Graduate J. Math. 4 (2019), 1 – 17

Diophantine approximation and coboundary equations

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Abstract

In this mainly, but not only, expository paper, revisiting work of M. Herman, A. Rozhdestvenskii and Y. Meyer, we investigate the regularity of functions h solutions to the coboundary equation $h(x+\alpha)-h(x)=f(x)$, in terms of the function-theoretic properties of the given function f and of the diophantine approximation properties of the irrational number α. We also revisit the equation $h(2x)-h(x)=f(x)$.

MSC 2010. Primary: 37A20, 37A30, 37A45; Secondary: 11K60, 11J70.

1 Introduction

This paper is mainly a survey paper on a vast topic only some aspects of which will be addressed here.

The main problem we are interested in relates to elementary difference equations in the context of 1-periodic functions, i.e. functions defined on the circle $\mathbf{T}=\mathbf{R}/\mathbf{Z}$, the multiplicative group of unimodular complex numbers or equivalently the additive group of real numbers modulo one, equipped with its normalized Haar measure m. More precisely, we focus on equations of the following form:

$$h(x+\alpha)-h(x)=f(x), \quad x\in\mathbf{T} \qquad (1.1)$$
$$f \text{ measurable, h unknown,}$$

involving the rotation $R_\alpha : x\mapsto x+\alpha$ on $\mathbf{T}$, $\alpha\in\mathbf{T}$. The equality in (1.1) is to be understood almost everywhere (with respect to the Haar measure of $\mathbf{T}$).

Those equations, also called "additive coboundary equations" or "cohomological equations", appear very naturally in ergodic theory when looking at the spectral properties of group extensions over some given rotation [25, 10, 16, 33]; they result also from the linearisation method in the famous conjugation problem of diffeomorphisms of the circle (for which the notion of rotation number has been introduced), initiated by Poincaré and Denjoy [14, 35].

The framework of this inverse type problem, of course, has to be made precise (nature of α, regularity of f) and the first question that comes to mind is the existence of *measurable* solutions to equation (1.1).

When $\alpha=p/q\in\mathbf{Q}$, $(p,q)=1$, a necessary and sufficient condition for equation (1.1) to admit a (measurable) solution is

$$S_q := \sum_{j=0}^{q-1} f\circ R_{j\alpha}=0,$$

and in this case, h has the same regularity as f, no more no less.

This condition is clearly necessary since, for any solution h,

$$\begin{aligned} S_q(x) &= \sum_{j=0}^{q-1} h(x+(j+1)\alpha)-h(x+j\alpha) \\ &= h(x+q\alpha)-h(x)=0. \end{aligned}$$

It is sufficient since, for a given f, the function

$$h := \frac{1}{q}\sum_{j=0}^{q-1}(j+1)(f\circ R_{j\alpha})$$

provides a solution as is easily checked.

Let us next assume α to be **irrational**. We may invoke basic facts of *ergodic theory* and recall in particular that a dynamical system $(X,\mathcal{B},T,\mu)$, T preserving the probability measure μ on $(X,\mathcal{B})$, is *ergodic* (or T itself is ergodic) if invariant sets are "trivial" ($\forall B\in\mathcal{B},\ T^{-1}(B)=B \Rightarrow \mu(B)=0$ or 1); equivalently, if invariant measurable functions are constant ($f\circ T=f\Rightarrow f=$ constant a.e.).

The irrational rotation R_α is ergodic, thus the measurable solutions to equation (1.1) (when they exist) are unique up to an additive constant. By iterating once more, we get now

$$\begin{aligned} S_n(f)(x) &:= f(x)+f(x+\alpha)+\cdots+f(x+(n-1)\alpha) \\ &= h(x+n\alpha)-h(x) \end{aligned}$$

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for every $n \geq 1$, so that, if h is a measurable solution we must have

$$S_{n_j}(f)(x) = h(x + n_j\alpha) - h(x) \to 0 \quad \text{in measure, as } n_j\alpha \to 0 \text{ mod } 1.$$

Let us detail the latter observation under the form of a simple lemma.

Lemma 1.1. *Let $h : \mathbf{T} \to \mathbf{C}$ be a measurable function, and (t_j) a sequence of reals tending to zero* mod 1. *Then, $h(x + t_j) - h(x) \to 0$ in measure.*

Proof. We have to show that

$$I_j := \int_{\mathbf{T}} \frac{|h(x + t_j) - h(x)|}{1 + |h(x + t_j) - h(x)|} dm(x) \to 0.$$

Let $\varepsilon > 0$. By Lusin's theorem ([5] p. 208), there exists a continuous function $g : \mathbf{T} \to \mathbf{C}$ and a measurable (even compact) subset E of $\mathbf{T}$ such that $h = g$ on E and $m(E^c) \leq \varepsilon$. Let $E_j = E \cap (E - t_j)$. Note that $E_j^c = E^c \cup (E^c - t_j)$ has measure $\leq 2\varepsilon$ and that $h(x + t_j) - h(x) = g(x + t_j) - g(x)$ on E_j. Hence

$$\begin{aligned} I_j &\leq \int_{E_j} \frac{|h(x + t_j) - h(x)|}{1 + |h(x + t_j) - h(x)|} dm(x) + 2\varepsilon \\ &= \int_{E_j} \frac{|g(x + t_j) - g(x)|}{1 + |g(x + t_j) - g(x)|} dm(x) + 2\varepsilon \\ &\leq \int_{\mathbf{T}} \frac{|g(x + t_j) - g(x)|}{1 + |g(x + t_j) - g(x)|} dm(x) + 2\varepsilon. \end{aligned}$$

This implies $\limsup_{j\to\infty} I_j \leq 2\varepsilon$, ending the proof since ε is arbitrary. □

Besides, observe that, if h is some measurable solution, a non-measurable solution also exists. Indeed, by adding to h a weak character (non-measurable) γ, which satisfies

$$\gamma(x + \alpha) = \gamma(x)\gamma(\alpha), \quad \gamma(\alpha) = 1,$$

we get the non-measurable solution $g = h + \gamma$. The existence of such weak characters is folklore in harmonic analysis. A detailed presentation can be found e.g. in ([29] chapter 1). The idea is: take $\gamma(x) = e^{2i\pi g(x)}$, where g is a non-measurable solution of the Cauchy equation on the real line

$$g(x + y) = g(x) + g(y),$$

rational-valued, with $g(1) = g(\alpha) = 1$.
But we will see that (1.1) may have NO measurable solutions at all, even with an analytic right-hand side f.
If we are looking now for integrable solutions (in L^1 or L^2) the necessary condition

$$\sup_{n\geq 1} ||S_n(f)||_{1,2} < \infty$$

emerges if such an h exists. The converse will be useful (Lemma 4.1).

