^p(K(\mathbb{Z},n);\mathbb{Q}) \otimes H^q(K(\mathbb{Z},n-1);\mathbb{Q})$. As $E_2^{*,*} \Rightarrow H^*(*;\mathbb{Q}) = \mathbb{Q}$, all terms but the $\mathbb{Z}$ in $(0,0)$ must disappear. If n is even then $H^*(K(\mathbb{Z},n-1);\mathbb{Q}) = \Lambda(z)$ with $|z| = n-1$. Moreover, the only chance of dying for z at position $(0,n-1)$ is by killing $d_n(z) = x$ at $(n,0)$. Then zx at $(n,n-1)$ must die killing $d_n(zx) = d_n(z)x + (-1)^{n-1}zd_n(x) = x^2$.



Pushing this argument further, one may conclude that $H^*(K(\mathbb{Z},n);\mathbb{Q}) = \mathbb{Q}[x]$ with $|x| = n$. Now assume that n is odd. Then $H^*(K(\mathbb{Z},n-1);\mathbb{Q}) = \mathbb{Q}[z]$ with $|z| = n-1$ and again z at $(0,n-1)$ must die killing $d_n(z) = x$ at $(n,0)$. Next, we deduce that $d_n(z^2) = d_n(z)z + (-1)^{n-1}zd_n(z) = 2zx$ and hence zx is killed by $\frac{1}{2}z^2$:



Following these arguments, one may conclude that $H^*(K(\mathbb{Z},n);\mathbb{Q}) = \Lambda(x)$ with $|x| = n$.

The next three exercises are meant to highlight the role of the base ring R. The computations are similar to those of Example 5.7 but with integral coefficients instead of rational coefficients.

Exercise 5.8. Determine the cohomology ring $H^*(K(\mathbb{Z},2);\mathbb{Z})$ using the pathspace fibration

$$\Omega K(\mathbb{Z},2) \to PK(\mathbb{Z},2) \to K(\mathbb{Z},2),$$

where we have $\Omega K(\mathbb{Z},2) \simeq K(\mathbb{Z},1)$ and $PK(\mathbb{Z},2) \simeq *$. Solution in 12.3.

Exercise 5.9. Determine the cohomology ring $H^*(K(\mathbb{Z},3);\mathbb{Z})$ using the pathspace fibration

$$\Omega K(\mathbb{Z},3) \to PK(\mathbb{Z},3) \to K(\mathbb{Z},3)$$

and the previous exercise, where we have $\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2)$ and $PK(\mathbb{Z},3) \simeq *$. Solution in 12.4.

Exercise 5.10. Determine the cohomology ring $H^*(\Omega S^n;\mathbb{Z})$ using the pathspace fibration $\Omega S^n \to PS^n \to S^n$, where $PS^n \simeq *$. Solution in 12.5.

6 Lifting problem

Recall the general setting for the lifting problem: we have a graded algebra H^* with a filtration F^* and then we consider the associated bigraded algebra E_∞ with the induced product as in (d). So, for $F^pH^{p+q} = F^p \cap H^{p+q}$, we have $E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$ and the product $E_\infty^{p,q} \otimes E_\infty^{p',q'} \to E_\infty^{p+p',q+q'}$ sends

$$\begin{aligned}&(a + F^{p+1}H^{p+q}) \cdot (b + F^{p'+1}H^{p'+q'})\\ &= (a \cdot b) + F^{p+p'+1}H^{p+q+p'+q'}.\end{aligned}$$

Example 6.1 ([17, Example 1.J, p. 23])**.** Consider the following two algebras:

$$H_1^* = \mathbb{Q}[x,y,z]/(x^2,y^2,z^2,xy,xz,yz)$$

with $|x| = 7$, $|y| = 8$, $|z| = 15$, and

$$H_2^* = \mathbb{Q}[u,v,w]/(u^2,v^2,w^2,uv-w,uw,vw)$$

with $|u| = 7$, $|v| = 8$, $|w| = 15$. The following decreasing filtrations F_1 and F_2 of H_1 and H_2 (respectively) are compatible with the respective products:

$$\begin{aligned}0 = F_1^8 \subset F_1^7 = F_1^6 = F_1^5 = \langle z\rangle \subset F_1^4 = F_1^3 = \langle y,z\rangle\\ \subset F_1^2 = F_1^1 = H_1,\\ 0 = F_2^8 \subset F_2^7 = F_2^6 = F_2^5 = \langle w\rangle \subset F_2^4 = F_2^3 = \langle v,w\rangle\\ \subset F_2^2 \subset F_2^1 = H_2.\end{aligned}$$

The next pictures depict the corresponding bigraded algebras $E_{\infty,1}$ and $E_{\infty,2}$:

	1	2	3	4	5	6	7	8
9	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	$\bar{z}$	0
7	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
5	0	$\bar{x}$	0	0	0	0	0	0
4	0	0	0	$\bar{y}$	0	0	0	0
3	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0

	1	2	3	4	5	6	7	8
9	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	$\bar{w}$	0
7	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
5	0	$\bar{u}$	0	0	0	0	0	0
4	0	0	0	$\bar{v}$	0	0	0	0
3	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0

We have $E_{\infty,1} \cong E_{\infty,2}$ as bigraded algebras because all products of generators are zero in both cases. In the former case, this is a consequence of the products being zero in the graded algebra H_1^*. In the latter case, the product $\bar{u}\bar{v}$ is zero because $|\bar{u}| = (2,5)$, $|\bar{v}| = (4,4)$, $|\bar{u}\bar{v}| = (6,9)$ and $E_{\infty,2}^{6,9} = 0$. What is happening is that $\bar{u}\bar{v} = 0 \in E_{\infty,2}^{6,9} = F_2^6H_2^{15}/F_2^7H_2^{15}$ implies that $uv \in F_2^7H_2^{15} = \langle w\rangle$. So we may deduce that $uv = \lambda w$ for some $\lambda \in \mathbb{Q}$, but we cannot know whether $\lambda = 0$ or not.

The following result clarifies the general situation.

Theorem 6.2 ([6, Theorem 2.1])**.** *If E_∞ is a finitely generated bigraded commutative algebra, then H^* is a finitely generated graded commutative algebra. Moreover, given a presentation of E_∞ by double homogeneous generators and double homogeneous relations:*

1. *Generators for H^* are given by lifting the given generators of E_∞, and they have the same total degree.*
2. *Relations for H^* are given by lifting the given relations of E_∞, and they have the same total degree.*

In Example 6.1, we have

$$E_{\infty,1}^{*,*} = \mathbb{Q}[\bar{x},\bar{y},\bar{z}]/(\bar{x}^2,\bar{y}^2,\bar{z}^2,\bar{x}\bar{y},\bar{x}\bar{z},\bar{y}\bar{z})$$

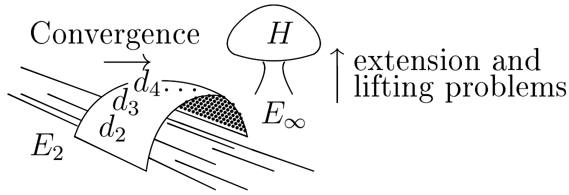
and these generators and relations lift to give $H_1^* = \mathbb{Q}[x,y,z]/(x^2,y^2,z^2,xy,xz,yz)$. For

$$E_{\infty,2}^{*,*} = \mathbb{Q}[\bar{u},\bar{v},\bar{w}]/(\bar{u}^2,\bar{v}^2,\bar{w}^2,\bar{u}\bar{v},\bar{u}\bar{w},\bar{v}\bar{w}),$$

the relation $\bar{u}\bar{v} = 0$ lifts as $uv = w$. This theorem has the following immediate consequence.

