



Rigidity of Discontinuous Groups for Threadlike Lie Groups

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Abstract

Let $G = G_n$ be the $(n + 1)$ -dimensional threadlike Lie group, H an arbitrary closed connected Lie subgroup of G and $\Gamma \subset G$ an abelian discontinuous group for the homogeneous space G/H . We provide in this work an explicit description of the parameter space $\mathcal{R}(\Gamma, G, H)$ and of the deformation space $\mathcal{T}(\Gamma, G, H)$ in the particular case where $n = 3$. Furthermore we discuss the local rigidity of $\mathcal{T}(\Gamma, G, H)$ and make a link with the Baklouti conjecture. This paper provides, through a concrete case study, a gentle initiation to the theory of deformations of Lie group actions.

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1 Introduction

Given a space M with an action of a connected Lie group G , a fundamental problem is the study of the geometry and dynamics of subgroups of G acting properly discontinuously on M . Topologizing the set of subgroups, or the set of subgroups isomorphic to a given group Γ , or the set of all closed subgroups of a topological group, etc, leads to notions of deformations and parameterizations of group actions. In our context, we will fix Γ to be a finitely generated discrete group, G a connected Lie group and we will consider all embeddings of Γ into G . An embedding is an injective homomorphism. We can then endow $\text{Emb}(\Gamma, G) \subset \text{Hom}(\Gamma, G)$ with the topology of point-wise convergence. If G acts on a space M , we then define the *parameter space*:

$$\mathcal{R}(\Gamma, G, M) := \{\varphi \in \text{Emb}(\Gamma, G) \mid \varphi(\Gamma) \text{ acts properly discontinuously on } G/H\}. \quad (1.1)$$

Generally we assume M to be Hausdorff and locally compact.

A key classical example is to consider a Lie group G acting on a Riemannian manifold M by isometry. In this case $\Gamma \subset G$ acts properly discontinuously on M if and only if it is a discrete subgroup. When M is a homogeneous space $X = G/H$, then $\mathcal{R}(\Gamma, G, X)$ becomes the parameter space of discontinuous group actions on X .

Definition 1.1. A discontinuous group of a homogeneous space X is a discrete group acting properly discontinuously on X .

The most basic but compelling example in the theory of discontinuous groups is the Teichmüller space. Here $X = \mathcal{H} := \{z \in \mathbb{C} : \text{Im} z > 0\}$ is the upper complex half-plane, and $G = PSL(2, \mathbb{R}) = \text{Aut}(\mathcal{H})$ is its group of biholomorphisms (or automorphisms). Fix Γ_g to be the finitely generated group

$$\Gamma_g = \{a_1, \dots, a_g, b_1, \dots, b_g \mid \prod [a_i, b_i] = 1\}$$

As we know Γ_g is isomorphic to the fundamental group of a compact Riemann surface of genus g . A discrete embedding of $\varphi : \Gamma_g \hookrightarrow G$ is called a Fuchsian group, and by uniformization the quotient of H by $\varphi(\Gamma_g)$ is a Riemann surface of genus g . Since any two conjugate subgroups of G yield two isomorphic Riemann surfaces, then we are led to consider the identification space

$$\mathcal{T}(\Gamma_g, G, \mathcal{H}) := \mathcal{R}(\Gamma_g, G, \mathcal{H}) / \sim$$

where $\varphi_1 \sim \varphi_2$ if and only if $\varphi_2 = g\varphi_1 g^{-1}$ for some $g \in G$.

Proposition 1.2. $\mathcal{T}(\Gamma_g, G, \mathcal{H})$ is homeomorphic to the Teichmüller space of Riemann surfaces of genus g . This is a ball of dimension $6g - 6$.

The study of the Teichmüller space, or equivalently the study of spaces of discontinuous co-compact Fuchsian groups, and their associated coarse moduli of Riemann surfaces, has been one of the most thriving fields of algebraic geometry, mathematical physics and algebraic topology in the last two decades. The Teichmüller space turns out to be in fact a connected component of

$$\text{Hom}(\pi_1(S), PSL(2, \mathbb{R})) / PSL(2, \mathbb{R})$$

the space of all homomorphism of $\pi_1(S)$, S a compact surface of genus g , up to conjugation. The study of the space $\text{Hom}(\pi_1(S), G) / G$ for more general Lie groups is the subject of a vast and rich literature. The notion of the Teichmüller space in the context of homogeneous

spaces has been defined by Kobayashi ([14]). As before, start with a finitely generated Γ , a connected Lie group G and H a closed connected subgroup of G . Then G acts on G/H by left translation. Suppose that Γ is discontinuous for the homogenous space G/H . The associated parameter space is denoted $\mathcal{R}(\Gamma, G, H)$, which we recall is again:

$$\mathcal{R}(\Gamma, G, H) := \left\{ \varphi \in \text{Hom}(\Gamma, G) \left| \begin{array}{l} \varphi \text{ is injective and } \varphi(\Gamma) \text{ acts} \\ \text{properly discontinuously} \\ \text{on } G/H \end{array} \right. \right\}. \quad (1.2)$$

Let now $\varphi \in \text{Hom}(\Gamma, G)$ and $g \in G$, we consider the element φ^g of $\text{Hom}(\Gamma, G)$ defined by $\varphi^g(\gamma) = g\varphi(\gamma)g^{-1}$, $\gamma \in \Gamma$ and we consider the action of G on $\text{Hom}(\Gamma, G)$ given by:

$$g \cdot \varphi = \varphi^g. \quad (1.3)$$

It is clear that $\mathcal{R}(\Gamma, G, H)$ is G -invariant, then we define the deformation space of the action of Γ on the homogeneous space G/H , denoted by $\mathcal{T}(\Gamma, G, H)$, by the orbit space under the action of G . Let then:

$$\mathcal{T}(\Gamma, G, H) = \mathcal{R}(\Gamma, G, H)/G.$$

Unlike the Teichmüller space, $\mathcal{T}(\Gamma, G, H)$ is generally not a manifold but has singularities. More particularly, it might have isolated points. A point $\varphi_0 : \Gamma \hookrightarrow G$ is isolated if its orbit under G is open in $\text{Hom}(\Gamma, G)$. This leads to the notion of *rigidity* of discontinuous groups first introduced by T. Kobayashi in [11].

Definition 1.3. For $\varphi \in \mathcal{R}(\Gamma, G, H)$, the discontinuous group $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* as a discontinuous group for G/H , if the orbit of φ through the inner conjugation is open in $\mathcal{R}(\Gamma, G, H)$.

In other words, a subgroup Γ of G is deformation rigid if for any continuous paths φ_t of embeddings of Γ into G starting with $\varphi_0 = id$, φ_t is conjugate to φ_0 . One of the most beautiful and potent results from the early sixties give a characterization of local rigidity for semi-simple Lie groups.

Theorem 1.4. ([8, 17]) *A cocompact discrete subgroup Γ in semi-simple Lie groups without compact nor $SL(2, \mathbb{R})$ nor $SL(2, \mathbb{C})$ local factors is deformation rigid.*

The notion of rigidity is crucial in understanding the local topology of deformation spaces. In this paper we study the local rigidity of a special family of Lie groups, the so-called threadlike groups. These are introduced in §3. We will show that local rigidity globally fails on the parameter space for threadlike Lie groups. This confirms in this case a general conjecture of Ali Baklouti [6]:

Conjecture 4.1. Let G be a connected simply connected nilpotent Lie group, let H be a connected subgroup of G , and let Γ be a non-trivial discontinuous

group for G/H . Then local rigidity globally fails to hold on the parameter space.

For particular situations of connected and simply connected nilpotent Lie groups, it was proved that the conjecture holds ([2, 3, 4, 5, 6, 7]).

The present paper is organized as follows. In §2 we summarize key properties of the parameter and deformation spaces associated to the action of a discontinuous subgroup on exponential homogeneous spaces. In §3, we record some basic results about threadlike Lie groups. In §4, we provide an explicit description of the parameter and deformation spaces for any discontinuous abelian subgroup acting on a homogeneous space of a four dimensional threadlike Lie group. The purpose of the final section is to study the local rigidity of the deformation space making use of all previous results.

2 Preliminaries

Let us first recall some notation and results which are of relevance to this work

2.1 Proper and free actions.

The notion of proper and free actions is fundamental to deformation theory and is of direct relevance to the important problems studied in Lie group theory.

We start by recording some definitions and useful results.