Actually, everything will depend on the **interplay** between the properties of the *function* f, when supposed to be integrable, and the diophantine properties of the irrational *number* α; more precisely, the situation will depend on the one hand on the regularity properties of f (restricted Fourier spectrum, or fast decay of the Fourier coefficients), and on the other hand on the speed of approximation of α by rationals with controlled denominator. Let us recall the classical notations in diophantine approximation: first of all, here $||\cdot|| = d(\cdot, \mathbf{Z})$ so that $4||x|| \leq |e(x) - 1| \leq 2\pi||x||$, where as usual we denote

$$e(x) = e^{2i\pi x}.$$

If (p_n/q_n) is the sequence of **convergents** to α given by the Continued Fraction expansion of α, it is well-known ([13] p.151 or [28] p. 74) that it provides the best rational approximations in the following strong sense:

$$\inf_{1\leq q<q_{n+1}} \|q\alpha\| = \|q_n\alpha\| \tag{1.2}$$

and that

$$q_n\Big|\alpha - \frac{p_n}{q_n}\Big| = \|q_n\alpha\| \leq \frac{1}{q_n}\cdot \tag{1.3}$$

Hence the speed above is completely described by the decay rate of $\|q_n\alpha\|$ and depends only on the sequence of **denominators** (q_n). Badly approximable numbers are those for which (1.3) is best possible i.e.

$$\alpha \in \mathbf{Bad} \iff \exists\ C > 0,\ q\|q\alpha\| \geq C, \quad q = 1, 2, \ldots. \tag{1.4}$$

Diophantine numbers α are those for which some constants $C, r > 0$ exist such that

$$\|q\alpha\| \geq Cq^{-r}, \quad q = 1, 2, \ldots;$$

the complementary set to diophantine numbers consists in **Liouville numbers**, explicitly those numbers α such that

$$\liminf_{q\to\infty}(q^r\|q\alpha\|) = 0 \quad \forall r > 0.$$

In this paper, we will mainly focus on four types of results, with simplified and extended proofs:

- Anosov type results: if $\int_{\mathbf{T}} f(x)dm(x) =: \widehat{f}(0) \neq 0$, no measurable solutions h exist.
- Herman type results: if $f \in L^2$ has restricted Fourier spectrum (in a sense to be precised), and if a measurable solution exists, then an L^2, and even better, solution exists, *whatever the decay rate of* $\|q_n\alpha\|$.
- Meyer-Rozhdestvenskii type results: if the decay rate of $\|q_n\alpha\|$ is slow (more precisely if $\alpha \in \mathbf{Bad}$ or if α is diophantine), then a *slight reinforcement* of the assumption $f' \in L^1$ will ensure the existence of highly integrable solutions, more precisely solutions $h \in \cap_{p<\infty} L^p$ with estimates on $\|h\|_p$ as $p \to \infty$.

- Fukuyama type results: here, we consider the new coboundary equation $g(2x) - g(x) = f(x)$. In this case, we can prove, under some restriction on the Fourier-Walsh spectrum of f, the existence of a square-integrable solution as soon as a measurable solution does exist.

A large number of contributions on coboundary equations has been ignored in this paper which is mainly organized around the notion of $\Lambda(p)$-set, due to W. Rudin [32]. In fact, this article arose from our reading of Herman's article [15] which cleverly and successfully mixed ergodic theory and harmonic analysis. It is easily observed that the suitable required hypothesis on the spectrum of f in Herman's main result was the $\Lambda(p)$-condition involving the Lebesgue L^p-spaces. Later on, generalizations of those L^p-spaces appeared in papers of Meyer [23] and Rozhdestvenskii [31] in connection with absolutely continuous coboundaries.

In this work, we make a tentative link between harmonic analysis (thin sets of integers like $\Lambda(p)$ or Sidon sets, ...) and diophantine approximation (badly approximable numbers, continued fraction expansion,...).

Throughout the paper, we will assume that $\alpha \notin \mathbf{Q}$. Details and references on ergodic theory can be found in [25], and on diophantine approximation in [28].

2 Existence of measurable solutions

A. Wintner [34] seems to have been the first mathematician to raise the question of the regularity of the possible solutions of equation (1.1). He made the now well-known observation, allowing the use of Fourier techniques, that if $f \in L^1(\mathbf{T})$ is a given integrable function, the existence of some *integrable* solution clearly implies that $\widehat{f}(0) = 0$. But this remains true as soon as (1.1) admits a *measurable* solution by pioneering work of Anosov [1]. Fourier arguments cannot help in looking for possible measurable solutions and, actually, the existence of some measurable h is not automatic at all, whatever the regularity of f.

Proposition 2.1. *Suppose $f \in L^1$. If (1.1) admits a measurable solution, then necessarily $\widehat{f}(0) = 0$.*

Proof. We prove the result in a more general context. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and suppose by contradiction that there exists h measurable with $h(Tx) - h(x) = f(x)$ (μ a.e.), while $\int_X f d\mu > 0$ (f can be assumed real); put

$$S_n = \sum_{0 \le j < n} f \circ T^j;$$

by the ergodic theorem, $S_n/n \to \int f d\mu > 0$ almost everywhere, whence $S_n \to \infty$ almost everywhere (w.r. to μ). If we denote

$$E_n = \{\inf_{p \ge n} S_p > 2\},$$

we can find n with $\mu(E_n) > 0$. We decompose $E_n = \bigsqcup_{\ell \in \mathbf{Z}} E_{n,\ell}$ where

$$E_{n,\ell} := \{h \in [\ell, \ell+1[\} \cap E_n$$

so that $\mu(E_{n,\ell}) > 0$ for some $\ell \in \mathbf{Z}$. The Poincaré recurrence theorem ensures that

$$\mu(T^{-k} E_{n,\ell} \cap E_{n,\ell}) > 0 \text{ for infinitely many } k.$$

This leads to a contradiction: actually, for x in such an intersection with $k > n$, we have $h(T^k x), h(x) \in [\ell, \ell+1[$ and $h(T^k x) - h(x) \in [0, 1[$; but then

$$h(T^k x) - h(x) = S_k(x) \ge \inf_{p \ge n} S_p(x) > 2$$

since $x \in E_n$, and the proposition is proved by reductio ad absurdum. □

As Wintner observed, it is easy to construct a zero-mean integrable function f and an irrational number α such that equation (1.1) has no integrable solution. Of course if a solution $h \in L^1$ exists, we appeal to the Fourier techniques; by identification of the Fourier coefficients, since

$$\int_{\mathbf{T}} f(x) e(-kx) dm(x) =: \widehat{f}(k) = (e(k\alpha) - 1)\widehat{h}(k), \ k \in \mathbf{Z}, \tag{2.1}$$

we would have $\sup_{k \in \mathbf{Z}} |\widehat{h}(k)| < \infty$. It is just enough to ensure that

$$\sup_{k \neq 0} |\widehat{f}(k)| / ||k\alpha|| = \infty \tag{2.2}$$

to get a contradiction, and this can be realized step by step. If now f is given and satisfies some regularity condition, this contrast between f and the solution h still holds by adjusting the number α.