Corollary 6.3 ([6, Theorem 2.1])**.** *If R is a finite field, the graded algebra structure of H^* is determined, up to a finite number of possibilities, by the bigraded algebra structure of E_∞.*

At this point, the original picture of the bridge and the tree should be clearer. That there is a bridge means that the spectral sequence converges, and you can cross it if you are able to figure out enough differentials. Climbing the tree implies solving extension and lifting problems. This may require extra information.



We exemplify Theorem 6.2 by means of the next example.

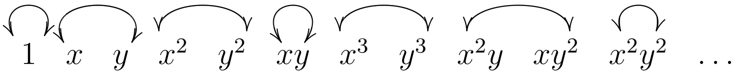
Example 6.4. Consider the dihedral group D_8 described as a wreath product:

$$D_8 = C_2 \wr C_2 = C_2 \times C_2 \rtimes C_2 = \langle a,b\rangle \rtimes \langle c\rangle,$$

where c interchanges a and b: ${}^ca = b$, ${}^cb = a$. The Lyndon-Hochschild-Serre spectral sequence 10.4 with coefficients in the field of two elements is

$$E_2^{p,q} = H^p(C_2; H^q(C_2 \times C_2; \mathbb{F}_2)) \Rightarrow H^{p+q}(D_8; \mathbb{F}_2).$$

The extension is not central and it is not immediate to describe E_2. Recall that $H^*(C_2 \times C_2; \mathbb{F}_2) = H^*(C_2;\mathbb{F}_2) \otimes H^*(C_2;\mathbb{F}_2) = \mathbb{F}_2[x,y]$, with $|x| = |y| = 1$. Here, $x = a^*$ and $y = b^*$ are the corresponding duals. This ring can be decomposed as a C_2-module into a direct sum of two type of C_2-modules: either the trivial C_2-module or the free transitive C_2-module:



Now, the cohomology of the trivial module has been already mentioned $H^*(C_2;\mathbb{F}_2)=\mathbb{F}_2[z]$ with $|z|=1$. The cohomology of the free transitive module M is given by the fixed points in degree 0, $H^0(C_2;M)=M^{C_2}$, and $H^n(C_2;M)=0$ for $n>0$ (as $0\to M\stackrel{id}{\to} M\to 0$ is a free $\mathbb{F}_2C_2$-resolution of M). To sum up, $\mathbb{F}_2$-generators for the corner of E_2 are as follows:

$x^3y+xy^3, x^4+y^4, x^2y^2$	x^2y^2z	$x^2y^2z^2$	$x^2y^2z^3$	$x^2y^2z^4$	$x^2y^2z^5$	$x^2y^2z^6$
x^3+y^3, x^2y+xy^2	0	0	0	0	0	0
x^2+y^2, xy	xyz	xyz^2	xyz^3	xyz^4	xyz^5	xyz^6
$x+y$	0	0	0	0	0	0
1	z	z^2	z^3	z^4	z^5	z^6

A similar description of E_2 and the general fact that the spectral sequence of a wreath product collapses in E_2 may be found in [3, IV, Theorem 1.7, p. 122]. So we have $E_\infty=E_2=\mathbb{F}_2[z,\sigma_1,\sigma_2]/(z\sigma_1)$ where $\sigma_1=x+y$ and $\sigma_2=xy$ are the elementary symmetric polynomials. So according to Theorem 6.2, we have $H^*(D_8;\mathbb{F}_2)=\mathbb{F}_2[w,\tau_1,\tau_2]/(R)$, where $|w|=1$, $|\tau_1|=1$, $|\tau_2|=2$ and R is the lift of the relation $z\sigma_1=0$.

Recall that $|z|=(1,0)$ and that $|\sigma_1|=(0,1)$. So $z\sigma_1=0\in E_\infty^{1,1}=F^1H^2/F^2H^2$, where F^* is some unknown filtration of $H^*=H^*(D_8;\mathbb{F}_2)$. So $w\tau_1\in F^2H^2$ and we need to know what is F^2H^2. For total degree 2, we have extension problems (4.1):

$$\begin{aligned} F^1H^2 \to& H^2 \to E_\infty^{0,2}=\langle x^2+y^2, xy\rangle=\mathbb{F}_2\oplus\mathbb{F}_2\\ F^2H^2\to& F^1H^2\to E_\infty^{1,1}=0\\ 0\to& F^2H^2\to E_\infty^{2,0}=\langle z^2\rangle=\mathbb{F}_2. \end{aligned}$$

So we deduce that $F^2H^2=F^1H^2=\langle z^2\rangle\subset H^2(D_8;\mathbb{F}_2)$. So we may take $w=z$ (see Remark 7.1) and then $w\tau_1=\lambda w^2$ for some $\lambda\in\mathbb{F}_2$. We cannot deduce the value of λ from the spectral sequence. A computation with the bar resolution shows that $\lambda=0$, and hence $H^2(D_8;\mathbb{F}_2)=\mathbb{F}_2[w,\tau_1,\tau_2]/(w\tau_1)$.

Exercise 6.5. Calculate the Poincaré series

$$P(t)=\sum_{i=0}^{\infty}\dim H^i(D_8;\mathbb{F}_2)t^i,$$

from the bigraded module E_∞ of Example 6.4. Solution is in 12.6.

Exercise 6.6. Fully describe the Lyndon-Hochschild-Serre spectral sequence with coefficients $\mathbb{F}_2$ 10.4 of the dihedral group D_8 described as a semi-direct product:

$$D_8=C_4\rtimes C_2=\langle a\rangle\rtimes\langle b\rangle,$$

with ${}^ba=a^{-1}$. Recall that by Example 6.4 $H^*(D_8;\mathbb{F}_2)=\mathbb{F}_2[w,\tau_1,\tau_2]/(w\tau_1)$. Solution is in 12.7.

7 Edge morphisms

Consider a cohomological type spectral sequence $E_2\Rightarrow H$ which is first quadrant, i.e., such that $E_2^{p,q}=0$ whenever $p<0$ or $q<0$. It is clear that there are monomorphisms and epimorphisms for all $p,q\geq 0$:

$$\begin{aligned} E_\infty^{0,q}\hookrightarrow\ldots\hookrightarrow E_r^{0,q}\hookrightarrow E_{r-1}^{0,q}\hookrightarrow\ldots\hookrightarrow E_3^{0,q}\hookrightarrow E_2^{0,q},\\ E_2^{p,0}\twoheadrightarrow E_3^{p,0}\twoheadrightarrow\ldots\twoheadrightarrow E_{r-1}^{p,0}\twoheadrightarrow E_r^{p,0}\twoheadrightarrow\ldots\twoheadrightarrow E_\infty^{p,0}. \end{aligned}$$

So $E_\infty^{0,q}$ is an R-submodule of $E_2^{0,q}$ and $E_\infty^{p,0}$ is a quotient R-module of $E_2^{p,0}$. Now, if we rewrite the extensions (4.1) together with $E_\infty^{a,b}\cong F^aH^{a+b}/F^{a+1}H^{a+b}$ we get

$$\begin{aligned} &F^1H^{a+b}\to H^{a+b}\to E_\infty^{0,a+b}\\ &F^2H^{a+b}\to F^1H^{a+b}\to E_\infty^{1,a+b-1}\\ &\ldots\\ &F^{a+b}H^{a+b}\to F^{a+b-1}H^{a+b}\to E_\infty^{a+b-1,1}\\ &0\to F^{a+b}H^{a+b}\to E_\infty^{a+b,0}. \end{aligned}$$

We deduce that $E_\infty^{0,q}$ is a quotient module of $H^q=F^0H^q$ and that $F^pH^p=E_\infty^{p,0}$ is a submodule of H^p. Summing up, we have morphisms, called *edge morphisms*:

$$\begin{aligned} H^q\twoheadrightarrow E_\infty^{0,q}\hookrightarrow E_2^{0,q},\\ E_2^{p,0}\twoheadrightarrow E_\infty^{p,0}\hookrightarrow H^p. \end{aligned}$$

For several spectral sequences, the edge morphisms may be explicitly described.