Definition 2.1. Let G be a connected Lie group and H be a closed connected subgroup of G . The action of a connected subgroup L of G on the homogeneous space G/H is said to be:

- (i) Proper if, for each compact subset $S \subset G$ the set $SHS^{-1} \cap L$ is compact.
- (ii) Free (or fixed point free) if, for each $g \in G$, the isotropy group $gHg^{-1} \cap L$ is trivial.
- (iii) Properly discontinuous if, L is discrete and the action of L on G/H is proper. If, moreover the action of L on G/H is free, we say that L is discontinuous for the homogeneous space G/H .

In [9], T. Kobayashi made a bridge between the action of a discrete group and that of a connected group by noticing that if Γ is a co-compact discrete subgroup of a connected subgroup L , then the action of L on G/H is proper if and only if the action of Γ on G/H is properly discontinuous. Using the notion of the synthetic hull, this fact greatly contributes to simplify the explicit description of the parameter space.

2.2 Characterization of the parameter and deformation spaces.

Let \mathfrak{g} denote a n -dimensional real exponential solvable Lie algebra, G will be the associated connected

and simply connected exponential Lie group. *Exponential* means that the exponential map $\exp : \mathfrak{g} \rightarrow G$ is a global C^∞ -diffeomorphism from \mathfrak{g} into G . Let \log denote the inverse map of \exp . The Lie algebra \mathfrak{g} acts on \mathfrak{g} by the adjoint representation ad , that is $\text{ad}_T(Y) = [T, Y]$, $T, Y \in \mathfrak{g}$. The group G acts on \mathfrak{g} by the adjoint representation Ad , defined by $\text{Ad}_g = \exp \circ \text{ad}_T$, $g = \exp T \in G$. Let $H = \exp \mathfrak{h}$ be a closed connected subgroup of G . Let Γ be an abelian discrete subgroup of G of rank k and define the parameter space $\mathcal{R}(\Gamma, G, H)$ as given in (1.2). Let L be the syndetic hull of Γ which is the smallest (and hence the unique) connected Lie subgroup of G which contains Γ cocompactly (see [2]). Recall that the Lie subalgebra \mathfrak{l} of L is the real span of the abelian lattice $\log \Gamma$, which is generated by $\{\log \gamma_1, \dots, \log \gamma_k\}$ where $\{\gamma_1, \dots, \gamma_k\}$ is a set of generators of Γ . The group G also acts on $\text{Hom}(\mathfrak{l}, \mathfrak{g})$ the set of groups homomorphisms from \mathfrak{l} to \mathfrak{g} , by:

$$g \star \psi = \text{Ad}_g \circ \psi. \quad (2.1)$$

The following useful result was obtained in [2].

Theorem 2.2. *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group, $H = \exp \mathfrak{h}$ a closed connected subgroup of G , Γ a discontinuous abelian subgroup for the homogeneous space G/H and $L = \exp \mathfrak{l}$ its syndetic hull. Then up to a homeomorphism, the parameter space $\mathcal{R}(\Gamma, G, H)$ is given by:*

$$\mathcal{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \{\psi \in \text{Hom}(\mathfrak{l}, \mathfrak{g}) : \dim \psi(\mathfrak{l}) = \dim \mathfrak{l} \text{ and } \exp(\psi(\mathfrak{l})) \text{ acts properly on } G/H\}.$$

The deformation space $\mathcal{T}(\Gamma, G, H)$ is likewise homeomorphic to the space $\mathcal{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \mathcal{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) / \text{Ad}$, where the action Ad of G is given as in (2.1). Furthermore, when G is completely solvable, the assumption on Γ to be abelian can be removed.

3 On threadlike Lie groups

Throughout this section and unless a specific mention, $\mathfrak{g} := \mathfrak{g}_n, n \geq 2$, designate the threadlike Lie algebra of dimension $n+1$ which admits a stratified basis $\mathcal{B} = \{X, Y_1, \dots, Y_n\}$ with non-trivial Lie brackets:

$$[X, Y_i] = Y_{i+1}, \quad i \in \{1, \dots, n-1\}. \quad (3.1)$$

The subspace $\mathfrak{g}_0 = \mathbb{R}\text{-span}\{Y_1, \dots, Y_n\}$ is clearly the one codimensional abelian ideal of \mathfrak{g} . The center $\mathfrak{z}(\mathfrak{g})$ of \mathfrak{g} is however one dimensional and it is the space $\mathbb{R}\text{-span}\{Y_n\}$.

Let G be the connected and simply connected Lie group associated to \mathfrak{g} and $\exp := \exp_G$ be the corresponding exponential map. To seek simple notation, we identify, from now on, the element $\exp(xX + y_1Y_1 + \dots + y_nY_n)$, $x, y_i \in \mathbb{R}, 1 \leq i \leq n$, of G by the column vector ${}^t(x, y_1, \dots, y_n)$.

3.1 On the structure of some Lie subgroups.

We first look at the case of closed connected subgroup of G . Thus, we only have to study the structure of the associated Lie subalgebra. Let then \mathfrak{h} be a p -dimensional

subalgebra of \mathfrak{g} , we are going to construct a strong Malcev basis $\mathcal{B}_{\mathfrak{h}}$ of \mathfrak{h} extracted from \mathcal{B} . Recall that a family of vectors $\{Z_1, \dots, Z_m\}$ is said to be a strong Malcev basis of a Lie algebra \mathfrak{l} , ($m = \dim \mathfrak{l}$) if $\mathfrak{l}_s = \mathbb{R}\text{-span}\{Z_1, \dots, Z_s\}$ is an ideal of \mathfrak{l} for all $s \in \{1, \dots, m\}$.

We denote by $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ and $\mathcal{J}_{\mathfrak{g}_0}^{\mathfrak{h}_0} = \{i_1 < \dots < i_{p_0}\}$ the set of indices $i \in \{1, \dots, n\}$ such that $\mathfrak{h}_0 + \mathfrak{g}^i = \mathfrak{h}_0 + \mathfrak{g}^{i+1}$, where $(\mathfrak{g}^i)_{1 \leq i \leq n+1}$ is the decreasing sequence of ideals of \mathfrak{g} given by:

$$\mathfrak{g}^i = \mathbb{R}\text{-span}\{Y_i, \dots, Y_n\}, i = 1, \dots, n \text{ and } \mathfrak{g}^{n+1} = \{0\}. \quad (3.2)$$

We note for all $i_s \in \mathcal{J}_{\mathfrak{g}_0}^{\mathfrak{h}_0}$, $\tilde{Y}_s = Y_{i_s} + \sum_{r=i_s+1}^n \alpha_{r,s} Y_r \in \mathfrak{h}$.

We get therefore that, if $\mathfrak{h} \subset \mathfrak{g}_0$, $\mathcal{B}_{\mathfrak{h}} = \{\tilde{Y}_1, \dots, \tilde{Y}_{p_0}\}$.

Otherwise, there exists $\tilde{X} = X + \sum_{r=1}^n x_r Y_r \in \mathfrak{h}$. Hence

$\mathcal{B}_{\mathfrak{h}} = \{\tilde{X}, \tilde{Y}_1, \dots, \tilde{Y}_{p_0}\}$. We define the matrix of \mathfrak{h} written in the basis \mathcal{B} , denoted by $M_{\mathfrak{h}, \mathcal{B}} \in M_{n+1,p}(\mathbb{R})$ as follows. We denote for every $s \in \{1, \dots, p_0\}$, by $M_{\mathfrak{h}}^s = {}^t(0, \dots, 0, 1, \alpha_{i_s+1,s}, \dots, \alpha_{n,s})$ and if $\mathfrak{h} \not\subset \mathfrak{g}_0$, we denote by $M_{\mathfrak{h}}^0 = {}^t(1, x_1, \dots, x_n)$. We put therefore:

- (i) If $\mathfrak{h} \subset \mathfrak{g}_0$, then $M_{\mathfrak{h}, \mathcal{B}} = [M_{\mathfrak{h}}^1, \dots, M_{\mathfrak{h}}^{p_0}]$, where this symbol merely designs the matrix constituted of the columns $M_{\mathfrak{h}}^1, \dots, M_{\mathfrak{h}}^{p_0}$.
- (ii) If $\mathfrak{h} \not\subset \mathfrak{g}_0$ then if $p = 1$ we have $M_{\mathfrak{h}, \mathcal{B}} = M_{\mathfrak{h}}^0$. Otherwise $M_{\mathfrak{h}, \mathcal{B}} = [M_{\mathfrak{h}}^0, \dots, M_{\mathfrak{h}}^{p_0}]$.

When \mathfrak{h} is not included in \mathfrak{g}_0 and $p \geq 2$, the following lemma, will be of use in the sequel and permits to construct a basis of \mathfrak{g} , said adapted to \mathfrak{h} , and which permits to have a particular form of the associated matrix.