Proposition 2.2. *For any continuous and zero mean function f, which is not a trigonometric polynomial, there exists an irrational number α such that (1.1) has no integrable solution.*

Proof. As above, assume that

$$\widehat{f}(k) = (e(k\alpha) - 1)\widehat{h}(k), \ k \in \mathbf{Z}, \quad \text{and} \quad \sup_{k \in \mathbf{Z}} |\widehat{h}(k)| < \infty.$$

It is then sufficient, (f, thus $(\widehat{f}(k))$ being given), to construct an α such that the sequence $\left(\widehat{f}(k)/||k\alpha||\right) \notin \ell^\infty$; but by our assumption on f, an infinite sequence (n_k) can be found with $\widehat{f}(n_k) \neq 0$; it remains to choose α as a Liouville-type number so that

$$|\widehat{f}(n_k)| / ||n_k \alpha|| \to \infty.$$

□

In the next two theorems, we are looking finally towards the existence of solutions, with a regularity as weak as possible. Since we are dealing with only measurable h, Fourier techniques are no more relevant.

Theorem 2.1. *For* every *irrational number α, there exists a continuous function f such that (1.1) admits (a.e.) a measurable solution which is not integrable.*

We can push up this result to the analytic case. Let us first recall the notation $C^\omega(\mathbf{T})$.

Definition 2.1. *The function $g \in C^\omega(\mathbf{T})$ (or g is said to be "analytic") if g can be analytically extended to some annulus $1-\gamma < |z| < 1+\gamma$, $\gamma > 0$, in $\mathbf{C}$.*

Theorem 2.2. *There exists a zero mean $f \in C^\omega(\mathbf{T})$ and* some *$\alpha \notin \mathbf{Q}$ such that (1.1) admits (a.e.) a measurable solution which is not integrable.*

We now give a detailed proof of these two theorems.

Proof. (of Theorem 2.1.) The number α (thus its denominators (q_n)) being fixed, we have to construct a continuous coboundary f and we express the candidate as a series.

Consider for every n the so-called triangle function Δ_n,

$$\Delta_n(x) = 2^n \max(0, 1 - 2^n|x|), \quad |x| \le 1/2$$

and extended by 1-periodicity; one has $\int_{\mathbf{T}} \Delta_n dm = 1$, and the measure of the set $\{\Delta_n \ge 2^{-n}\}$ is clearly less than 2^{-n+1}. Since

$$\sum_{n=1}^{\infty} m(\Delta_n(q_{k_n}x) > 2^{-n}) = \sum_{n=1}^{\infty} m(\Delta_n > 2^{-n}) < \infty,$$

the series

$$h(x) := \sum_{1}^{\infty} \Delta_n(q_{k_n}x)$$

converges almost everywhere for any subsequence (q_{k_n}) of (q_n). Such a non-negative function h, defined a.e., cannot belong to L^1:

$$\int_{\mathbf{T}} h dm = \sum_{1}^{\infty} \int_{\mathbf{T}} \Delta_n dm = \infty.$$

We are left with proving that h is a solution (a.e.) of the coboundary equation with the continuous righthand-side f; for that, we construct step by step the subsequence (q_{k_n}) in order to control the modulus of uniform continuity of Δ_n:

$$\omega_n(h) = \sup_{|x-y] \le h} |\Delta_n(x) - \Delta_n(y)|;$$

more precisely we adjust the q_{k_n}'s so as to get

$$\sum_{n=1}^{\infty} \omega_n(1/q_{k_n+1}) < \infty. \tag{2.3}$$

In this way, the series with general term

$$\varphi_n(x) = \Delta_n(q_{k_n}x + q_{k_n}\alpha) - \Delta_n(q_{k_n}x)$$

is normally convergent on $\mathbf{T}$, since

$$|\varphi_n(x)| \le \omega_n(||q_{k_n}\alpha||) \le \omega_n(1/q_{k_n+1}).$$

Let us now put $f(x) = \sum_{n=1}^{\infty} \varphi_n(x)$; this function f is continuous, by construction $\int_{\mathbf{T}} f dm = 0$, and it satisfies almost everywhere

$$f(x) = h(x+\alpha) - h(x),$$

which was to be proved. □

Proof. (of Theorem 2.2.) This time, we have to exhibit a function f, once more under the form of a series $\sum_n \phi_n$, and a sequence of denominators (q_n), in such a way that f is a coboundary for the rotation R_α and can be analytically extended to some neighbourhood of the circle. We start by producing a sequence of **trigonometric polynomials,** (ψ_n), which take place of Δ_n, and satisfy

1. $\psi_n \ge 0$
2. $\sum_{n=1}^{\infty} m(\psi_n > 2^{-n}) < \infty$
3. $\int_{\mathbf{T}} \psi_n dm \ge 1/2$.

For that, let us approach each function Δ_n by a trigonometric polynomial $\psi_n = \Delta_n * K_N$, where K_N is the Fejér kernel of order N, $N = N(n)$ being chosen so as to satisfy:

$$||\psi_n - \Delta_n||_\infty \le 2^{-n}.$$

Thus, $\psi_n(x) > 2^{-n}$ implies $|x| \le 2^{-n+1}$ ensuring 2., while conditions 1. and 3. are fulfilled as well. We now denote by

$$\omega_n(h) = \sup_{|x-y| \le h} |\psi_n(x) - \psi_n(y)|$$

the modulus of uniform continuity of ψ_n and we construct step by step a sequence (q_n), with $q_0 = 1$, $q_1 = a$, in order that

$$\sum_{n=1}^{\infty} \omega_n(1/q_{n+1}) < \infty \tag{2.4}$$

and

$$q_n \text{ divides } q_{n+1} - q_{n-1}. \tag{2.5}$$

(Hence, (q_n) will be the sequence of denominators of some irrational number α). As above, we put

$$h(x) = \sum_{n=1}^{\infty} \psi_n(q_n x) \ge 0;$$

since $\sum_{n=1}^{\infty} m(\psi_n(q_n x) > 2^{-n}) < \infty$, h is defined almost everywhere, nevertheless $h \notin L^1$ since

$$\int_{\mathbf{T}} h dm = \sum_{n=1}^{\infty} \int_{\mathbf{T}} \psi_n = \infty \text{ by 3.}$$

The series with general term

$$\varphi_n(x) = \psi_n(q_n x + q_n\alpha) - \psi_n(q_n x)$$

is normally convergent on $\mathbf{T}$ thanks to (2.4), and h satisfies (a.e.)

$$h(x+\alpha) - h(x) = f(x) := \sum_{n=1}^{\infty} \varphi_n(x).$$

We have one last task left: the analytic extension of the function f to the annulus $C := \{1/2 < |z| < 3/2\}$, up to a better choice of the (q_n). It is easy to extend analytically each trigonometric polynomial ψ_n to the punctured complex plane $\mathbf{C} \setminus \{0\}$ by putting

$$\psi_n(z) := \sum_{k\in\mathbf{Z}} \widehat{\psi_n}(k) z^k$$

then

$$\phi_n(z) = \psi_n(z^{q_n} e(q_n\alpha)) - \psi_n(z^{q_n}).$$

To proceed, we consider

$$\omega_{n,C}(h) := \sup_{\substack{|z-w|\le h\\ z,w\in C}} |\psi_n(z) - \psi_n(w)|$$

and, during the inductive construction of the q_n's, we replace the constraint (2.4) (choice of q_{n+1}) by a stronger one implying the following:

$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \omega_{n,C}(1/q_{n+1}) < \infty, \tag{2.6}$$

which is always possible. Since

$$\sup_C |\phi_n(z)| \le \left(\frac{3}{2}\right)^n \omega_{n,C}(||q_n\alpha||)$$

the function $f = \sum_{n=1}^{\infty} \phi_n$ is now well defined and analytic on C, which is what was left to prove. □

In the next sections, we refer to this last property (theorem (2.2)) as the "Anosov's phenomenon".