For the Lyndon-Hochschild-Serre spectral sequence 10.4 of a short exact sequence of groups, $N\stackrel{\iota}{\to}G\stackrel{\pi}{\to}Q$, we have $E_2^{0,q}=H^0(Q;H^q(N;R))=H^q(N;R)^Q$ and the edge morphism,

$$H^q(G;R)\twoheadrightarrow E_\infty^{0,q}\hookrightarrow E_2^{0,q}=H^q(N;R)^Q,$$

coincides with the restriction in cohomology

$$H^q(\iota)\colon H^q(G;R)\to H^q(N;R)^Q.$$

We also have $E_2^{p,0}=H^p(Q;H^0(N;R))=H^p(Q;R^N)$ and the edge morphism,

$$H^p(Q;R^N)=E_2^{p,0}\twoheadrightarrow E_\infty^{p,0}\hookrightarrow H^p(G;R),$$

is exactly the inflation

$$H^p(\pi)\colon H^p(Q;R^N)\to H^p(G;R).$$

Remark 7.1. We have seen that $E_\infty^{p,0}\subseteq H^p(G;R)$ as an R-module for all $p\geq 0$ but in fact $E_\infty^{*,0}\subseteq H^*(G;R)$ as a subalgebra because $E_\infty^{p,0}\cdot E_\infty^{p',0}\subseteq E_\infty^{p+p',0}$.

Exercise 7.2. Prove that in Example 5.2 we can deduce the algebra structure of $H^*(C_4;\mathbb{F}_2)$ from the E_∞-page. Solution in 12.8.

Consider the Serre spectral sequence of a fibration $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ 10.2 with B simply connected and F connected. The edge morphism

$$H^q(E;R) \twoheadrightarrow E_\infty^{0,q} \hookrightarrow E_2^{0,q} = H^q(F;R),$$

is

$$H^q(\iota)\colon H^q(E;R) \to H^q(F;R).$$

The edge morphism

$$H^p(B;R) = E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(E;R),$$

coincides with

$$H^p(\pi)\colon H^p(B;R) \to H^p(E;R).$$

Remark 7.3. We have seen that $E_\infty^{p,0} \subseteq H^p(E;R)$ as an R-module for all $p \geq 0$ but in fact $E_\infty^{*,0} \subseteq H^*(E;R)$ as a subalgebra because $E_\infty^{0,q} \cdot E_\infty^{0,q'} \subseteq E_\infty^{0,q+q'}$.

Exercise 7.4. Figure out what are the edge morphisms for the Serre and the Lyndon-Hochschild-Serre homological spectral sequences.

The next two examples show two particular situations where the edge morphisms give much information.

Example 7.5 ([8, p. 246])**.** We consider oriented bordism Ω_*^{SO}, which is a generalized cohomology theory. For a space X, $\Omega_n^{SO}(X)$ is, roughly speaking, the abelian group of oriented-bordism classes of n-dimensional closed manifolds mapping to X. Then we have the Atiyah-Hirzebruch spectral sequence 10.5 for the fibration

$$* \xrightarrow{\iota} X \xrightarrow{id} X,$$

where X is any topological space. It states that

$$H_p(X;\Omega_q^{SO}(*)) \Rightarrow \Omega_{p+q}^{SO}(X).$$

Also we have that

$$\Omega_q^{SO}(*) = \begin{cases} 0, \; q<0 \text{ or } q=1,2,3, \\ \mathbb{Z}, \text{ for } q=0,4. \end{cases}$$

In this case, the edge morphism,

$$\Omega_q^{SO}(*) \twoheadrightarrow H_0(X;\Omega_q^{SO}(*)) = E^2_{0,q} \twoheadrightarrow E^\infty_{0,q} \hookrightarrow \Omega_q^{SO}(X),$$

is given by $\Omega_q^{SO}(\iota)$. Note that the constant map $c\colon X \to *$ satisfies $c \circ \iota = 1_*$. Hence, $\Omega_q^{SO}(*) \circ \Omega_q^{SO}(\iota) = 1_{\Omega_q^{SO}(*)}$ and the edge morphism $\Omega_q^{SO}(\iota)$ is injective. In particular, $\Omega_q^{SO}(*) = H_0(X;\Omega_q^{SO}(*))$ and $E^2_{0,q} = E^\infty_{0,q}$ for all q. This last condition implies that all differentials arriving or emanating from the vertical axis must be zero:

4	$\Omega_4^{SO}(*)$	$H_1(X;\mathbb{Z})$	$H_2(X;\mathbb{Z})$	$H_3(X;\mathbb{Z})$	$H_4(X;\mathbb{Z})$
3	0	0	0	0	0
2	0	0	0	0	0
1	0	0	0	0	0
0	$\Omega_0^{SO}(*)$	$H_1(X;\mathbb{Z})$	$H_2(X;\mathbb{Z})$	$H_3(X;\mathbb{Z})$	$H_4(X;\mathbb{Z})$
	0	1	2	3	4

In particular, we deduce the oriented bordism of X in low dimensions:

$$\Omega_q^{SO}(X) = \begin{cases} H_q(X;\mathbb{Z}), \text{ for } q=0,1,2,3, \\ \mathbb{Z} \oplus H_4(X;\mathbb{Z}), \text{ for } q=0,4. \end{cases}$$

Example 7.6 ([10, Proposition 7.3.2])**.** Consider an extension of groups $N \xrightarrow{\iota} G \xrightarrow{\pi} Q$ which is split, i.e., such that G is the semi-direct product $G = N \rtimes Q$. We consider the Lyndon-Hochschild-Serre spectral sequence 10.4 with coefficients in a ring R with trivial G-action:

$$E_2^{p,q} = H^p(Q;H^q(N;R)) \Rightarrow H^{p+q}(G;R).$$

We know that the edge morphism

$$H^p(Q;R) = E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(G;R),$$

is exactly the inflation

$$H^p(\pi)\colon H^p(Q;R) \to H^p(G;R).$$

Because the extension is split, there is a homomorphism $s\colon Q \to G$ with $\pi \circ s = 1_Q$. Hence, $H^p(s) \circ H^p(\pi) = 1_{H^p(Q)}$, the inflation $H^p(\pi)$ is injective and we deduce that $E_2^{p,0} = E_\infty^{p,0}$. So all differentials arriving to the horizontal axis must be zero and we have an inclusion of algebras $H^*(Q;R) \subseteq H^*(G;R)$, see Remark 7.1. This situation already occurred in Example 6.4 and appears again in Exercise 6.6.

8 Different spectral sequences with same target

In Examples 6.4 and Exercise 6.6 we have seen two different spectral sequences converging to the same target $H^*(D_8;\mathbb{F}_2)$. In this section, we present another two examples of this phenomenon, the first with target $H^*(3_+^{1+2};\mathbb{F}_3)$ and the second with target $H_*(S^3;\mathbb{Z})$. From these examples, one sees that the price one pays for having an easily described E_2-page is more complicated differentials.

Example 8.1. We denote by $S = 3_+^{1+2}$ the extraspecial group of order 27 and exponent 3. It has the following presentation

$$S = \langle A,B,C | A^3 = B^3 = C^3 = [A,C] = [B,C] = 1, \; [A,B] = C \rangle$$

and it fits in the central extension

$$\langle C \rangle = C_3 \to 3_+^{1+2} \to C_3 \times C_3 = \langle \bar{A}, \bar{B} \rangle.$$

Leary describes in [15] the Lyndon-Hochschild-Serre spectral sequence of this extension. Its second page is given by

$$\begin{aligned} E_2^{*,*} &= H^*(C_3;\mathbb{F}_3) \otimes H^*(C_3 \times C_3;\mathbb{F}_3) \\ &= \Lambda(u) \otimes \mathbb{F}_3[t] \otimes \Lambda(y_1,y_2) \otimes \mathbb{F}_3[x_1,x_2], \end{aligned}$$

with $|u| = |y_1| = |y_2| = 1$ and $|t| = |x_1| = |x_2| = 2$. The differentials in E_* are the following, and the spectral sequence collapses at E_6.