Lemma 3.1. [4] *Let \mathfrak{g} be a threadlike Lie algebra and let \mathfrak{h} be a p -dimensional Lie subalgebra of \mathfrak{g} such that $\mathfrak{h} \not\subset \mathfrak{g}_0$ and $p \geq 2$. Then there exists a basis $\mathcal{B} = \{X, X_1, \dots, X_n\}$ of \mathfrak{g} such that:*

$$[X, X_i] = X_{i+1} \text{ for all } i = 1, \dots, n-1,$$

$$[X_i, X_j] = 0 \text{ for all } i, j = 1, \dots, n$$

$$\text{and } \mathfrak{h} = \mathbb{R}\text{-span}\{X, X_{n-p+2}, \dots, X_n\}.$$

The following Theorem, proved in [1], will be used later and deals with the proper action on threadlike nilpotent homogeneous spaces.

Theorem 3.2. *Let G be a connected simply connected special nilpotent Lie group, H and K are closed connected subgroups of G . Then the following assertions are equivalent:*

- i. K acts properly on G/H .
- ii. The action of K on G/H has the fixed point property, that is $K \cap gHg^{-1} = \{e\}$ for any $g \in G$.
- iii. $\mathfrak{k} \cap \text{Ad}_g \mathfrak{h} = 0$ for any $g \in G$. Here \mathfrak{h} and \mathfrak{k} are the Lie algebras of H and K respectively.

4 Description of the deformation space

Let $G = G_3$ be a threadlike Lie group, H a closed connected Lie subgroup of G and Γ an abelian discontinuous group for the homogeneous space G/H . This section gives a complete description of the parameter space $\mathcal{R}(\Gamma, G, H)$ and the deformation space $\mathcal{T}(\Gamma, G, H)$

4.1 Description of $\text{Hom}(\Gamma, G)$

Our main result in this subsection consists in giving an explicit description of $\text{Hom}(\Gamma, G)$. Moreover, note that the set of all injective homomorphisms from Γ to G denoted by $\text{Hom}^0(\Gamma, G)$, rather than $\text{Hom}(\Gamma, G)$ itself will be of interest in the next section, merely because it is involved in deformations. The following results accurately determines the stratification of such sets.

Towards such a purpose, we fix from now on a basis $\mathcal{B} = \{X, Y_1, Y_2, Y_3\}$ of \mathfrak{g} with non-trivial Lie brackets defined in (3.1). Let Γ be a discrete subgroup of G of rank $k \in \{1, \dots, 4\}$ and $\{\gamma_1, \dots, \gamma_k\}$ a set of generators of Γ . Then any $\varphi \in \text{Hom}(\Gamma, G)$ is determined by $\varphi(\gamma_j)$, $j = 1, \dots, k$. We obtain the injective map $\Psi : \text{Hom}(\Gamma, G) \rightarrow \mathfrak{g}^k = \mathfrak{g} \times \dots \times \mathfrak{g}$ defined by:

$$\Psi(\varphi) = (\varphi_{\mathfrak{g}}(\gamma_1), \dots, \varphi_{\mathfrak{g}}(\gamma_k)) \quad (4.1)$$

where $\varphi_{\mathfrak{g}} = \log \circ \varphi$. So, we reduce our problem of the description of $\text{Hom}(\Gamma, G)$ to the determination of the image of Ψ . From now on, we identify any element $T = xX + \sum_{i=1}^3 y_i Y_i \in \mathfrak{g}$ by the column vector ${}^t(x, y_1, y_2, y_3)$, and $\mathfrak{g}^k = \mathfrak{g} \times \dots \times \mathfrak{g}$ to the space $M_{4,k}(\mathbb{R})$.

We consider the sets:

$$H_{0,k} = \left\{ \begin{pmatrix} \vec{0} \\ N \end{pmatrix} \in M_{4,k}(\mathbb{R}) : N \in M_{3,k}(\mathbb{R}) \right\} \simeq M_{3,k}(\mathbb{R}), \quad (4.2)$$

and for any $j \in \{1, \dots, k\}$:

$$H_{j,k} = \left\{ \begin{pmatrix} \lambda_1 T \dots \lambda_{j-1} T & T & \lambda_{j+1} T \dots \lambda_k T \\ z_1 & \dots & z_{j-1} & z_j & z_{j+1} & \dots & z_k \end{pmatrix} \in M_{4,k}(\mathbb{R}) \mid \begin{array}{l} {}^t T \in \mathbb{R}^* \times \mathbb{R}^2 \\ (z_1, \dots, z_k) \in \mathbb{R}^k, \\ (\lambda_1, \dots, \lambda_j, \dots, \lambda_k) \in \mathbb{R}^{k-1} \end{array} \right\} \quad (4.3)$$

Proposition 4.1. *With the same notation and hypotheses, we have:*

$$\text{Hom}(\Gamma, G) = \bigcup_{j=0}^k H_{j,k}.$$

Proof. We begin by proving the following:

Lemma 4.2. *$\text{Hom}(\Gamma, G)$ is homeomorphic to*

$$M'_{4,k}(\mathbb{R}) = \{[T_1, \dots, T_k] \in M_{4,k}(\mathbb{R}) : [T_s, T_r] = 0, \\ 1 \leq r, s \leq k\}.$$

Proof. Since $\varphi \in \text{Hom}(\Gamma, G)$ satisfies:

$$\varphi(\gamma_r)\varphi(\gamma_s) = \varphi(\gamma_s)\varphi(\gamma_r) \text{ for all } 1 \leq r, s \leq k, \quad (4.4)$$

we get that:

$$[\varphi_{\mathfrak{g}}(\gamma_r), \varphi_{\mathfrak{g}}(\gamma_s)] = 0, \text{ for all } 1 \leq r, s \leq k.$$

This implies that $\Psi(\text{Hom}(\Gamma, G)) \subset M'_{4,k}(\mathbb{R})$. Let conversely $[T_1, \dots, T_k] \in M'_{4,k}(\mathbb{R})$, we can define a groups homomorphism:

$$\varphi : \Gamma \rightarrow G; \varphi\left(\sum_{s=1}^k m_s \gamma_s\right) = \exp(m_1 T_1) \cdots \exp(m_k T_k),$$

$m_i \in \mathbb{Z}, i = 1, \dots, k$, in such a way that $\Psi(\varphi) = [T_1, \dots, T_k]$. Thus, $\Psi(\text{Hom}(\Gamma, G)) = M'_{4,k}(\mathbb{R})$.

By identifying $\mathfrak{g}^k = \mathfrak{g} \times \dots \times \mathfrak{g}$ to the space $M_{n+1,k}(\mathbb{R})$, we can easily see the continuity of Ψ . As for the converse, we take $[T_1^j, \dots, T_k^j]_{j \in \mathbb{N}}$ a sequence of elements of $\Psi(\text{Hom}(\Gamma, G))$ which converges to $[T_1, \dots, T_k]$ and let $\varphi_j, \varphi \in \text{Hom}(\Gamma, G)$ such that $\Psi(\varphi_j) = [T_1^j, \dots, T_k^j]$ and $\Psi(\varphi) = [T_1, \dots, T_k]$. So, for any $r \in \{1, \dots, k\}$ and any $j \in \mathbb{N}$, we have $\varphi_j(\gamma_r) = \exp T_r^j$ which gives that $\exp T_r = \varphi(\gamma_r)$. We thus arrive to the continuity of Ψ^{-1} . \square

The second step consists in giving an explicit description of $M'_{4,k}(\mathbb{R})$. Let $M = \begin{pmatrix} x_1 & \dots & x_k \\ y_{11} & \dots & y_{1k} \\ y_{21} & \dots & y_{2k} \\ y_{31} & \dots & y_{3k} \end{pmatrix} \in M_{4,k}(\mathbb{R})$. Recall from equation (4.4) that $M \in M'_{4,k}(\mathbb{R})$

if and only if $[T_s, T_r] = 0$ where $T_s = x_s X + \sum_{i=1}^3 y_{is} Y_i$ and $T_r = x_r X + \sum_{i=1}^3 y_{ir} Y_i$ for any $1 \leq s, r \leq k$, which gives rise to the following equation:

$$x_s y_{ir} - x_r y_{is} = 0 \text{ for all } 1 \leq r, s \leq k \text{ and all } i \in \{1, 2\}. \quad (4.5)$$

In order to find the solutions of (4.5), we shall discuss the two following dichotomous cases. Assume in a first time that the first line of M is zero, that is $x_j = 0$, $1 \leq j \leq k$. In this case M satisfies (4.5) and obviously belongs to $M'_{4,k}(\mathbb{R})$. Suppose now that the first line of M is not zero. There exists then $j \in \{1, \dots, k\}$ satisfying $x_j \neq 0$. So, $M \in M'_{4,k}(\mathbb{R})$ if and only if there exists $\Lambda_j = (\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_k) \in \mathbb{R}^{k-1}$ such that for $s \neq j$, we have $T_s = \lambda_s T_j$. According to this discussion, we get that $M'_{4,k}(\mathbb{R}) = \bigcup_{j=0}^k H_{j,k}$ where $H_{j,k}, 0 = 1, \dots, k$ are as determined by equations (4.2) and (4.3). This achieves the proof of the proposition. \square

Now, we determine a stratification of $\text{Hom}^0(\Gamma, G)$. We prove the following.