3 Reminders of harmonic analysis

We recall in this section some basic facts on "thin sets" in harmonic analysis to be used in subsequent sections.

3.1 Fourier and harmonic Analysis

Let G be a compact abelian group, equipped with its normalized Haar measure m. The character group Γ of G is the (discrete) group of continuous homomorphisms of G to $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ (see for example [32] or [22]).

A first example, which is essential for our diophantine purposes, is the already defined group $G = \mathbf{T}$ itself with m the arc-length measure. The dual group of $\mathbf{T}$ is the group $\mathbf{Z}$ of integers, with the action $n \mapsto e_n$ where

$$e_n(t) = [e(t)]^n = \exp(2i\pi nt),\ n \in \mathbf{Z},\ t \in [0,1[. \tag{3.1}$$

Another interesting example, which will show up in this work, is the discrete group $\Omega = \{-1,1\}^{\mathbf{N}}$ of choices of signs $\omega = (r_n(\omega))_{n\ge1}$, with Haar measure denoted P. The functions $\omega \mapsto r_n(\omega)$ appear as P-independent, identically distributed random variables, the so-called Rademacher variables. The dual group $\widehat{\Omega}$ of Ω is formed by the so-called Walsh functions, namely

$$w_\emptyset = 1,\ w_A = \prod_{n\in A} r_n,\ A \subset \mathbf{N},\ A \text{ finite}.$$

These Walsh functions can be realized as functions on $\mathbf{T}$. If the regular binary expansion of $x \in \mathbf{T} \sim [0,1)$ is $x = \sum_{j=1}^{\infty} x_j 2^{-j}$, we set:

$$r_0 = 1,\ r_j(x) = 1 - 2x_j = (-1)^{x_j},\ j \ge 1;$$

the (r_j) are now the *Rademacher functions on* $(\mathbf{T}, m)$. Let $n = \sum_1^{\infty} n_j 2^{j-1}$ be the binary decomposition of n; the Walsh function (w_n) (with respect to the Paley ordering), are defined by

$$w_0 = 1,\ w_n(x) = \prod_{j=1}^{\infty} (-1)^{n_j x_j},\ n \ge 1.$$

If $n-1 = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}$ with $0 \le k_1 < k_2 < \cdots < k_s$, we recover the notation $w_n = w_A = \prod_{j\in A} r_j$ with $A = \{k_1+1, k_2+1, \ldots, k_s+1\}$, so that $|A| =: [n] = s$. The mapping $\varphi : \omega \in \Omega \mapsto \sum_1^{\infty} \frac{1+\omega_j}{2} 2^{-j} \in \mathbf{T}$ exchanges the Haar measure on Ω and the Lebesgue measure on $\mathbf{T}$; the spaces $L^1(\Omega)$ and $L^1(\mathbf{T})$ are thus isometrically isomorphic. Since φ is continuous and onto, $f \mapsto f\circ\varphi$ is an isometry from $C(\mathbf{T})$ to $C(\Omega)$ and the dual mapping $M(\Omega) \to M(\mathbf{T})$ is onto. We thus identify the two points of view.

The subset

$$\Lambda_s := \{w_A : |A| = s\} = \{w_n : [n] = s\} \subset \widehat{\Omega} \tag{3.2}$$

will play an important role in the final section. Note that Λ_1 is the set of Rademacher variables.

If μ is a complex Borel measure on G, its Fourier transform $\widehat{\mu} : \Gamma \to \mathbf{C}$ is defined by

$$\widehat{\mu}(\gamma) = \int_G \gamma(-t) d\mu(t).$$

The (Fourier) spectrum $sp(\mu)$ of μ is by definition:

$$sp(\mu) = \{\gamma \in \Gamma : \widehat{\mu}(\gamma) \ne 0\}. \tag{3.3}$$

If $f \in L^1$, the Fourier transform of f at γ is that of the absolutely continuous measure $\mu = f dm$. If $\Lambda \subset \Gamma$ and $X \subset L^1$ is a Banach space of integrable functions on G, we denote by X_Λ the subspace of X formed by those functions with Fourier spectrum contained in Λ, that is

$$f \in X_\Lambda \Longleftrightarrow f \in X \text{ and } \widehat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda. \tag{3.4}$$

Often, assuming that $f \in X$ has a restricted Fourier spectrum, namely $f \in X_\Lambda$ with Λ small, implies that

f is better than the generic element of X. An extreme example is $\Lambda = \emptyset$, then $f \in X_\Lambda \Rightarrow f = 0$! Since we can hardly imagine a thinner set than the empty one, sets Λ such that X_Λ is formed by much better behaved functions will be called, by extension, "thin sets". For more on this topic, we refer to [21]. We will essentially concentrate here on one type of thin sets, the $\Lambda(p)$-sets (of which Sidon sets are a basic example).

3.2 $\Lambda(p)$-sets

3.2.1 Definitions

Let p a real number with $2 < p < \infty$. *A subset* Λ *of* Γ *is called a* $\Lambda(p)$*-set if there exists a constant* λ *such that*

$$\|f\|_p \leq \lambda \|f\|_2$$

for each trigonometric polynomial f *with spectrum contained in* Λ.

We denote by $\lambda_p(\Lambda)$ the best possible constant λ. Bourgain [3] proved that, for each $2 < p < \infty$, there exists a "true" $\Lambda(p)$-set, i.e. a set Λ which is $\Lambda(p)$, but not $\Lambda(q)$ for any $q > p$. Whether there exist "true" $\Lambda(2)$-sets is an open question [3].

An important property of $\Lambda(p)$-sets with $p > 2$ is given by the following key lemma, which utilizes the equivalence of norms on L^2_Λ and L^p_Λ to prove that each set B with large measure is "associate", meaning that the L^2-norm of $\varphi \in L^2_\Lambda$ is essentially computable on B, up to a constant. The proof is a variant of a lemma due to Paley and Zygmund ([18] p. 31), and plays a key role in what follows.