(i) $d_2(u) = y_1y_2$, $d_2(t) = 0$,

(ii) $d_3(t) = x_1y_2 - x_2y_1$,

(iii) $d_4(t^iu(x_1y_2 - x_2y_1)) = it^{i-1}(x_1x_2^2y_2 - x_1^2x_2y_1)$, $d_4(t^2y_i) = u(x_1y_2 - x_2y_1)x_i$,

(iv) $d_5(t^2(x_1y_2 - x_2y_1)) = x_1^3x_2 - x_1x_2^3$, $d_5(ut^2y_1y_2) = ku(x_1^3y_2 - x_2^3y_1)$, $k \neq 0$.

A long and intricate computation leads from E_2 to E_6 [9]. The following table contains representatives of classes that form an $\mathbb{F}_3$-basis of $E_6^{n,m}$ for $0 \leq n \leq 6$ and $0 \leq m \leq 5$:

5							
4							
3			uty_1y_2				
2		ty_1, ty_2		ty_1x_1, ty_1x_2 ty_2x_2		$tx_1^2y_1, tx_1^2y_2$ $tx_2^2y_1, tx_2^2y_2$	
1		uy_1, uy_2	uy_1y_2	uy_1x_1, uy_1x_2 uy_2x_1, uy_2x_2		$ux_1^2y_1, ux_1^2y_2$ $ux_2^2y_1, ux_2^2y_2$	
0	1	y_1, y_2	x_1, x_2	y_1x_1, y_1x_2 y_2x_2	x_1^2, x_2^2 x_1x_2	$x_1^2y_1, x_1^2y_2$ $x_2^2y_1, x_2^2y_2$	x_1^3, x_2^3 $x_1^2x_2, x_1x_2^2$
	0	1	2	3	4	5	6

It turns out that this description of the corner of E_6 determines the rest of E_6 as there are both vertical and horizontal periodicities. More precisely, there are $\mathbb{F}_3$-isomorphisms $E_6^{n,m} \cong E_6^{n,m+6}$ for $n, m \geq 0$ and $E_6^{n,m} \cong E_6^{n+2,m}$ for $n \geq 5$ and $m \geq 0$. The extraspecial group $S = 3_+^{1+2}$ also fits in an extension

$$\langle B, C \rangle = C_3 \times C_3 \to 3_+^{1+2} \to C_3 = \langle \bar{A} \rangle,$$

where the action is given by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. In this case, the page $E_2^{*,*} = H^*(C_3; H^*(C_3 \times C_3; \mathbb{F}_3))$ is quite complicated as the action of C_3 is far from being trivial. It is described in [20], where the author also shows that $E_2 = E_\infty$. The page E_2 has 9 generators, $x_1, \gamma_1, x_2, y_2, \gamma_2, x_3, x_6, z_2, z_3$, and they lie in the following positions:

6	x_6			
5				
4				
3	x_3			
2	x_2, y_2	z_3		
1	x_1	z_2		
0		γ_1	γ_2	
	0	1	2	3

Compare to the E_2-page in Equation (8.1), which has 6 generators all of which lie on the axes.

Example 8.2. In Example 3.1, we saw that the Hopf fibration $S^1 \to S^3 \to S^2$ gives rise to a spectral sequence converging to $H_*(S^3)$. Consider now the fibration

$$C_2 \to S^3 \to \mathbb{RP}^3,$$

where C_2 acts by the antipodal map and $\mathbb{RP}^3$ is the projective space of dimension 3. Because the fiber is discrete, the associated Serre spectral sequence 10.1:

$$E^2_{p,q} = H_p(\mathbb{RP}^3; H_q(C_2; \mathbb{Z})) \Rightarrow H_{p+q}(S^3; \mathbb{Z})$$

collapses in the horizontal axis of E^2: We have $H_0(C_2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ with $C_2 = \pi_1(\mathbb{R}P^3)$ interchanging the two $\mathbb{Z}$'s, and $H_q(C_2; \mathbb{Z}) = 0$ for $q > 0$. Hence, $E^2_{p,0}$ is the twisted homology $H_p(\mathbb{RP}^3; \mathbb{Z} \oplus \mathbb{Z})$ and $E^2_{p,q} = 0$ for $q > 0$. We deduce that the twisted homology $H_p(\mathbb{RP}^3; \mathbb{Z} \oplus \mathbb{Z})$ must be $\mathbb{Z}$ for $p = 0, 3$ and 0 otherwise.

Let us directly compute the twisted homology $H_p(\mathbb{RP}^3; \mathbb{Z} \oplus \mathbb{Z})$ via the universal cover S^3 of $\mathbb{RP}^3$ [11, 3.H]. The sphere S^3 has a structure of free C_2-complex with cells

$$S^3 = e^3_+ \cup e^3_- \cup e^2_+ \cup e^2_- \cup e^1_+ \cup e^1_- \cup e^0_+ \cup e^0_-,$$

where C_2 interchanges each pair of cells on every dimension. This gives rise to the following chain complex over $\mathbb{Z}C_2$:

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

with $d(1,0) = (1,-1)$, $d(0,1) = (-1,1)$ in dimensions 1 and 3, $d(1,0) = d(0,1) = (1,1)$ in dimension 2 and $d(1,0) = d(0,1) = 1$ in dimension 0. Set M to be equal to the $\mathbb{Z}C_2$-module $H_0(C_2; \mathbb{Z})$. Then we tensor every term of the chain complex above by $\otimes_{C_2} M$ and we compute the resulting homology.

Setting $a = (1,0) \otimes (1,0) = (0,1) \otimes (0,1)$ and $b = (1,0) \otimes (0,1) = (0,1) \otimes (1,0)$ on every dimension, it turns out that $d(a) = a - b$, $d(b) = b - a$ in dimensions 1 and 3, $d(a) = d(b) = a + b$ in dimension 2 and $d(a) = d(b) = 0$ in dimension 0. Thus we obtain the desired homology $H_*(S^3; \mathbb{Z})$.

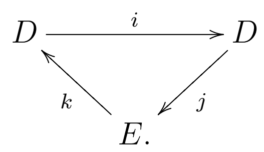
Exercise 8.3. Do similar computations to those of Example 8.2 for the fibration

$$I \to S^3 \to X,$$

where I is the binary icosahedral group of order 120 and X is the Poincaré homology sphere. In this case, you should recover the homology groups $H_*(S^3; \mathbb{Z})$ as the twisted homology of X with coefficients in a module for the group $\pi_1(X) = I$.

9 A glimpse into the black box

In this section, we show how a filtration gives rise to a spectral sequence. An equivalent approach to construct spectral sequences is exact couples [17, p. 37]. An *exact couple* consist of three module homomorphisms between graded or bigraded modules such that the following diagram is exact at each module:



The notion of *derived couple* makes it very natural to derive a spectral sequence from a given exact couple. Moreover, sometimes exact couples are the natural setting to construct certain spectral sequences, e.g., the Bockstein spectral sequence. Nevertheless, here we focus on the approach via filtrations.