Proposition 4.3. *Let $k \in \{1, \dots, 4\}$. If $k = 4$, then $\text{Hom}^0(\Gamma, G) = \emptyset$. Otherwise, there exists a finite set I_k such that $0 \in I_k$ and*

$$\text{Hom}^0(\Gamma, G) = \coprod_{j \in I_k} K_{j,k},$$

where for any $k \in \{1, 2, 3\}$ we have:

$$K_{0,k} = \left\{ \begin{pmatrix} \vec{0} \\ N \end{pmatrix} : N \in M_{3,k}^0(\mathbb{R}) \right\} \simeq M_{3,k}^0(\mathbb{R}). \quad (4.6)$$

Here $M_{n,m}^0(\mathbb{R})$ denotes the set of all matrix of n rows, m columns and of maximal rank. Moreover,

i. if $k = 3$, then $I_3 = \{0\}$

ii. If $k = 2$ then $I_2 = \{0, 1, 2, 3\}$ and

$$K_{1,2} = \left\{ \begin{pmatrix} x & 0 \\ y_1 & 0 \\ y_2 & 0 \\ y_3 & y'_3 \end{pmatrix} \in M_{4,2}(\mathbb{R}) : xy'_3 \neq 0 \right\},$$

$$K_{2,2} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y_1 \\ 0 & y_2 \\ y'_3 & y_3 \end{pmatrix} \in M_{4,2}(\mathbb{R}) : xy'_3 \neq 0 \right\},$$

$$K_{3,2} = \left\{ \begin{pmatrix} x & \lambda x \\ y_1 & \lambda y_1 \\ y_2 & \lambda y_2 \\ y_3 & y'_3 \end{pmatrix} \in M_{4,2}(\mathbb{R}) : \lambda x \neq 0 \right. \\ \left. \lambda y_3 - y'_3 \neq 0 \right\}.$$

iii. If $k = 1$, then $I_1 = \{0, 1\}$ and

$$\begin{aligned} K_{1,1} &= H_{1,1} \\ &= \left\{ \begin{pmatrix} x \\ \vec{y} \end{pmatrix} : x \in \mathbb{R}^*, {}^t\vec{y} \in \mathbb{R}^3 \right\} \\ &\simeq \mathbb{R}^* \times \mathbb{R}^3 \end{aligned}$$

Proof. First observe that for all $M \in \text{Hom}(\Gamma, G)$, $\text{rank}(M) \leq 3$. This gives that for $k = 4$, $\text{Hom}^0(\Gamma, G) = \emptyset$. Let now $k \in \{1, 2, 3\}$, $j \in \{1, \dots, k\}$ and $M \in H_{j,k}$. It is not hard to check that $\text{rank}(M) \leq 2$. In addition, M is of maximal rank if and only if $\text{rank}(M) = k$. So, it appears clear that if $k = 3$, we have $\text{Hom}^0(\Gamma, G) \cap H_{j,3} = \emptyset$ for all $j \in \{1, 2, 3\}$ and then $I_3 = \{0\}$. Suppose now that $k = 2$ and choose in a first time $M = \begin{pmatrix} x & \lambda x \\ \vec{y} & \lambda \vec{y} \\ y_3 & y'_3 \end{pmatrix} \in H_{1,2}$, where $x \in \mathbb{R}^*$, ${}^t\vec{y} = (y_1, y_2) \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$ and $(y_3, y'_3) \in \mathbb{R}^2$. The condition M is of maximal rank is equivalent to $\lambda y_3 - y'_3 \neq 0$. If furthermore we choose $M = \begin{pmatrix} \lambda x & x \\ \lambda \vec{y} & \vec{y} \\ y_3 & y'_3 \end{pmatrix} \in H_{2,2}$ for some $x \in \mathbb{R}^*$, ${}^t\vec{y} = (y_1, y_2) \in \mathbb{R}^2$ and $(y_3, y'_3) \in \mathbb{R}^2$, we

get M is of maximal rank if and only if $\lambda y'_3 - y_3 \neq 0$. Therefore:

$$\text{Hom}^0(\Gamma, G) \cap H_{1,2} = \left\{ M = \begin{pmatrix} x & \lambda x \\ \vec{y} & \lambda \vec{y} \\ y_3 & y'_3 \end{pmatrix} \in H_{1,2} : \lambda y_3 - y'_3 \neq 0 \right\}$$

and

$$\text{Hom}^0(\Gamma, G) \cap H_{2,2} = \left\{ M = \begin{pmatrix} \lambda x & x \\ \lambda \vec{y} & \vec{y} \\ y_3 & y'_3 \end{pmatrix} \in H_{2,2} : \lambda y'_3 - y_3 \neq 0 \right\}.$$

It is then easy to see that $\text{Hom}^0(\Gamma, G) \cap (H_{1,2} \cup H_{2,2})$ is equal to the disjoint union of the sets $K_{j,2}$, $j = 1, 2, 3$ defined above. So we end up with the following decomposition:

$$\text{Hom}^0(\Gamma, G) = \coprod_{j=0}^3 K_{j,2}.$$

Finally, if $k = 1$ then any homomorphism in $H_{1,1}$ is injective, therefore $K_{1,1} = H_{1,1}$ and then

$$\text{Hom}^0(\Gamma, G) = K_{0,1} \coprod K_{1,1}.$$

□

4.2 Description of the parameter space

$\mathcal{R}(\Gamma, G, H)$.

The most important problem in the study of the deformation space of discontinuous groups is the description of the parameter set $\mathcal{R}(\Gamma, G, H)$ given by equation (1.2). Let G act on $M'_{4,k}(\mathbb{R})$ by:

$$g \cdot M = \text{Ad}_{g^{-1}} \cdot M, \quad g \in G, \quad M \in M'_{4,k}(\mathbb{R}). \quad (4.7)$$

Here we view $\text{Ad}_{g^{-1}}$ as a real valued matrix for any $g \in G$. In light of Theorem 2.2 and Theorem 3.2, the following result is immediate:

Lemma 4.4. *Let $G = \exp \mathfrak{g}$ be the four-dimensional threadlike Lie group, $H = \exp \mathfrak{h}$ be a closed connected subgroup of G and let $L = \exp \mathfrak{l}$ be the syndetic hull of an abelian discrete subgroup Γ of G . Then the set $\mathcal{R}(\Gamma, G, H)$ is homeomorphic to:*

$$\begin{aligned} \mathcal{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) &= \{M \in M'_{4,k}(\mathbb{R}) : \\ &\text{rank}(M \uplus g \cdot M_{\mathfrak{h}, \mathcal{B}}) = k + p \text{ for all } g \in G\}, \end{aligned}$$

where $\mathcal{B} = \{X, Y_1, Y_2, Y_3\}$ and the symbol \uplus merely means the superposition of the matrices written through \mathcal{B} .

Proof. If $M \in \mathcal{R}(\Gamma, G, H)$ then $M \in M'_{4,k}(\mathbb{R})$ which gives that $\text{rank}(M) = k$. Now using Theorem 3.2, the proper action of L on G/H is equivalent to the fact

that $\mathfrak{l} \cap \text{Ad}_g \mathfrak{h} = \{0\}$ for any $g \in G$ which means that $\text{rank}(M \uplus g \cdot M_{\mathfrak{h}, \mathcal{B}}) = k + p$. Note finally that the condition $\text{rank}(M) = k$ is irrelevant at this stage which gives the result as was to be shown. \square

Proposition 4.5. *Let G be the four-dimensional thread-like Lie group and H a connected Lie subgroup of G . Then:*

$$\mathcal{R}(\Gamma, G, H) = \coprod_{j \in I_k} R_{j,k}$$

where $R_{j,k} = \mathcal{R}(\Gamma, G, H) \cap K_{j,k}$. More precisely, one has:

- i. If $k = 3$, then $\mathcal{R}(\Gamma, G, H) = R_{0,3}$.
- ii. If $k = 2$, then $\mathcal{R}(\Gamma, G, H) = \coprod_{j=0}^3 R_{j,2}$.
- iii. If $k = 1$, then $\mathcal{R}(\Gamma, G, H) = R_{0,1} \coprod R_{1,1}$.