Lemma 3.1. *Let* $p > 2$ *and let* Λ *be a* $\Lambda(p)$*-set with constant* $\lambda_p(\Lambda)$*. Then, there exists constants* $0 < b < 1$ *and* C*, both depending only on* $\lambda_p(\Lambda)$*, such that if* B *is a measurable subset of* Γ **with** $m(B) \geq b$**,** *one has for all* $\varphi \in L^2_\Lambda$

$$\|\varphi\|_2 \leq C\Big(\int_B |\varphi|^2\Big)^{1/2}.$$

Proof. Let $\varphi \in L^2_\Lambda$ with $\|\varphi\|_2 = 1$; we will show that $\int_B |\varphi|^2 \geq 1/C^2$ for some C, which implies the lemma. We fix a number $0 < a < 1$, set $E = \{|\varphi| > a\}$, $\lambda = \lambda_p(\Lambda)$ and first show that $m(E)$ is large when a is small. Precisely, if $r = p/2$ and r' is the conjugate exponent, we see that:

$$1 = \|\varphi\|_2^2 = \int_{|\varphi|\leq a} |\varphi|^2 + \int_E |\varphi|^2 \leq a^2 + \int_E |\varphi|^2$$

$$\leq a^2 + \|\varphi\|_p^2 \Big(m(E)\Big)^{1/r'} \leq a^2 + \lambda^2 \|\varphi\|_2^2 \Big(m(E)\Big)^{1/r'}$$

by Hölder and the assumption on Λ; it ensues that

$$1 \leq a^2 + \lambda^2 \Big(m(E)\Big)^{1/r'}$$

or again that

$$m(E) \geq \Big(\frac{1-a^2}{\lambda^2}\Big)^{r'}.$$

Next, choose $b = 1 - \frac{\left(\frac{1-a^2}{\lambda^2}\right)^{r'}}{2} \geq 1 - \frac{m(E)}{2}$. Assume that $m(B) \geq b$; then

$$m(B \cap E) \geq m(B) + m(E) - 1 \geq \frac{m(E)}{2} \geq \frac{\left(\frac{1-a^2}{\lambda^2}\right)^{r'}}{2},$$

which entails

$$\int_B |\varphi|^2 \geq \int_{B\cap E} |\varphi|^2 \geq \frac{a^2}{2}\Big(\frac{1-a^2}{\lambda^2}\Big)^{r'} =: 1/C^2.$$

□

Remarks. **1.** The preceding lemma claims that every Borel set B of $\mathbf{T}$ with Haar measure *sufficiently close to* 1 is associate for L^2_Λ with a constant depending only on $m(B)$. If $\Lambda = (\lambda_n)_{n\geq 1}$ is a Hadamard *lacunary set* of positive integers (meaning that $\lambda_{n+1}/\lambda_n \geq q > 1$), it can be proved that *every* set B with positive Haar measure $b > 0$ is C_b-associate, with a constant C_b depending only on b. But this is a much deeper result due to Nazarov, Nishry and Sodin [24], which will not be needed here.

2. Since we make use of it once in the end, the following "extrapolation" property is worth noting: if Λ is a $\Lambda(p)$-set with $p > 2$, there exists $\delta > 0$ such that, for every $f \in \mathcal{P}_\Lambda$,

$$\|f\|_1 \geq \delta \|f\|_2.$$

Indeed, if $\lambda = \lambda_p(\Lambda)$, Hölder's inequality, with $1/2 = (1-\theta)/1 + \theta/p$, gives

$$\|f\|_2 \leq \|f\|_1^{1-\theta} \|f\|_p^\theta \leq \|f\|_1^{1-\theta} \lambda^\theta \|f\|_2^\theta$$

whence

$$\|f\|_2 \leq \lambda^{\frac{\theta}{1-\theta}} \|f\|_1.$$

3.2.2 Examples

In the next subsection, we will define the Sidon sets and comment on the fact that they are $\Lambda(p)$-sets for all $2 < p < \infty$. Here are two other examples (indeed extensions of the Sidon case) in the framework of $\mathbf{Z}$ [22].

Theorem 3.1. *Let* $p_1, \ldots, p_s$ *be* s *distinct prime numbers. Then, the following sets are* $\Lambda(p)$*-sets in* $\mathbf{Z}$ *with* $\lambda_p(\Lambda) \lesssim p^{s/2}$ *for the first one and* $\lesssim p^s$ *for the second:*

1. $\Lambda = \{\lambda_n = \sum_{j=1}^s p_j^{n_j}, \quad n_j = 1, 2, \ldots\}$
2. $\Lambda = \{\lambda_n = \prod_{j=1}^s p_j^{n_j}, \quad n_j = 1, 2, \ldots\}$

A typical example of the first class is the set of integers $2^i + 3^j$. A typical example of the second class (less well known) is the so-called Fürstenberg sequence, the increasing rearrangement (f_n) of integers $2^i \times 3^j$. This example has been studied by Gundy and Varopoulos in this context [11].

We now give examples in the framework of Walsh functions, to be used in the final Section. We first have the classical ([22], Vol.1, p. 30):

Theorem 3.2. *Let $S = \sum_{n=1}^{\infty} z_n r_n(\omega)$ a finite sum, with $z_n \in \mathbf{C}$. Then*

$$\|S\|_p \le \sqrt{p}\,\|S\|_2 = \sqrt{p}\Big(\sum_{n=1}^{\infty} |z_n|^2\Big)^{1/2} \text{ for all } p \ge 2.$$

This $\Lambda(p)$-property of Rademacher functions extends to the set Λ_s (3.2):

Theorem 3.3. *Let $p > 2$ and $(a_k)_{k\ge 0}$ be a sequence of complex numbers. Then, for every integer $s \ge 1$:*

$$\| \sum_{k,[k]=s} a_k w_k \|_p \le (p-1)^{s/2} \| \sum_{k,[k]=s} a_k w_k \|_2.$$

Actually, Bonami and Borell ([2], [4], see also [22]) proved a more precise hypercontractivity result:

Theorem 3.4. *Let $1 < q < p$, $\lambda = \sqrt{\frac{q-1}{p-1}}$ and $(a_k)_{k\ge 0}$ be a sequence of complex numbers. Then*

$$\Big\| \sum_{k=0}^{\infty} \lambda^{[k]} a_k w_k \Big\|_p \le \Big\| \sum_{k=0}^{\infty} a_k w_k \Big\|_q.$$

Take $q = 2,\ p > 2$ and $a_k = 0$ if $[k] \ne s$ to recover the previous result.

3.3 Sidon sets

3.3.1 Definition and stability properties

We briefly define the Sidon sets (which are typical examples of $\Lambda(p)$-sets), even though they play a marginal role in this work.

A subset $\Lambda = \{\gamma_n\}_{n\ge 1}$ of Γ is called a Sidon set if any continuous function $f : G \to \mathbf{C}$ with Fourier spectrum in Λ has an absolutely convergent Fourier series, equivalently if there exists a constant C such that, for any trigonometric polynomial $f(x) = \sum_{n=1}^{\infty} a_n \gamma_n(x)$ with spectrum in Λ, it holds

$$\sum_{n=1}^{\infty} |a_n| \le C\|f\|_\infty. \tag{3.5}$$

The best constant in (3.5) is called the Sidon constant of Λ and is denoted $S(\Lambda)$.