So let (C, d) be a chain complex and let F_*C be an increasing filtration of (C, d), i.e., an ordered family of chain subcomplexes:

$$\ldots \subseteq F_{n-1}C \subseteq F_nC \subseteq \ldots \subseteq C.$$

We may define a filtration of the homology H of C via the inclusions: $F_nH = \operatorname{Im}(H(F_nC, d) \longrightarrow H(C, d))$. Thus we have an ordered family of graded modules:

$$\ldots \subseteq F_{n-1}H \subseteq F_nH \subseteq \ldots \subseteq H.$$

On each total degree $p+q$, we obtain a filtration of the modules C_{p+q} and H_{p+q}:

$$\begin{array}{l} \ldots \subseteq F_{p-1}C_{p+q} \subseteq F_pC_{p+q} \subseteq \ldots \subseteq C_{p+q} \\ \ldots \subseteq F_{p-1}H_{p+q} \subseteq F_pH_{p+q} \subseteq \ldots \subseteq H_{p+q}. \end{array}$$

We will assume that the filtration is bounded, i.e., that for each total degree n, there exist s and r such that $F_sC_n = 0$ and $F_tC_n = C_n$. Then we have:

$$\begin{array}{rcl} 0 = F_sC_{p+q} \subseteq \ldots & \subseteq & F_{p-1}C_{p+q} \subseteq F_pC_{p+q} \\ & \subseteq & \ldots \subseteq F_rC_{p+q} = C_{p+q} \quad (9.1) \\ 0 = F_sH_{p+q} \subseteq \ldots & \subseteq & F_{p-1}H_{p+q} \subseteq F_pH_{p+q} \\ & \subseteq & \ldots \subseteq F_rH_{p+q} = H_{p+q} \quad (9.2) \end{array}$$

Because of the boundedness assumption, there exists a spectral sequence $\{E^r_{*,*}, d_r\}_{r\geq 0}$ that converges to H in the sense explained in (a) and (b). More precisely, $E^0_{*,*}$ is the *associated graded object*, i.e., it is obtained by taking quotients on (9.1):

$$E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q},$$

and the stable term $E^\infty_{p,q}$ at (p, q) is also obtained as an asociated graded object, i.e., it is isomorphic to considering quotients on (9.2):

$$E^\infty_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}.$$

This correspond to the Equations (3.1). So the problem is to define the rest of pages $E^r_{*,*}$ and their differentials and see how they fit together.

Define $Z^r_{p,q} = F_pC_{p+q} \cap d^{-1}(F_{p-r}C_{p+q-1})$ and $B^r_{p,q} = F_pC_{p+q} \cap d(F_{p+r}C_{p+q+1})$:

$$F_{p-r}C_{p+q-1} \xleftarrow{d} F_pC_{p+q} \xleftarrow{d} F_{p+r}C_{p+q+1}$$

Then we set

$$E^r_{p,q} = \frac{Z^r_{p,q}}{Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}},$$

and one has to check that:

(e) The differential d induces a differential

$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

(f) When taking homology we have

$$H_{p,q}(E^r_{*,*}, d^r) \cong E^{r+1}_{p,q}.$$

Exercise 9.1. Compute the modules $Z^r_{p,q}$'s, $B^r_{p,q}$'s and $E^r_{p,q}$'s for Example 2.1.

These two facts are consequence of intricate diagram chasings, and it is here where things become delicate. We do not reproduce the details (that may be found in [17, 2.2] or [21, 5.4] for instance), but instead we explain the crucial step.

The aforementioned chasing arguments reduce to the following situation: We are given a 3×3-diagram of modules and inclusions,

$$\begin{array}{ccccc} A & \hookrightarrow & B & \hookrightarrow & C \\ \downarrow d & & \downarrow d & & \downarrow d \\ D & \hookrightarrow & E & \hookrightarrow & F \\ \downarrow d & & \downarrow d & & \downarrow d \\ G & \hookrightarrow & H & \hookrightarrow & I, \end{array}$$

where $d \circ d = 0$, and we need to prove (g) and (h) below. We define E_1 as taking quotient and homology (in this order):

$$\begin{aligned} E^E_1 &= H(B/A \to E/D \to H/G), \\ E^G_1 &= H(D \to G \to 0), \\ E^C_1 &= H(0 \to C/B \to F/E), \end{aligned}$$

and E_∞ as taking homology and quotient (in this order):

$$E^E_\infty = \frac{\operatorname{Im}(H(B \to E \to H) \to H(C \to F \to I))}{\operatorname{Im}(H(A \to D \to G) \to H(C \to F \to I))}.$$

Then the following hold:

(g) The differential d induces a differential

$$E^C_1 \xrightarrow{d^1} E^E_1 \xrightarrow{d^1} E^G_1.$$

(h) When taking homology we have

$$H(E^C_1 \xrightarrow{d^1} E^E_1 \xrightarrow{d^1} E^G_1) \cong E^E_\infty.$$

Exercise 9.2. Prove that (g) and (h) hold.

The situation for a decreasing filtration of a cochain complex is similar. Considering a product in the cochain complex gives rise to a spectral sequence of algebras, see [17, 2.3].

10 Appendix I

There are predictably a large number of interesting spectral sequences in the literature. Classically, the most used and the most well-known are: (i) the Leray-Serre spectral sequence which manages the intertwining relationships between the (co)homology groups of base, fiber and total spaces of a fibration, (ii) the Lyndon-Hochschild spectral sequence which is used in computing the (co)homology of groups, (iii) the Atiyah-Hirzebruch spectral sequence, and (iv) the Adams spectral sequence for computing the stable homotopy groups of spheres. We discuss the three first spectral sequences in this appendix, and we briefly discuss the Adams spectral sequence in Subsection 11.4.

Theorem 10.1 (Serre spectral sequence for homology, [17, Theorem 5.1])**.** *Let M be an abelian group and let $F \to E \to B$ be a fibration with B path-connected. Then there is a first quadrant homological spectral sequence*

$$E^2_{p,q} = H_p(B; H_q(F; M)) \Rightarrow H_{p+q}(E; M),$$

where we are considering homology of B with local coefficients in the homology of the fiber F.

Theorem 10.2 (Serre spectral sequence for cohomology, [17, Theorem 5.2])**.** *Let R be a ring and let $F \to E \to B$ be a fibration with B path-connected. Then there is a first quadrant cohomological spectral sequence of algebras and converging as an algebra*

$$E_2^{p,q} = H^p(B; H^q(F; R)) \Rightarrow H^{p+q}(E; R),$$

where we are considering cohomology of B with local coefficients in the cohomology of the fiber F.

Homology (cohomology) with local coefficients is also known as twisted homology (cohomology). Being “twisted” means that an action of the fundamental group of the space on the module is taken into account. It can be understood through the universal cover, see [11, 3.H], and this is the approach we take in Example 8.2. If this action turns out to be trivial then there is no “twist” and we recover ordinary homology (cohomology) of the space over the module. For instance, in Theorems 10.1 and 10.2, if $\pi_1(B)$ acts trivially on $H_*(F; M)$ or $H^*(F; R)$, then the second page is given by ordinary homology of B or ordinary cohomology of B respectively. Of course, the action is trivial whenever $\pi_1(B) = 1$. In particular, in the latter theorem, if $\pi_1(B) = 1$ and R is a field we have

$$E_2^{p,q} = H^p(B; R) \otimes H^q(F; R) \Rightarrow H^{p+q}(E; R).$$

Sketch of proof of Theorem 10.2. We follow [17, Chapter 5]. Denote by $F \to E \xrightarrow{\pi} B$ the given fibration and consider the skeletal filtration of B

$$\emptyset \subseteq B^0 \subseteq B^1 \subseteq B^2 \ldots \subseteq B.$$

Then we have a filtration of E given by $J^s = \pi^{-1}(B^s)$

$$\emptyset \subseteq J^0 \subseteq J^1 \subseteq J^2 \ldots \subseteq E.$$

Consider the cochain complex of singular cochains on E with coefficients in R, $C^*(E; R)$, given by