Proof. This result stems immediately from Proposition 4.3 which describes the structure of $\text{Hom}^0(\Gamma, G)$. \square

Our main upshot in this section is the explicit description of the parameter space $\mathcal{R}(\Gamma, G, H)$. For such a study, we will state our results separately according to the values of $k = \text{rank}(\Gamma)$, p and to the position of H inside G . Firstly, we can suppose that $k \notin \{0, 4\}$ and $p \notin \{0, 4\}$. Now, we introduce the matrix $A(t)$, $t \in \mathbb{R}$ of $(\text{Ad}_{\exp tX})|_{\mathfrak{g}_0}$ written through the basis $\mathcal{B}_0 = \{Y_1, Y_2, Y_3\}$. So a routine computation shows that:

$$A(t) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{t^2}{2} & t & 1 \end{pmatrix}. \quad (4.8)$$

The following proposition deals with the description of $R_{0,k}$ which coincides with the parameter space in the case where $k = 3$.

Proposition 4.6. *We keep the same hypotheses and notation as before. Then:*

(i) if $\mathfrak{h} \not\subset \mathfrak{g}_0$, then:

$$\begin{aligned} R_{0,k} &= \left\{ \begin{pmatrix} \vec{0} \\ N_1 \\ N_2 \end{pmatrix} \in H_{0,k} : N_1 \in M_{4-p,k}^0(\mathbb{R}), N_2 \in M_{p,k}(\mathbb{R}) \right\} \\ &\simeq M_{4-p,k}^0(\mathbb{R}) \times M_{p,k}(\mathbb{R}). \end{aligned} \quad (4.9)$$

(ii) If $\mathfrak{h} \subset \mathfrak{g}_0$, then if $k = 3$, then $R_{0,3} = \emptyset$. Otherwise:

$$\begin{aligned} R_{0,k} &= \left\{ \begin{pmatrix} \vec{0} \\ N \end{pmatrix} \in H_{0,k} : \text{rank}(A(t)N \uplus M_{\mathfrak{h}, \mathcal{B}_0}) = k + p, \right. \\ &\quad \left. \text{for any } t \in \mathbb{R} \right\}. \end{aligned} \quad (4.10)$$

Proof. Recall that:

$$\begin{aligned} M \in \mathcal{R}(\Gamma, G, H) &\Leftrightarrow \\ \text{rank}(M \uplus g \cdot M_{\mathfrak{h}, \mathcal{B}}) &= k + p \text{ for all } g \in G. \end{aligned} \quad (4.11)$$

So it appears clear that if $k > 4 - p$, we have:

$$\mathcal{R}(\Gamma, G, H) = \emptyset.$$

We can from now on suppose that $k \leq 4 - p$. Suppose in first time that $\mathfrak{h} \not\subset \mathfrak{g}_0$. We note $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ which is an ideal of \mathfrak{g} . Let $M = \begin{pmatrix} \vec{0} \\ N \end{pmatrix} \in H_{0,k}$. If $p = 1$, then equation (4.11) is equivalent to $\text{rank}(M) = k$ and then $N \in M_{4-p,k}^0(\mathbb{R})$. Suppose now that $p > 1$. So, equation (4.11) is equivalent to $\text{rank}(M \uplus M_{\mathfrak{h}_0, \mathcal{B}}) = k + p - 1$ which is in turn equivalent to the fact that:

$$\text{rank}(N \uplus M_{\mathfrak{h}_0, \mathcal{B}_0}) = k + p - 1. \quad (4.12)$$

According to our choice of the basis \mathcal{B} , we get that $M_{\mathfrak{h}_0, \mathcal{B}_0} = \begin{pmatrix} (0) \\ I_{p-1} \end{pmatrix} \in M_{3,p-1}(\mathbb{R})$ where I_{p-1} designates the identity matrix of $M_{p-1}(\mathbb{R})$. We now write $M = \begin{pmatrix} \vec{0} \\ N_1 \\ N_2 \end{pmatrix}$ where $N_1 \in M_{4-p,k}^0(\mathbb{R})$ and $N_2 \in M_{p-1,k}(\mathbb{R})$, we get that equation (4.12) is equivalent to have $\text{rank}(N_1) = k$. We hence end up with the fact that:

$$\begin{aligned} R_{0,k} &= \mathcal{R}(\Gamma, G, H) \cap H_{0,k} \\ &= \left\{ \begin{pmatrix} \vec{0} \\ N_1 \\ N_2 \end{pmatrix} : N_1 \in M_{4-p,k}^0(\mathbb{R}), N_2 \in M_{p-1,k}(\mathbb{R}) \right\}. \end{aligned}$$

We now treat the case where $\mathfrak{h} \subset \mathfrak{g}_0$. So, it is not hard to see that $\text{Ad}_G \mathfrak{h} = \bigcup_{t \in \mathbb{R}} \text{Ad}_{\exp tX} \mathfrak{h}$. Let $M = \begin{pmatrix} \vec{0} \\ N \end{pmatrix} \in K_{0,k}$. We get therefore that:

$$\begin{aligned} M \in \mathcal{R}(\Gamma, G, H) &\Leftrightarrow \text{rank}(M \uplus \exp tX \cdot M_{\mathfrak{h}, \mathcal{B}}) = k + p, \text{ for all } t \in \mathbb{R} \\ &\Leftrightarrow \text{rank}(\exp tX \cdot N \uplus M_{\mathfrak{h}, \mathcal{B}_0}) = k + p, \forall t \in \mathbb{R} \\ &\Leftrightarrow \text{rank}(A(t)N \uplus M_{\mathfrak{h}, \mathcal{B}_0}) = k + p, \forall t \in \mathbb{R}. \end{aligned}$$

When $k = 3$, we have $p = 1$ and $A(t)N \uplus M_{\mathfrak{h}, \mathcal{B}_0} \in M_{3,4}$ which gives that $\text{rank}(A(t)N \uplus M_{\mathfrak{h}, \mathcal{B}_0}) < k + p$. This completes the proof of the theorem. \square

We assume henceforth that $k \in \{1, 2\}$. We will be dealing with these subsequent cases separately. The following upshot exhibits an accurate description of the parameter space when $k = 2$.

Proposition 4.7. *Let $k = 2$ and $R_{0,2}$ as in Proposition ???. Then:*

(i) if $\mathfrak{h} \not\subset \mathfrak{g}_0$, then:

i₁. If $p = 1$, then:

$$R_{1,2} = \left\{ \begin{pmatrix} x & 0 \\ y_1 & 0 \\ y_2 & 0 \\ y_3 & y'_3 \end{pmatrix} M_{4,2}(\mathbb{R}) : y_1 x y'_3 \neq 0 \right\},$$

$$R_{2,2} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y_1 \\ 0 & y_2 \\ y'_3 & y_3 \end{pmatrix} \in M_{4,2}(\mathbb{R}) : y_1 x y'_3 \neq 0 \right\}$$

and

$$R_{3,2} = \left\{ \begin{pmatrix} x & \lambda x \\ y_1 & \lambda y_1 \\ y_2 & \lambda y_2 \\ y_3 & y'_3 \end{pmatrix} : \lambda x (\lambda y_3 - y'_3) y_1 \neq 0 \right\}.$$

i_2 . If $p > 1$, then $p = 2$ and:

$$R_{j,2} = \emptyset, \quad j = 1, 2, 3.$$

(ii). If $\mathfrak{h} \subset \mathfrak{g}_0$, then $R_{1,2} = R_{2,2} = R_{3,2} = \emptyset$ if ever $Y_3 \in \mathfrak{h}$. Otherwise,

$$R_{i,2} = K_{i,2}, \quad i = 1, 2, 3.$$

Proof. Let $M = \begin{pmatrix} x & \lambda x \\ y_1 & \lambda y_1 \\ y_2 & \lambda y_2 \\ y_3 & y'_3 \end{pmatrix} \in \text{Hom}^0(\Gamma, G)$ such that $\lambda y_3 - y'_3 \in \mathbb{R}^*$.