A basic theorem due to Rudin (Rudin's transference principle) is:

Theorem 3.5. *Let $\Lambda \subset \Gamma$ be a Sidon set. Then, Λ is a $\Lambda(p)$-set for all $p > 2$ and moreover*

$$\lambda_p(\Lambda) \le S(\Lambda)\sqrt{p}.$$

We refer to [22], Vol. 2 p. 146, for a proof. A deep theorem of Pisier ([26], see also [22] Vol. 2 p. 146) claims that the converse is true: if Λ is $\Lambda(p)$ for each $2 < p < \infty$ with $\lambda_p(\Lambda) \le C\sqrt{p}$ for some constant C, then Λ is a Sidon set.

We finally enunciate a fundamental result due to Drury [6].

Theorem 3.6. *Let Λ_1 and Λ_2 be two Sidon sets. Then, their union is again a Sidon set.*

3.3.2 Examples

The basic example of a Sidon set is that of a Hadamard set (also called a *lacunary* set and already mentioned), namely a set $\Lambda = (\lambda_n)_{n\ge 1}$ of positive integers with

$$\frac{\lambda_{n+1}}{\lambda_n} \ge q > 1.$$

Theorem 3.7. *If Λ is a Hadamard set, it is a Sidon set and $S(\Lambda) \le C_q$ where C_q only depends on q.*

By Drury's theorem for example, it follows that every finite union of lacunary sets is itself Sidon. There are others, but they will not be used in this work. We now switch to applications.

4 Lacunary coboundaries

After Anosov, Herman studied in [15] the coboundary equation (1.1), focusing on the heredity of the L^2-integrability property : a priori, $f \in L^1(\mathbf{T})$ has zero mean and the expected solution h is measurable. If $f \in L^2$ (with zero mean), it may happen, as observed in section 2, that (1.1) admits no measurable solution, but Herman proved that in case such a solution h exists, then $h \in L^2$, PROVIDED f has a lacunary spectrum, in other words, an "Anosov phenomenon" cannot occur with lacunary right-handside.

This theorem does not involve arithmetical properties of $\alpha \notin \mathbf{Q}$, but rests on thin sets theory in harmonic analysis, also on ergodic theory, with slight improvements that we present here. We stick to the notations (3.3) and (3.4) of section 3.

Theorem 4.1. *Let $\Lambda \subset \mathbf{Z}$ be a $\Lambda(p)$-set for some $p > 2$ and fix $f \in L^2_\Lambda$. If the coboundary equation*

$$h(x+\alpha) - h(x) = f(x)$$

has a measurable solution, then it has a solution $\in L^2_\Lambda$.

Remark. If one wishes to have non-lacunary examples, the theorem applies with the sets

$$\Lambda = \{\lambda_n = \sum_{j=1}^{s} p_j^{n_j},\ n_j = 1, 2, \ldots\} \quad \text{as well as}$$

$$\Lambda = \{\lambda_n = \prod_{j=1}^{s} p_j^{n_j},\ n_j = 1, 2, \ldots\}.$$

Proof. (of Theorem 4.1). The starting point consists in using a classical criterion for a function φ to be a coboundary in L^2 i.e. to ensure that equation (1.1) with $f = \varphi$ has an L^2-solution [25]:

Lemma 4.1. *Let T be a contraction of a Hilbert space H to itself. Given $\varphi \in H$, a necessary and sufficient condition for the existence of ψ such that $\psi - T(\psi) = \varphi$ is: $\sup_{n\ge 1} \|S_n\| =: M < \infty$, where $S_n = S_n(\varphi) = \sum_{j=0}^{n} T^j(\varphi)$.*

Proof. The condition is obviously necessary as seen in the introduction, with $\|S_n\| \leq 2\|\psi\|$. To see that it is sufficient, we first consider $\lambda \in]0,1[$ and the cocycle equation

$$\psi - \lambda T(\psi) = \varphi$$

whose unique solution is the Neumann series $\psi_\lambda = \sum_{n=0}^\infty \lambda^n T^n(\varphi)$. Thanks to an Abel summation, we have as well

$$\psi_\lambda = \sum_{n=0}^\infty (\lambda^n - \lambda^{n+1}) S_n,$$

$$\text{and} \quad \|\psi_\lambda\| \leq M \sum_{n=0}^\infty (\lambda^n - \lambda^{n+1}) = M.$$

The family (ψ_λ) being norm-bounded in H, there exists a sequence (λ_j), going to 1, such that the sequence (ψ_{λ_j}) converges weakly to some $\psi \in H$, so that $T(\psi_{\lambda_j})$ converges weakly to $T(\psi)$; passing to the weak limit in the equation

$$\psi_{\lambda_j} - \lambda_j T(\psi_{\lambda_j}) = \varphi$$

gives us

$$\psi - T(\psi) = \varphi,$$

as claimed. □

Remark. This proof works in the framework of reflexive Banach spaces, whose unit ball is weakly compact; it can be seen as a weak form of the Markov-Kakutani theorem.

Let us now go back to Theorem 4.1; suppose that $f \in L^2_\Lambda$ and let h be a measurable solution to equation (1.1); by iterating,

$$h(x + n\alpha) - h(x) = f_n(x) := \sum_{j=0}^{n-1} f(x + j\alpha).$$

The coboundary equation has a solution in L^2 if the sequence (f_n) is bounded in L^2 (Lemma 4.1 with $H = L^2$ and $Tf = f \circ R_\alpha$). To check that last point, fix $\varepsilon > 0$; by Lusin's property, h coincides with a continuous function g on some compact set $K \subset \mathbf{T}$ with measure $\geq 1 - \varepsilon$. One has immediately

$$\int_{K \cap R_{-n\alpha}(K)} |f_n(x)|^2 dx \leq 4 \sup_K |g(x)|^2 < \infty. \tag{4.1}$$

Also this set $K \cap R_{-n\alpha}(K)$ is big enough for many integers n, this is a consequence of the following "recurrence" theorem due to Khintchine ([25]):

Lemma 4.2. *Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system, with a bijective and measure-preserving transformation T of X and let E be a measurable subset of X. Then for every $\varepsilon > 0$, the set*

$$A(E) = \{n \in \mathbf{Z} : \ \mu(E \cap T^{-n}E) \geq \mu(E)^2 - \varepsilon\}$$

is relatively dense (namely has gaps bounded by k for some $k = k_\varepsilon > 0$).