$$C^s(E; R) = \{\text{functions } f\colon C_s(E) \to R\},$$

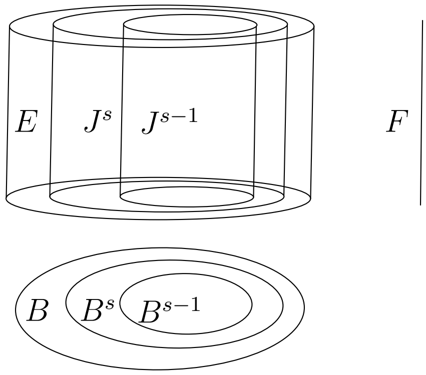
where $C_s(E) = \{\sigma\colon \Delta^s \to E \text{ continuous}\}$ are the singular s-chains in E. Now define a decreasing filtration of $C^*(E; R)$ by

$$F^sC^*(E; R) = \ker(C^*(E; R) \to C^*(J^{s-1}; R)),$$

i.e., the singular cochains of E that vanish on chains in J^{s-1}. It turns out that

$$E_1^{p,q} = H^{p+q}(J^p, J^{p-1}; R),$$

the relative cohomology of the pair (J^p, J^{p-1}).



The keystep of the proof relies in showing that

$$H^{p+q}(J^p, J^{p-1}; R) \cong C^p(B; H^q(F; R)),$$

the twisted p-cochains of B with coefficients in the q-cohomology of the fiber, which are defined as follows:

$$\{\text{functions } f\colon C_p(B) \to \textstyle\bigcup_{b\in B} H^q(\pi^{-1}(b); R)$$

$$\text{such that } f(\sigma) \in H^q(\pi^{-1}(\sigma(v_0)); R)\}.$$

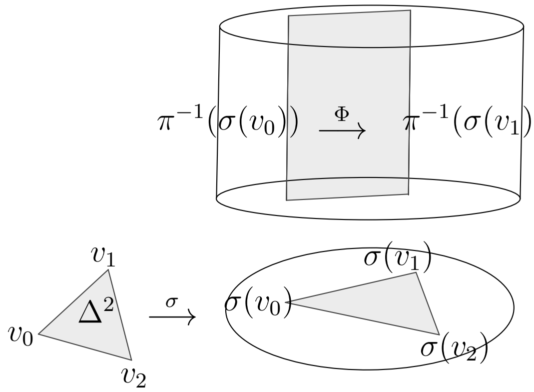
Note that, as B is connected, $\pi^{-1}(b)$ has the same homotopy type (F) for all $b \in B$. Also, we assume that $\Delta^p \subseteq \mathbb{R}^p$ is spanned by the vertices $\{v_0, \ldots, v_{p+1}\}$. The (twisted) differential

$$C^p(B; H^q(F; R)) \to C^{p+1}(B; H^q(F; R))$$

is given by

$$\partial(f)(\sigma) = H^q(\Phi; R)(f(\sigma_0)) + \sum_{i=1}^{p+1} (-1)^i f(\sigma_i),$$

where σ_i is the restriction of σ to the i-th face of Δ^p, $\{v_0, \ldots, \hat{v}_i, \ldots, v_{p+1}\}$, and the map $\Phi\colon \pi^{-1}(\sigma(v_0)) \to \pi^{-1}(\sigma(v_1))$ is a “lift” of the path $\sigma_{[v_0,v_1]}$ starting in v_0 and ending in v_1.



Then $E_2^{*,*}$ is obtained as the cohomology of $C^p(B; H^q(F;R))$, and this gives the description in the statement of the theorem. □

Theorem 10.3 (Lyndon-Hochschild-Serre spectral sequence for homology, [21, 6.8.2])**.** *Let $N \to G \to Q$ be a short exact sequence of groups and let M be a G-module. Then there is a first quadrant homological spectral sequence*

$$E^2_{p,q} = H_p(Q; H_q(N;M)) \Rightarrow H_{p+q}(G;M).$$

Theorem 10.4 (Lyndon-Hochschild-Serre spectral sequence for cohomology, [21, 6.8.2])**.** *Let $N \to G \to Q$ be a short exact sequence of groups and let R be a ring. Then there is a first quadrant cohomological spectral sequence of algebras converging as an algebra*

$$E_2^{p,q} = H^p(Q; H^q(N;R)) \Rightarrow H^{p+q}(G;R).$$

In the latter theorem, if $N \leq Z(G)$ and R is a field we have

$$E_2^{p,q} = H^p(Q;R) \otimes H^q(N;R) \Rightarrow H^{p+q}(G;R).$$

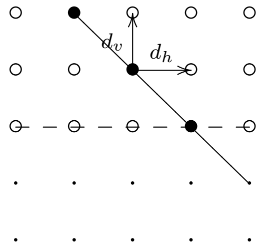
Sketch of proof of Theorem 10.4. We follow [16, XI, Theorem 10.1] but consult [12] for a direct filtration on the bar resolution of G. We do not provide details about the multiplicative structure. Consider the double complex

$$\begin{aligned} C^{p,q} &= \mathrm{Hom}_{RQ}(B_p(Q), \mathrm{Hom}_{RN}(B_q(G), R) \\ &\cong \mathrm{Hom}_{RG}(B_p(Q) \otimes B_q(G), R), \end{aligned}$$

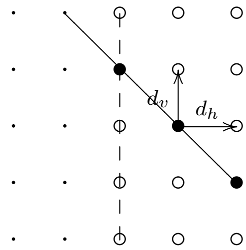
where $B_*(Q)$, $B_*(G)$ denote the corresponding bar resolutions and RQ, RN and RG denote group rings. So $B_p(Q)$ is the free R-module with basis Q^{p+1} and with differential the R-linear extension of $\partial(q_0, \dots, q_p) = \sum_{i=0}^p (-1)^i (q_0, \dots, \hat{q}_i, \dots, q_p)$. Then the horizontal and vertical differentials of $C^{*,*}$ are given by

$$\begin{aligned} \partial_h(f)(b \otimes b') &= (-1)^{p+q+1} f(\partial(b) \otimes b'), \text{ and} \\ \partial_v(f)(b \otimes b') &= (-1)^{q+1} f(b \otimes \partial(b')). \end{aligned}$$

There are two filtrations of the graded differential R-module $(\mathrm{Total}(C), \partial_h + \partial_v)$ obtained by considering either all rows above a given row or all columns to the right of a given column:



Filtration by rows



Filtration by columns

These two filtrations give rise to spectral sequences converging to the cohomology of $(\mathrm{Total}(C), \partial_h + \partial_v)$ [21, Section 5.6],

$$\begin{aligned} {}^r E_2^{p,q} &= H_v^p H_h^q(C) \Rightarrow H^{p+q}(\mathrm{Total}(C)), \text{ and} \\ {}^c E_2^{p,q} &= H_h^p H_v^q(C) \Rightarrow H^{p+q}(\mathrm{Total}(C)), \end{aligned}$$

where the subscripts h and v denote taking cohomology with respect to ∂_h or ∂_v respectively. For the former spectral sequence, we obtain ${}^r E_2^{p,q} = H^p(G;R)$ for $q = 0$ and $E_2^{p,q} = 0$ for $q > 0$ by [16, XI, Lemma 9.3]. For the latter spectral sequence we get

$${}^c E_1^{p,q} = \mathrm{Hom}_{RQ}(B_p(Q), H^q(N;R)),$$

as $B_p(Q)$ is a free RQ-module and hence commutes with cohomology, and then

$${}^c E_2^{p,q} = H^p(Q; H^q(N;R)).$$

□

Theorem 10.5 (Atiyah-Hirzebruch spectral sequence, [8, Theorem 9.22])**.** *Let h be a generalized cohomology theory and let $F \to E \to B$ be a fibration with B path-connected. Assume $h^q(F) = 0$ for q small enough. Then there is a "half-plane" cohomological spectral sequence*

$$E_2^{p,q} = H^p(B; h^q(F)) \Rightarrow h^{p+q}(E),$$

where we are considering cohomology of B with local coefficients.