We tackle first the case where $\mathfrak{h} \not\subset \mathfrak{g}_0$. In the case where $\mathfrak{h} = \mathbb{R}X$, a simple computation shows that $\text{Ad}_G \mathfrak{h} = \mathbb{R}X + [X, \mathfrak{g}_0]$. Hence:

$$\begin{aligned} \text{rank}(M \uplus g \cdot M_{\mathfrak{h}, \mathcal{B}}) &= 3, \text{ for any } g \in G \\ \Leftrightarrow \text{rank}(M \uplus {}^t(1, 0, \alpha_2, \alpha_3)) &= 3, \forall \alpha_2, \alpha_3 \in \mathbb{R} \\ \Leftrightarrow y_1 &\in \mathbb{R}^* \end{aligned}$$

Therefore:

$$M \in R_{1,2} \cup R_{3,2} \Leftrightarrow y_1 \in \mathbb{R}^*.$$

Similar computations show that for $M = \begin{pmatrix} 0 & x \\ 0 & y_1 \\ 0 & y_2 \\ y'_3 & y_3 \end{pmatrix} \in$

$K_{2,2}$, one gets that $M \in R_{2,2}$ if and only if $y_1 \in \mathbb{R}^*$. Suppose now that $\mathbb{R}X \subsetneq \mathfrak{h}$, that is $p = 2$, we have that the vector $Y_3 = {}^t(0, 0, 0, 1) \in \mathfrak{h}$ and it is a linear combination of the columns of M . So $\text{rank}(M \uplus M_{\mathfrak{h}, \mathcal{B}}) < p + 2$ and then:

$$\mathcal{R}(\Gamma, G, H) \cap K_{j,2} = R_{j,2} = \emptyset, \quad j = 1, 2, 3.$$

Suppose finally that $\mathfrak{h} \subset \mathfrak{g}_0$. If $Y_3 \in \mathfrak{h}$ then $\text{rank}(M \uplus M_{\mathfrak{h}, \mathcal{B}}) < 2 + p$, which gives us $\mathcal{R}(\Gamma, G, H) \cap K_{j,2} = \emptyset, j = 1, 2, 3$. Otherwise,

$$\begin{aligned} M &\in \mathcal{R}(\Gamma, G, H) \\ \Leftrightarrow \text{rank}(M \uplus \exp tX \cdot M_{\mathfrak{h}, \mathcal{B}}) &= 2 + p, \forall t \in \mathbb{R} \\ \Leftrightarrow \text{rank}(M) &= 2 \\ \Leftrightarrow M &\in \text{Hom}^0(\Gamma, G). \end{aligned}$$

Thus, we have:

$$R_{j,2} = K_{j,2}, \quad j = 1, 2, 3,$$

which completes the proof in this case. \square

Proposition 4.8. Assume that $k = 1$ and let $R_{0,1}$ as in Proposition 4.6. Then

i. if $\mathfrak{h} \not\subset \mathfrak{g}_0$, then:

$$\begin{aligned} R_{1,1} &= \left\{ \begin{pmatrix} x \\ y_1 \\ y \end{pmatrix} : x \in \mathbb{R}^*, y_1 \in \mathbb{R}^*, \vec{y} \in \mathbb{R}^2 \right\} \\ &\simeq (\mathbb{R}^*)^2 \times \mathbb{R}^2. \end{aligned}$$

ii. If $\mathfrak{h} \subset \mathfrak{g}_0$, then:

$$R_{1,1} = H_{1,1}.$$

Proof. We have in this case that $\Gamma \simeq \mathbb{Z}$. Suppose in a first time that $\mathfrak{h} \not\subset \mathfrak{g}_0$ and let $M \in K_{1,1}$. So clearly:

$$\text{rank}(M \uplus g \cdot M_{\mathfrak{h}, \mathcal{B}}) = p + 1, \text{ for any } g \in G \Leftrightarrow (4.13)$$

$$\text{rank}(g \cdot M \uplus M_{\mathfrak{h}, \mathcal{B}}) = p + 1, \text{ for any } g \in G,$$

which gives when writing M as $\begin{pmatrix} x \\ y \end{pmatrix} \in K_{1,1}$ for $x \in \mathbb{R}^*$ and $\vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$, that (4.13) holds if and only if $y_1 \neq 0$.

If now $\mathfrak{h} \subset \mathfrak{g}_0$, we have:

$$R_{1,1} = K_{1,1} = H_{1,1}.$$

\square

4.3 Description of the deformation space $\mathcal{T}(\Gamma, G, H)$.

This section aims to describe the deformation space of the action of an abelian discrete subgroup $\Gamma \subset G$ on a threadlike homogeneous space G/H . This description strongly relies on the comprehensive details about the parameter space provided in the previous section. Taking into account the action of G on $\text{Hom}(\mathfrak{l}, \mathfrak{g})$ and on $M'_{4,k}(\mathbb{R})$ defined in (4.7) and (2.1), the following lemma is immediate.

Lemma 4.9. The map Ψ defined in (4.1), is G -equivariant. That is, for any $\psi \in \text{Hom}(\mathfrak{l}, \mathfrak{g})$ and $g \in G$, we have:

$$\Psi(g \star \psi) = g \cdot \Psi(\psi).$$

Now we can state the following.

Proposition 4.10. Let G be a connected and simply connected nilpotent four dimensional threadlike Lie group, H be a closed connected subgroup of G and let Γ be an abelian discrete subgroup of G . The disjoint components $R_{j,k}$ involved through the description of the parameter space $\mathcal{R}(\Gamma, G, H)$ are G -invariant. More precisely, we have the following:

- i. If $k = 3$, then $\mathcal{T}(\Gamma, G, H) = (R_{0,3}/G)$.
- ii. If $k = 2$, then $\mathcal{T}(\Gamma, G, H) = \coprod_{j=0}^3 (R_{j,2}/G)$.
- iii. If $k = 1$, then $\mathcal{T}(\Gamma, G, H) = (R_{0,1}/G) \coprod (R_{1,1}/G)$.

Proof. Let us first prove that the set $R_{0,k}$ is G -stable. It is clear that the G -action on $R_{0,k}$ is reduced to the action of $\exp \mathbb{R}X$. Let $M = \begin{pmatrix} \vec{0} \\ N \end{pmatrix} \in R_{0,k}$ and $t \in \mathbb{R}$, then:

$$\exp tX \cdot M = \begin{pmatrix} \vec{0} \\ A(t)N \end{pmatrix}. \quad (4.14)$$

From the G -invariance of $\mathcal{R}(\Gamma, G, H)$, we get that $\exp tX \cdot M \in \mathcal{R}(\Gamma, G, H)$, so we are done in this case.

Suppose now that $k = 2$. Let $M = \begin{pmatrix} x & \lambda x \\ y_1 & \lambda y_1 \\ y_2 & \lambda y_2 \\ y_3 & y'_3 \end{pmatrix} \in R_{3,2}$

and let $g = \exp(tX + a_1Y_1 + a_2Y_2 + a_3Y_3) \in G$ for some $t, a_1, a_2, a_3 \in \mathbb{R}$, then a routine computation shows that:

$$g \cdot M = \begin{pmatrix} x & \lambda x \\ y_1 & \lambda y_1 \\ y_2(g) & \lambda y_2(g) \\ y_3(g) & y'_3(g) \end{pmatrix}$$

where

$$y_2(g) = y_2 + (ty_1 - xa_1),$$

$$y_3(g) = y_3 + \frac{t}{2}(ty_2 - xa_2) + (ty_1 - xa_1)$$

and

$$y'_3(g) = y'_3 + \frac{t}{2}(ty_2 - xa_2) + (ty_1 - xa_1),$$

As $\lambda y_3(g) - y'_3(g) = \lambda y_3 - y'_3$, we get that $g \cdot M \in R_{3,2}$ as was to be shown. We opt for the same arguments to show that $R_{1,2}$ and $R_{2,2}$ are G -invariant as well. Suppose finally that $k = 1$. Let $M = \begin{pmatrix} x \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \in R_{1,1}$ and

$g \in G$ as above. Then:

$$g \cdot M = \begin{pmatrix} x \\ y_1 \\ y_2 + (ty_1 - xa_1) \\ y_3 + \frac{t}{2}(ty_2 - xa_2) + (ty_1 - xa_1) \end{pmatrix}$$

which gives rise to the G -invariance of $R_{1,1}$. Finally, using Proposition 4.5 which describes the structure of the parameter space $\mathcal{R}(\Gamma, G, H)$, we can deduce that:

$$\mathcal{T}(\Gamma, G, H) = \coprod_{j \in I_k} (R_{j,k}/G).$$

This completes the proof of the proposition. \square

We are now ready to give an explicit description of the deformation space $\mathcal{T}(\Gamma, G, H)$. Towards this purpose, we can divide the task into three parts as in the previous section. More precisely, we shall define a cross-section of $R_{j,k}/G$ denoted by $\mathcal{T}_{j,k}$ for any $j \in I_k$ and $k \in \{1, 2, 3\}$. Recall that if $4 - p < k$ and $k \geq 2$, then we got $\mathcal{R}(\Gamma, G, H) = \emptyset$. We suppose then that