We now use that $f \in L^p_\Lambda$ and refer to the key Lemma 3.1: if $b > 0$ is the constant associated to the $\Lambda(p)$-set Λ and if K is as above, we consider

$$A_1 = \{n \in \mathbf{Z} : \ m(K \cap R_{-n\alpha}K) \geq b\};$$

then we choose $\varepsilon > 0$ in order to get

$$m(K)^2 - \varepsilon \geq (1 - \varepsilon)^2 - \varepsilon \geq b.$$

Thus $A_1 \supset A(K)$ is k-dense where k is given by Lemma 4.2 with $E = K,\ T = R_\alpha,\ \mu = m$. Hence, every integer n can be decomposed into $n = n_1 + n_2$ with $n_1 \in A_1$ and $n_2 \in A_2 := \{-k, \dots, k\}$; next, since (cocycle law)

$$f_n = f_{n_1 + n_2} = f_{n_1} \circ R_{n_2\alpha} + f_{n_2},$$

we get

$$\sup_n \|f_n\|_2 \leq \sup_{n \in A_1} \|f_n\|_2 + \sup_{n \in A_2} \|f_n\|_2 \leq C_1 + C_2;$$

indeed, for $n \in A_1$, by combining Lemma 3.1 and (4.1), we can bound

$$\|f_n\|_2 \leq C\Big(\int_{K \cap R_{-n\alpha}K} |f_n|^2\Big)^{1/2} \leq 2C \sup_K |g|;$$

the second inequality for $n \in A_2$ is trivial and the proof is complete. □

Here are consequences of Theorem 4.1.

Corollary 4.1. *There exist a zero mean square-integrable function f and an irrational number α such that equation (1.1) has no measurable solution.*

Take for that a function f with a lacunary spectrum (thus $\Lambda(p)$, $\forall p > 2$) and a number α such that (1.1) has no solution in L^2.

Corollary 4.2. *Let Λ be a $\Lambda(p)$-set, $p > 2$. If $f \in L^2_\Lambda$, $\alpha \notin \mathbf{Q}$ and if h is a measurable solution of $h \circ R_\alpha - h = f$, then, actually, h belongs to L^p.*

Indeed, we proved that some solution $h \in L^2$ exists. But by construction $h \in L^2_\Lambda$, and hence $h \in L^p_\Lambda$. □.

Remark. Theorem 4.1 is in some sense optimal. Actually, it is rather easy to produce some function $f \in C^\omega$ with a lacunary spectrum and $\alpha \notin \mathbf{Q}$ such that equation (1.1) has no solution in L^∞ (even though it has one in $\cap_{p<\infty} L^p$). We choose first α with an hyper-lacunary sequence of denominators (q_n), for example

$$||q_n\alpha|| \leq e^{-q_n}; \tag{4.2}$$

and afterwards we construct the function f. For that purpose, given $(c_n) \in \ell^2 \backslash \ell^1$, $c_0 = 0$, we consider

$$f(x) = \sum_{n=1}^\infty c_n (e(q_n\alpha) - 1) e(q_n x);$$

By (4.2), $f \in C^\omega(\mathbf{T})$, $\int_\mathbf{T} f(x)dx = 0$, and $h(x) = \sum_{n=1}^\infty c_n e(q_n x)$ is a L^p- solution to the coboundary equation for each $p < \infty$. But $h \notin L^\infty$, otherwise, we should get $\sum_{n=1}^\infty |c_n| < \infty$ since h is a lacunary series too, and would have to belong to the Wiener algebra.

5 Orlicz or BMO-type coboundaries

Section 4 provides conditional results on the existence of integrable solutions to (1.1): under some suitable spectral hypothesis on f, such ones exist *as soon as* a measurable solution exists; but up to now, no concrete answer is still given to the question "for which f and α does an integrable solution exist ?".

Regarding relation (2.1), it is easy to check that for $\alpha \in \mathbf{Bad}$ (resp. α r-diophantine) and $f' \in L^2$ (resp. $f^{(r)} \in L^2$), our equation (1.1) has solutions in L^2. Can we improve this result? In this section, we shall see that, α being diophantine, a *slight reinforcement* of the assumption $f' \in L^1$ will ensure the existence of highly integrable solutions.

5.1 Orlicz spaces

5.1.1 General definition

Some results to come (as well as some previous results on $\Lambda(p)$ and Sidon sets) are best formulated in the language of Orlicz spaces. Accordingly, we devote some room to their definition and basic properties. Those spaces are an extension of the Lebesgue spaces $L^p(\mathbf{T})$, the Orlicz functions replacing the power functions $x \mapsto x^p, 1 < p < \infty$. For Lebesgue spaces, Minkowski's inequality in L^p is proved by observing that the set $B = \left\{u : \int_{\mathbf{T}} |u|^p dm \leq 1\right\}$ is convex. And its gauge is the usual L^p-norm. All this generalizes to Orlicz spaces L^ψ, which also represent a scale of interpolation between L^p and L^∞. Good references are [20] and [30].

Definition 5.1. *An Orlicz function is a function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ which is increasing, convex, with $\psi(0) = 0$ and $\lim_{x\to\infty} \frac{\psi(x)}{x} = \infty$.*

Definition 5.2. *The Orlicz space $L^\psi = L^\psi(\mathbf{T})$ attached to the Orlicz function ψ is the Banach space of measurable functions $u : \mathbf{T} \to \mathbf{C}$ such that $\int \psi(|u|/a)dm < \infty$ for some constant $a > 0$ and the associated (Luxemburg) norm $\|u\|_\psi$ of $u \in L^\psi$ is*

$$\|u\|_\psi = \inf\{a > 0 \ : \ \int \psi(|u|/a)dm \leq 1\}.$$

This definition is better understood as follows: let

$$B = \left\{u : \int_{\mathbf{T}} \psi(|u|)dm \leq 1\right\};$$

since ψ is convex, the set B is convex and balanced, and its gauge defines a norm, which is nothing but the Luxemburg norm above.
If φ is an Orlicz function, the conjugate Orlicz function (or Legendre transform) ψ of φ is defined by

$$\psi(x) = \sup_{y\geq 0}(xy - \varphi(y)).$$

Observe that $\psi(x) < \infty$ since $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$, and that φ is the conjugate of ψ. Another approach to duality, on which we shall not dwell, is the following. One writes $\varphi(x) = \int_0^x u(t)dt$ where $u : \mathbf{R}^+ \to \mathbf{R}^+$ is a function vanishing at 0 and increasing to infinity, with inverse function v. Then, $\psi(x) = \int_0^x v(t)dt$.
Since what matters is the behaviour at infinity, it is good to coin a definition.

Definition 5.3. *Let φ_1 and φ_2 be two Orlicz functions. We say that φ_1 is* dominated *by φ_2, and write $\varphi_1 \prec \varphi_2$, if there exist positive constants x_0, k such that*

$$\varphi_1(x) \leq \varphi_2(kx) \text{ for all } x \geq x_0. \tag{5.1}$$

We say that φ_1 is equivalent *to φ_2 if $\varphi_1 \prec \varphi_2$ and $\varphi_2 \prec \varphi_1$.*

A simple example of a pair of conjugate Orlicz functions is: $\varphi(x) = \frac{x^p}{p}$, $\psi(x) = \frac{x^q}{q}$, where q is the conjugate exponent of p.

A second, typical example, of such a pair is: $\psi_2(x) = e^{x^2} - 1$, and $\varphi_2(x) = x\sqrt{\log(x + 1)}$, up to equivalence. More generally, for $r > 0$, the functions $\psi_r(x) = e^{x^r} - 1$, and $\varphi_r(x) = x\Big(\log(x+1)\Big)^{1/r}$ are conjugate, even if, for $r < 1$, the function ψ_r is convex only for large x.