Again, if $\pi_1(B)$ acts trivially on $h^*(F)$, then the second page is given by ordinary cohomology of B.

11 Appendix II

11.1 Finitely generated homology.

Let X be a space. We say that X is of *finite type* if $H_i(X;\mathbb{Z})$ is a finitely generated abelian group for all $i \geq 0$. The following result is a nice and easy application of the Serre spectral sequence 10.1.

Theorem 11.1. *Let $F \to E \to B$ be a fibration with B simply connected and F path-connected. If two of the spaces F, E or B are of finite type, then so is the third.*

A proof may be found in [17, Example 5.A] or [13, Lemma 1.9]. From this, we may derive the following two consequences.

Corollary 11.2. *The loop space of a simply connected finite CW-complex is of finite type.*

This corollary is obtained from the previous theorem using the pathspace fibration of a space X:

$$\Omega X \to PX \to X.$$

Recall that $PX \simeq *$ and that, if X is a finite CW-complex, then cellular homology gives directly that X is of finite type. By Theorem 11.1, the loop space of X, ΩX, is also of finite type.

Corollary 11.3. *If A is a finitely generated abelian group then $K(A,n)$ is of finite type for all $n \geq 1$.*

Here, $K(A,n)$ is an Eilenberg-MacLane space. The proof is as follows: We consider the pathspace fibration

$$\Omega K(A,n) \to PK(A,n) \to K(A,n),$$

where $PK(A,n) \simeq *$ and $\Omega K(A,n) \simeq K(A,n-1)$. Then, by Theorem 11.1 and induction, it is enough to deal with the case $n = 1$, i.e., to compute the homology of the group A. Moreover, as A is a product of cyclic groups, Theorem 11.1 reduces the problem to consider just one of the cyclic factors. Finally, we know that the homology of either the infinite cyclic group $\mathbb{Z}$ or a finite cyclic group $\mathbb{Z}_m$ is finitely generated on every degree.

11.2 Homotopy groups of spheres.

The Serre spectral sequence may be used to obtain general results about homotopy groups of spheres. For instance, the cohomology Serre spectral sequence with coefficients $\mathbb{Q}$ is the tool needed for the following result.

Theorem 11.4 ([13, Theorem 1.21][8, Theorem 10.10]). *The groups $\pi_i(S^n)$ are finite for $i > n$, except for $\pi_{4n-1}(S^{2n})$ which is the direct sum of $\mathbb{Z}$ with a finite group.*

Using as coefficients the integers localized at p, $\mathbb{Z}_{(p)}$, i.e., the subring of $\mathbb{Q}$ consisting of fractions with denominator relatively prime to p, yields the next result.

Theorem 11.5 ([13, Theorem 1.28] 25)[8, Corollary 10.13]). *For $n \geq 3$ and p a prime, the p-torsion subgroup of $\pi_i(S^n)$ is zero for $i < n + 2p - 3$ and C_p for $i = n + 2p - 3$.*

The Serre spectral sequence may be further exploited to compute more homotopy group of spheres [18, Chapter 12] but it does not give a full answer. The EHP spectral sequence is another tool to compute homotopy group of spheres, see for instance [13, p. 43].

11.3 Cohomology operations and Steenrod algebra.

Cohomology operations of type $(n, m, \mathbb{F}_p, \mathbb{F}_p)$ are exactly the natural transformations between the functors $H^n(-;\mathbb{F}_p)$ and $H^m(-;\mathbb{F}_p)$:

$$\theta\colon H^n(-;\mathbb{F}_p) \Rightarrow H^m(-;\mathbb{F}_p).$$

Among cohomology operations, we find stable cohomology operations, i.e., those families of cohomology operations of fixed degree m,

$$\{\theta_n\colon H^n(-;\mathbb{F}_p) \to H^{n+m}(-;\mathbb{F}_p)\}_{n\geq 0},$$

that commute with the suspension isomorphism:

$$\begin{array}{ccc} H^n(X;\mathbb{F}_p) & \xrightarrow{\theta_n} & H^{n+m}(X;\mathbb{F}_p) \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ H^{n+1}(\Sigma X;\mathbb{F}_p) & \xrightarrow{\theta_{n+1}} & H^{n+m+1}(\Sigma X;\mathbb{F}_p). \end{array}$$

These stable cohomology operations for mod p cohomology are assembled together to form the Steenrod algebra $\mathcal{A}_p$. Moreover, cohomology operations of a given type $(n, m, \mathbb{F}_p, \mathbb{F}_p)$ are in bijection with the cohomology group $H^m(K(\mathbb{F}_p, n);\mathbb{F}_p)$. The cohomology ring $H^*(K(\mathbb{F}_p, n);\mathbb{F}_p)$ may be determined via the Serre spectral sequence for all n and for all primes p, although this computation is much harder than the already seen in Example 5.7. Hence, information about $H^*(K(\mathbb{F}_p, n);\mathbb{F}_p)$ provides insight into the Steenrod algebra $\mathcal{A}_p$, which in turn is needed in the Adams spectral sequence 11.4. See [13, Theorem 1.32], [17, Theorem 6.19], [18, Chapter 9] and [8, Section 10.5].

11.4 Stable homotopy groups of spheres.

These groups are defined as follows:

$$\pi_k^S = \varinjlim_{n\to\infty} (\ldots \to \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1}) \to \ldots),$$

where the homomorphisms are given by suspension:

$$S^{n+k} \xrightarrow{f} S^n \rightsquigarrow \Sigma S^{n+k} = S^{n+k+1} \xrightarrow{\Sigma f} S^{n+1} = \Sigma S^n.$$

In fact, by Freudenthal's suspension theorem, $\pi_k^S = \pi_{n+k}(S^n)$ for $n > k+1$, i.e., all morphisms become isomorphism for n large enough. The Adams spectral sequence is the tool to compute stable homotopy groups of spheres: For each prime p, there is a spectral sequence which second page is

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$$

and converging to π_*^S modulo torsion of order prime to p. Here, $\mathcal{A}_p$ is the Steenrod algebra, see 11.3. The E_2-page is so complicated that the May spectral sequence is used to determine it. See [18, Chapter 18], [14] and [17, Chapter 9].

11.5 Hopf invariant one problem

Adams invented and used his spectral sequence to solve this problem (although later on he gave another proof via secondary cohomology operations).

The Hopf invariant of a map $S^{2n-1} \xrightarrow{f} S^n$ is the only integer $H(f)$ satisfying:

$$x^2 = H(f)y,$$

where x and y are the generators in degrees n and $2n$ of the integral cohomology $H^*(X;\mathbb{Z})$ of certain space X. This space obtained by attaching a $2n$-dimensional cell to S^n via f:

$$X = S^n \cup_f D^{2n}.$$

The Hopf invariant one problem consists of determining for which values of n there exists a map f with $H(f) = 1$, and its solution is that $n \in \{1, 2, 4, 8\}$. The problem can also be phrased as determining for which n does $\mathbb{R}^n$ admit a division algebra structure. See [11, 4.B] and [17, Theorem 9.38] or Adam's original papers [1], [2].