$k \leq 4 - p$. Let $m, n \in \mathbb{N}$ and denote for any $1 \leq r \leq n$ and any $1 \leq s \leq m$ by $M_{n,m}(r, s, \mathbb{R})$ the subset of $M_{n,m}(\mathbb{R})$ defined by:

$$M_{n,m}(r, s, \mathbb{R}) = \left\{ \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & x_{rs} & * & \dots & * \\ * & \dots & * & * & * & \dots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \dots & * & * & * & \dots & * \end{pmatrix} \in M_{n,m}(\mathbb{R}) : x_{rs} \in \mathbb{R}^* \right\},$$

and

$$M'_{n,m}(r, s, \mathbb{R}) = \{M \in M_{n,m}(r, s, \mathbb{R}) : x_{(r+1)s} = 0\}.$$

We now consider the sets:

$$R_{0,k}(r, s) = R_{0,k} \cap M_{4,k}(r, s, \mathbb{R}).$$

For any $k \in \{1, 2, 3\}$, let J_k designate the set of all $(r, s) \in \{1 \leq r \leq 4\} \times \{1 \leq s \leq k\}$ for which $R_{0,k}(r, s) \neq \emptyset$, then $(1, s) \notin J_k$ and we have:

$$\mathcal{R}(\Gamma, G, H) = \coprod_{(r,s) \in J_k} R_{0,k}(r, s)$$

whenever $k = 3$. Moreover, it is not hard to see that $R_{0,k}(r, s)$ is G -invariant. That is:

$$R_{0,k}/G = \coprod_{(r,s) \in J_k} (R_{0,k}(r, s)/G).$$

Let for any $(r, s) \in J_k$:

$$\mathcal{T}_{0,k}(r, s) = R_{0,k}(r, s) \cap M'_{4,k}(r, s, \mathbb{R}) = R_{0,k} \cap M'_{4,k}(r, s, \mathbb{R}),$$

so, with the above in mind, we have the following:

Proposition 4.11. *We keep all our notation as above. Then we have:*

i. $\mathcal{T}_{0,k}(r, s)$ is homeomorphic to $R_{0,k}(r, s)/G$ for any $(r, s) \in J_k$.

ii. $\mathcal{T}_{0,k} = \coprod_{(r,s) \in J_k} \mathcal{T}_{0,k}(r, s)$.

If in particular $k = 3$, then $\mathcal{T}(\Gamma, G, H) \simeq \mathcal{T}_{0,3}$.

Proof. Let $(r, s) \in J_k$. We show that $\mathcal{T}_{0,k}(r, s)$ is a cross-section of all adjoint orbits of $R_{0,k}(r, s)$. It is clear that the G -action on $R_{0,k}(r, s)$ is reduced to the action of

$\exp \mathbb{R}X$. More precisely, let $M = \begin{pmatrix} \vec{0} \\ N \end{pmatrix} \in R_{0,k}(r, s)$, then:

$$G \cdot M = [M] = \left\{ \begin{pmatrix} \vec{0} \\ A(t)N \end{pmatrix} : t \in \mathbb{R} \right\}. \quad (4.15)$$

Noting $N = \{(a_{i,j}), 1 \leq i \leq 3, 1 \leq j \leq k\}$, we get $a_{r-1,s} \neq 0$. Let:

$$t_M = \begin{cases} -\frac{a_{r,s}}{a_{r-1,s}} & \text{if } r < 4, \\ 0 & \text{if } r = 4. \end{cases}$$

We can then show that:

$$\left\{ \begin{pmatrix} \vec{0} \\ A(t_M)N \end{pmatrix} \right\} = G \cdot M \cap \mathcal{T}_{0,k}(r, s). \quad (4.16)$$

Remark that if $r = 4$ then $k = 1$ and $G \cdot M = M$, so (4.16) holds. Suppose now that $r \leq 3$. It is then clear that $\begin{pmatrix} \vec{0} \\ A(t_M)N \end{pmatrix} \in G \cdot M \cap \mathcal{T}_{0,k}(r, s)$ using the G -invariance of the layer $R_{0,k}(r, s)$. Conversely, if $\begin{pmatrix} \vec{0} \\ A(t)N \end{pmatrix} \in \mathcal{T}_{0,k}(r, s)$ then, by an easy computation, we can see that $a_{r,s} + ta_{r-1,s} = 0$, which gives that $t = t_M$. The next step consists in showing that the map:

$$\begin{aligned} (\Phi_{0,k})_{(r,s)} : R_{0,k}(r, s)/G &\rightarrow \mathcal{T}_{0,k}(r, s) \\ [M] &\mapsto \begin{pmatrix} \vec{0} \\ A(t_M)N \end{pmatrix} \end{aligned} \quad (4.17)$$

is bijective. First of all, it is clear that $(\Phi_{0,k})_{(r,s)}$ is well defined. In fact, let $M_1, M_2 \in R_{0,k}(r, s)$ such that $[M_1] = [M_2]$. Then $(\Phi_{0,k})_{(r,s)}([M_2]) = G \cdot M_2 \cap \mathcal{T}_{0,k}(r, s) = G \cdot M_1 \cap \mathcal{T}_{0,k}(r, s) = (\Phi_{0,k})_{(r,s)}([M_1])$. For $M_1 = \begin{pmatrix} \vec{0} \\ N_1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} \vec{0} \\ N_2 \end{pmatrix} \in R_{0,k}(r, s)$ such that $(\Phi_{0,k})_{(r,s)}([M_1]) = (\Phi_{0,k})_{(r,s)}([M_2])$, we have:

$$\begin{pmatrix} \vec{0} \\ A(t_{M_1})N_1 \end{pmatrix} = \begin{pmatrix} \vec{0} \\ A(t_{M_2})N_2 \end{pmatrix} \in G \cdot M_1 \cap G \cdot M_2.$$

It follows therefore that $[M_1] = [M_2]$, which leads to the injectivity of $(\Phi_{0,k})_{(r,s)}$. Now, to see that $(\Phi_{0,k})_{(r,s)}$ is surjective, it is sufficient to verify that for all $M \in \mathcal{T}_{0,k}(r, s)$, we have $(\Phi_{0,k})_{(r,s)}([M]) = M$ as $G \cdot M \cap \mathcal{T}_{0,k}(r, s) = \{M\}$. To achieve the proof, we prove that $(\Phi_{0,k})_{(r,s)}$ is bi-continuous. Let $(\pi_{0,k})_{(r,s)} : R_{0,k}(r, s) \rightarrow R_{0,k}(r, s)/G$ be the canonical surjection. Thus, we can easily see the continuity of $(\tilde{\Phi}_{0,k})_{(r,s)} = (\Phi_{0,k})_{(r,s)} \circ (\pi_{0,k})_{(r,s)}$ which is equivalent to the continuity of $(\Phi_{0,k})_{(r,s)}$. Finally, it is clear that

$$((\Phi_{0,k})_{(r,s)})^{-1} = ((\pi_{0,k})_{(r,s)})|_{\mathcal{T}_{0,k}(r,s)},$$

so the bi-continuity follows. \square

Proposition 4.12. Assume that $k = 2$, then $\mathcal{T}_{0,2}$ is described in Proposition 4.11 and $R_{j,2}/G$ is homeomorphic to $\mathcal{T}_{j,2}$ for $j = 1, 2, 3$ given as follows:

(i). if $\mathfrak{h} \not\subset \mathfrak{g}_0$, then we have the following subcases:

i_1 . if $p = 2$, then $\mathcal{T}_{j,2} = \emptyset, j = 1, 2, 3$.

i_2 . if $p = 1$, then:

$$\mathcal{T}_{1,2} = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \\ 0 & 0 \\ 0 & \beta \end{pmatrix} \in M_{4,2}(\mathbb{R}) : x, \beta, y \in \mathbb{R}^* \right\}, \quad (4.18)$$

$$\mathcal{T}_{2,2} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \\ 0 & 0 \\ \beta & 0 \end{pmatrix} \in M_{4,2}(\mathbb{R}) : x, \beta \in \mathbb{R}^*, y \in \mathbb{R} \right\} \quad (4.19)$$

and

$$\mathcal{T}_{3,2} = \left\{ \begin{pmatrix} x & \lambda x \\ y & \lambda y \\ 0 & 0 \\ 0 & \beta \end{pmatrix} \in M_{4,2}(\mathbb{R}) : x \in \mathbb{R}^*, y \in \mathbb{R}, \beta \in \mathbb{R}^*, \lambda \in \mathbb{R}^* \right\}. \quad (4.20)$$

(ii) If $\mathfrak{h} \subset \mathfrak{g}_0$, then:

ii_1 . if $Y_3 \in \mathfrak{h}$, then $\mathcal{T}_{j,2} = \emptyset, j = 1, 2, 3$.

ii_2 . If $Y_3 \notin \mathfrak{h}$, then $\mathcal{T}_{j,2}, j = 1, 2, 3$ are given by:

$$\mathcal{T}_{1,2} = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \\ 0 & 0 \\ 0 & \beta \end{pmatrix} \in M_{4,2}(\mathbb{R}) : x, \beta \in \mathbb{R}^*, y \in \mathbb{R} \right\}, \quad (4.21)$$

$$\mathcal{T}_{2,2} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \\ 0 & 0 \\ \beta & 0 \end{pmatrix} \in M_{4,2}(\mathbb{R}) : x \in \mathbb{R}^*, \beta \in \mathbb{R}^* \right\} \quad (4.22)$$

and

$$\mathcal{T}_{3,2} = \left\{ \begin{pmatrix} x & \lambda x \\ y & \lambda y \\ 0 & 0 \\ 0 & \beta \end{pmatrix} \in M_{4,2}(\mathbb{R}) : (x, \beta, \lambda) \in (\mathbb{R}^*)^3 \right\} \quad (4.23)$$

Proof. It is clear that whenever $R_{j,2} = \emptyset, j = 1, 2, 3$ we have $\mathcal{T}_{j,2} = \emptyset, j = 1, 2, 3$ and therefore $\mathcal{T}(\Gamma, G, H) \simeq \mathcal{T}_{0,2}$. So, we only have to treat the case when $\mathfrak{h} \not\subset \mathfrak{g}_0$ for $p = 1$ and the case when $Y_3 \notin \mathfrak{h}$ and $\mathfrak{h} \subset \mathfrak{g}_0$. Let in a

first time $M = \begin{pmatrix} x & \lambda x \\ y_1 & \lambda y_1 \\ y_2 & \lambda y_2 \\ y_3 & \lambda y_3 + \beta \end{pmatrix} \in R_{3,2}$. We get that:

$$G \cdot M = \left\{ \begin{pmatrix} x & \lambda x \\ y_1 & \lambda y_1 \\ a & \lambda a \\ b & \lambda b + \beta \end{pmatrix} : b \in \mathbb{R}, a \in \mathbb{R} \right\}. \quad (4.24)$$

We define then a subset $\mathcal{T}_{3,2}$ of $R_{3,2}$ as in (4.20), we get:

$$G \cdot M \cap \mathcal{T}_{3,2} = \left\{ \begin{pmatrix} x & \lambda x \\ y_1 & \lambda y_1 \\ 0 & 0 \\ 0 & \beta \end{pmatrix} \right\}.$$

The case where $M \in R_{1,2}$ is settled with the same way as above by taking $\lambda = 0$. We have:

$$G \cdot M \cap \mathcal{T}_{1,2} = \left\{ \begin{pmatrix} x & 0 \\ y_1 & 0 \\ 0 & 0 \\ 0 & \beta \end{pmatrix} \right\}.$$

Finally, similar computations show:

$$G \cdot M \cap \mathcal{T}_{2,2} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y_1 \\ 0 & 0 \\ \beta & 0 \end{pmatrix} \right\}.$$

We now see earlier that for $j = 1, 2, 3$, the set $\mathcal{T}_{j,2}$ is homeomorphic to $R_{j,2}/G$. This achieves the proof of the proposition. \square

Proposition 4.13. *Assume that $k = 1$. The layer $\mathcal{T}_{0,1}$ being described in Proposition 4.11), we have that $R_{1,1}/G$ is homeomorphic to $\mathcal{T}_{1,1}$, where:*

i. if $\mathfrak{h} \not\subset \mathfrak{g}_0$, then:

$$\mathcal{T}_{1,1} = \left\{ \begin{pmatrix} x \\ y_1 \\ 0 \\ 0 \end{pmatrix} \in M_{4,1}(\mathbb{R}) : x \in \mathbb{R}^*, y_1 \in \mathbb{R}^* \right\}.$$

ii. If $\mathfrak{h} \subset \mathfrak{g}_0$, then:

$$\mathcal{T}_{1,1} = \left\{ \begin{pmatrix} x \\ y_1 \\ 0 \\ 0 \end{pmatrix} \in M_{4,1}(\mathbb{R}) : x \in \mathbb{R}^*, y_1 \in \mathbb{R} \right\}.$$

Proof. For $M = \begin{pmatrix} x \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \in R_{1,1}$, we have

$$G \cdot M = \left\{ \begin{pmatrix} x \\ y_1 \\ a \\ b \end{pmatrix} : (a, b) \in \mathbb{R}^2 \right\}.$$

Hence, for $\mathcal{T}_{1,1}$ as above, we get:

$$G \cdot M \cap \mathcal{T}_{1,1} = \left\{ \begin{pmatrix} x \\ y_1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Moreover it is clear that the sets $R_{1,1}/G$ and $\mathcal{T}_{1,1}$ are homeomorphic, which completes the proof of the proposition. \square

Summarizing section 4 and 5, we have then given a proof to the following:

Theorem 4.14. *Let G be a connected and simply connected threadlike nilpotent Lie group, H a closed connected subgroup of G and $\Gamma \simeq \mathbb{Z}^k$ a discrete subgroup of G . Then the deformation space $\mathcal{T}(\Gamma, G, H)$ is described as follows:*

$$\mathcal{T}(\Gamma, G, H) = \coprod_{(r,s) \in J_k} \mathcal{T}_{0,k}^{r,s}(\Gamma, G, H) \coprod_{j \in I_k \setminus \{0\}} \mathcal{T}_{j,k}(\Gamma, G, H),$$

where $\mathcal{T}_{0,k}^{r,s}(\Gamma, G, H)$ is homeomorphic to $\mathcal{T}_{0,k}(r, s)$ for any $(r, s) \in J_k$ and $\mathcal{T}_{j,k}(\Gamma, G, H)$ to $\mathcal{T}_{j,k}$ for any $j \in I_k \setminus \{0\}$.

5 The rigidity problem

This section is devoted to the study of the topological property of the parameter space namely the local rigidity. We keep the same notation agreed on, in the previous sections. Our main result in this section is the following:

Theorem 5.1. *Let G be the four-dimensional threadlike Lie group, H be a connected subgroup of G , and Γ be a non-trivial discontinuous subgroup for G/H . Then local rigidity globally fails to hold on the parameter space.*

Proof. We have to show that the G -orbit of any $M \in \mathcal{R}(\Gamma, G, H)$ is not open in the parameter space. Assume first that $k = 3$ and let $M = \begin{pmatrix} \vec{0} \\ N \end{pmatrix} \in R_{0,3}$.

Writing $N = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \in M_{3,3}^0(\mathbb{R})$, the sequence $(M_s)_{s \in \mathbb{N}^*}$ given by

$$M_s = \begin{pmatrix} 0 & 0 & 0 \\ y_{11} & y_{12} & y_{13} \\ y_{21} + \frac{1}{s}y_{11} & y_{22} + \frac{1}{s}y_{12} & y_{23} + \frac{1}{s}y_{13} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

belongs to $\mathcal{R} \setminus G \cdot M$ and converges to M . Now for $k = 2$, thus either we are in the context where M belongs to $R_{0,2}$ and then the last arguments apply or

$$M = \begin{pmatrix} \lambda_1 x & \lambda_2 x \\ \lambda_1 y_1 & \lambda_2 y_1 \\ \lambda_1 y_2 & \lambda_2 y_2 \\ \lambda_1 y_3 & \lambda_2 y_3 + \beta \end{pmatrix} \in R_{j,2}.$$

As

$$G \cdot M = \left\{ \begin{pmatrix} \lambda_1 x & \lambda_2 x \\ \lambda_1 y_1 & \lambda_2 y_1 \\ \lambda_1 a & \lambda_2 a \\ b & \lambda b + \beta \end{pmatrix} : b \in \mathbb{R}, a \in \mathbb{R} \right\}.$$

Then it suffices to consider the sequence $(M_s)_{s \in \mathbb{N}^*}$ given by

$$M_s = \begin{pmatrix} \lambda_1 x & \lambda_2 x \\ \lambda_1 y_{1s} & \lambda_2 y_{1s} \\ \lambda_1 y_2 & \lambda_2 y_2 \\ \lambda_1 y_3 & \lambda_2 y_3 + \beta \end{pmatrix} \in \mathcal{R} \setminus G \cdot M,$$

where

$$y_{1s} = \begin{cases} y_1 + \frac{1}{s} & \text{if } y_1 > 0, \\ y_1 - \frac{1}{s} & \text{otherwise.} \end{cases}$$

Finally for $k = 1$, the same argument as in case $k = 2$ applies. \square

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