11.6 Segal's conjecture.

This conjecture states that for any finite group G, the natural map from the completion of the Burnside ring of G at its augmentation ideal to the stable cohomotopy of its classifying space is an isomorphism:

$$A(G)^\wedge \to \pi_s^0(BG).$$

This statement relates the pure algebraic object of the left hand side to the pure geometric object of the right hand side. The Burnside ring of G is the Grothendieck group of the monoid of isomorphism classes of finite G-sets. The sum is induced by disjoint union of G-sets and the product by direct product of G-sets with diagonal action. The augmentation ideal is the kernel of the augmenation map $A(G) \to \mathbb{Z}$. Cohomotopy groups are defined as maps to spheres instead of from spheres:

$$\pi_s^0(BG) = [\Sigma^\infty BG_+, \mathbb{S}] = [\Sigma^\infty BG \vee \mathbb{S}, \mathbb{S}],$$

where $\mathbb{S} = \Sigma^\infty S^0$ is the sphere spectrum. The map $A(G) \to \pi_s^0(BG)$ (before completion) takes the transitive G-set G/H for $H \leq G$ to the map

$$\Sigma^\infty BG_+ \xrightarrow{tr_H} \Sigma^\infty BH_+ \to \Sigma^\infty *_+ = \mathbb{S},$$

where tr_H is the transfer map. The conjecture was proven by Carlsson [7], who reduced the case of p-groups to the case of p-elementary abelian groups. The general case had already been reduced to p-groups by McClure. Ravenel, Adams, Guanawardena and Miller used the Adams spectral sequence to do the computations for p-elementary abelian groups.

12 Solutions

Solution 12.1 (Solution to 3.7)**.** This is [8, p. 243].

Solution 12.2 (Solution to 5.4)**.** The Lyndon-Hochschild-Serre spectral sequence of the central extension $C_3 \to C_9 \to C_3$ is

$$\begin{aligned} E_2^{p,q} &= H^p(C_3;\mathbb{F}_3) \otimes H^q(C_3, \mathbb{F}_3) \\ &= \Lambda(y) \otimes \mathbb{F}_3[x] \otimes \Lambda(y') \otimes \mathbb{F}_3[x'] \Rightarrow H^{p+q}(C_9;\mathbb{F}_3), \end{aligned}$$

where $|y| = (1,0)$, $|x| = (2,0)$, $|y'| = (0,1)$ and $|x'| = (0,2)$. So the corner of E_2 has the following generators:

x'^2	x'^2y	x'^2x	x'^2yx	x'^2x^2
$y'x'$	$y'x'y$	$y'x'x$	$y'x'yx$	$y'x'x^2$
x'	$x'y$	$x'x$	$x'yx$	$x'x^2$
y'	$y'y$	$y'x$	$y'yx$	$y'x^2$
1	y	x	yx	x^2

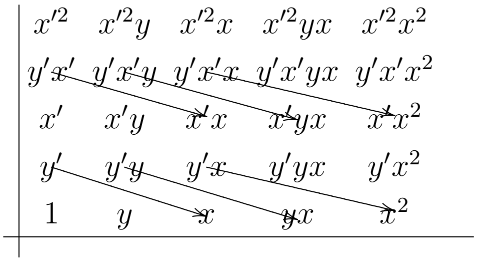
By elementary group theory, $H^1(C_9;\mathbb{F}_3) = \mathbb{F}_3$ and hence y' must die killing $d_2(y) = x$ (or $-x$). Then all odd rows disappear in E_2 as, for instance,

$$\begin{aligned} d_2(y'y) &= d_2(y')y - y'd_2(y) = xy = yx, \\ d_2(y'x) &= d_2(y')x - y'd_2(x) = x^2. \end{aligned}$$

Also, we must have $d_2(x') = 0$ because $d_2(x') \in \langle y'x \rangle$, $d_2(y'x) = x^2$ and $d_2 \circ d_2 = 0$. Then we deduce that, for instance,

$$d_2(x'y) = d_2(x')y' + x'd_2(y) = 0,$$

and in fact d_2 is zero on even rows. Summing up,



E_3 is as follows,

x'^2	x'^2y	0	0	0
0	0	0	0	0
x'	$x'y$	0	0	0
0	0	0	0	0
1	y	0	0	0

and the spectral sequence collapses at the free graded commutative algebra $E_3 = E_\infty = \Lambda(y) \otimes \mathbb{F}_3[x']$. Then by Theorem 5.5, we have $H^*(C_9; \mathbb{F}_3) = \Lambda(z) \otimes \mathbb{F}_3[z']$ with $|z| = 1$ and $|z'| = 2$.

Solution 12.3 (Solution to 5.8)**.** This is [13, Example 1.15] and the arguments are already described in 5.7.

Solution 12.4 (Solution to 5.9)**.** This is [13, Example 1.19] or [4, p. 245].

Solution 12.5 (Solution to 5.10)**.** This is [13, Example 1.16] or [4, p. 204]

Solution 12.6 (Solution to 6.5)**.** The answer is

$$P(t) = 1 + 2t + 3t^2 + 4t^3 + \ldots = \frac{1}{(t-1)^2}.$$

Solution 12.7 (Solution to 6.6)**.** The Lyndon-Hochschild-Serre spectral sequence is:

$$H^p(C_2; H^q(C_4; \mathbb{F}_2)) \Rightarrow H^{p+q}(D_8; \mathbb{F}_2).$$

The cohomology ring $H^*(C_4; \mathbb{F}_2) = \Lambda(y) \otimes \mathbb{F}_2[x]$ with $|y| = 1$, $|x| = 2$ was described in Example 5.2. On each degree q, the $\mathbb{F}_2$-module $H^q(C_4; \mathbb{F}_2)$ is equal to $\mathbb{F}_2$, and hence it must be trivial as a C_2-module. So, although the given extension is not central, the E_2-page is still equal to

$$E_2^{*,*} = H^*(C_4; \mathbb{F}_2) \otimes H^*(C_2; \mathbb{F}_2) = \Lambda(y) \otimes \mathbb{F}_2[x] \otimes \mathbb{F}_2[z],$$

where z is a generator of $H^1(C_2; \mathbb{F}_2)$. Generators in the corner of E_2 lie as follows:

x^2	x^2z	x^2z^2	x^2z^3
yx	yxz	yxz^2	yxz^3
x	xz	xz^2	xz^3
y	yz	yz^2	yz^3
1	z	z^2	z^3

What are the differentials $d_2, d_3, \ldots$? It is straightforward that the Poincaré series of $E_2^{*,*}$ is

$$P(t) = 1 + 2t + 3t^2 + 4t^3 + \ldots = \frac{1}{(t-1)^2}.$$

This coincides with the Poincaré series of $H^*(D_8; \mathbb{F}_2)$ by Solution 12.6. So all terms in the E_2-page must survive, all differentials must be zero and $E_\infty = E_2 = \Lambda(y) \otimes \mathbb{F}_2[x] \otimes \mathbb{F}_2[z]$. By Theorem 6.2, $H^*(D_8; \mathbb{F}_2) = \mathbb{F}_2[z, \tau, \tau']/(R)$ where z, τ, τ' are lifts of z, y, x respectively (see Remark 7.1) and R is a lift of the relation $y^2 = 0$. This lift must be of the form

$$\tau^2 = \lambda \tau z + \lambda' z^2,$$

and we know that $\lambda = 1$ and $\lambda' = 0$.

Solution 12.8 (Solution to 7.2)**.** Recall that we have $E_\infty = \mathbb{F}_2[x, x']/(x^2) = \Lambda(x) \otimes \mathbb{F}_2[x']$ with $|x| = (1, 0)$ and $|x'| = (0, 2)$. Hence, by Theorem 6.2, we have $H^*(C_4; \mathbb{F}_2) = \mathbb{F}_2(z, z')/(R)$ with $|z| = 1$, $|z'| = 2$ and where R is a lift of the relation $x^2 = 0$. Note that $x^2 \in E_\infty^{2,0} = 0 = F^2H^2 \subset H^2(D_8; \mathbb{F}_2)$ by 4.1. If z is a lift of x, then the lift of the relation $x^2 = 0$ must be $z^2 = 0$ and we are done.

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