We have the important inequality, called Young's inequality in the approach by inverse functions (see [20], Chapter I), and which is a definition in the Legendre approach:

$$xy \leq \varphi(x) + \psi(y) \text{ for all } x, y \geq 0. \tag{5.2}$$

This generalizes the famous Hölder inequality $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$. An important consequence of (5.2) is the following duality principle:

Proposition 5.1. *Let φ, ψ be conjugate Orlicz functions and let $f \in L^\varphi$, $g \in L^\psi$. Then $fg \in L^1$ and*

$$\left|\int fg\right| \leq 2\|f\|_\varphi \|g\|_\psi.$$

Proof. We can assume that $\|f\|_\varphi = \|g\|_\psi = 1$. We then use (5.2) to get

$$\int |fg| \leq \int \varphi(|f|) + \int \psi(|g|) = 2.$$

This finishes the proof. □

An easily checked, but important fact (use Stirling's formula) is

Proposition 5.2. *If $r > 0$, then*

$$\|f\|_{\psi_r} \approx \sup_{p>2} \frac{\|f\|_p}{p^{1/r}}.$$

With the language of Orlicz spaces, Theorems 3.3, 3.1 and 3.5 can be gathered in the following proposition.

Proposition 5.3. *The following three properties hold:*
a) *If* $S = \sum_{|A|=s} z_A\, w_A$, *then* $\|S\|_{\psi_{2/s}} \lesssim \|S\|_2$.
b) *If* $\Lambda = \{\lambda_n = \prod_{j=1}^s p_j^{n_j}, \quad n_j = 1, 2, \dots\}$ *and* $S \in \mathcal{P}_\Lambda$, *then one has* $\|S\|_{\psi_{1/s}} \lesssim \|S\|_2$.
c) *If* Λ *is a Sidon set, then* $\|f\|_{\psi_2} \lesssim \|f\|_2$ *for each* $f \in \mathcal{P}_\Lambda$.

5.1.2 An instructive example

Consider the function g defined by

$$g(x) =: \sum_{q\ge 1} \varepsilon_q \cos 2\pi q x, \quad \varepsilon_q = \frac{1}{\sqrt{\log(q+1)}}.$$

We will see that g is on the verge of belonging to L^{φ_2} (this example will be reconsidered later). More precisely, $g(x) \gtrsim \frac{1}{x \log^{3/2}(1/x)}$ as $x \to 0^+$ (this is the right order of growth near the origin, but we insist on the lower bound). Let us prove this. Using two Abel summations, we first get:

$$\begin{aligned}
& (2\sin \pi x) g(x) \\
= \ & \sum_{q\ge 1} \varepsilon_q [\sin(2q+1)\pi x - \sin(2q-1)\pi x] \\
= \ & -\varepsilon_1 \sin(\pi x) + \sum_{q\ge 2} (\varepsilon_{q-1} - \varepsilon_q)[\sin(2q-1)\pi x] \\
= \ & -\varepsilon_1 \sin(\pi x) + (\varepsilon_2 - \varepsilon_1)\sin(\pi x) \\
& \quad + \sum_{q\ge 2} (\varepsilon_{q-1} + \varepsilon_{q+1} - 2\varepsilon_q) S_q(x)
\end{aligned}$$

where $S_q(x) = \sum_{k=1}^q \sin(2k-1)\pi x = \sin^2 \pi q x / \sin \pi x \ge 0$ on $(0, 1]$. We next observe that:

$$S_q(x) \gtrsim q^2 x \text{ for } qx \le 1/2$$

$$\text{and} \quad \varepsilon_{q-1} + \varepsilon_{q+1} - 2\varepsilon_q \approx \frac{1}{q^2 \log^{3/2} q}.$$

Indeed, $S_q(x) = \sin^2 \pi q x / \sin \pi x \gtrsim q^2 x$ and the second estimate comes from Taylor's formula using the second derivative of the function $t \mapsto 1/\sqrt{\log t}$. Therefore, using also $S_q(x) \ge 0$, we get for any small $x > 0$ and the integer $N = [1/x]$:

$$\begin{aligned}
x|g(x)| \ & \gtrsim \sum_{2\le q\le N} (\varepsilon_{q-1} + \varepsilon_{q+1} - 2\varepsilon_q) S_q(x) \\
& \gtrsim \sum_{2\le q\le N} \frac{q^2 x}{q^2 \log^{3/2} q} \\
& \gtrsim x \frac{N}{\log^{3/2} N}.
\end{aligned}$$

We thus get for x near 0:

$$x|g(x)| \gtrsim \frac{1}{\log^{3/2}(1/x)},$$

which is the lower bound claimed in Section 3, and the analogous upper bound

$$x|g(x)| \lesssim \frac{1}{\log^{3/2}(1/x)},$$

for x near 0, can be proved similarly. This implies $g \notin L^{\varphi_2}$ since

$$\begin{aligned}
& \int_0^1 |g(x)| \sqrt{\log |g(x)|}\, dx \\
\approx \ & \int_0^1 \frac{1}{x \log^{3/2}(1/x)} \sqrt{\log(1/x)}\, dx \\
= \ & \int_0^1 \frac{1}{x \log(1/x)}\, dx = \infty.
\end{aligned}$$

Remark. This example is "sharp" since, in some sense, g is quite close to belonging in L^{φ_2}! Indeed, the previous pointwise (upper) estimate shows that $g \in L^{\varphi_r}$ for all $r > 2$. An alternative proof would be the following: $g(x) = \Re G(x)$ where $G(x) = \sum_{q\neq 0} \varepsilon_{|q|} e^{2i\pi q x}$. A double Abel summation by parts now gives

$$G = \sum_{q=1}^{\infty} q\Big(\varepsilon_{q-1} + \varepsilon_{q+1} - 2\varepsilon_q\Big) K_q \tag{5.3}$$

where $K_q = \sum_{|j|\le q} (1 - |j| q^{-1}) e_q$ is the Fejér kernel of order q (see [19] p. 24). But since $\|K_q\|_\infty = q$ and $\|K_q\|_1 = 1$, we see that

$$\begin{aligned}
\|K_q\|_{\varphi_r} \ & \lesssim \int_{\mathbf{T}} K_q \cdot (\log(1 + K_q))^{1/r} \\
& \lesssim (\log q)^{1/r} \int_{\mathbf{T}} K_q = (\log q)^{1/r}.
\end{aligned}$$

So that the series defining G in (5.3) is absolutely convergent in L^{φ_r} for $r > 2$, since

$$q\Big(\varepsilon_{q-1} + \varepsilon_{q+1} - 2\varepsilon_q\Big) \|K_q\|_{\varphi_r} \lesssim \frac{1}{q (\log q)^{3/2 - 1/r}}$$

with $3/2 - 1/r > 1$.

5.2 The space BMO

An interesting related space is the Banach space BMO of integrable functions on $\mathbf{T}$ with bounded mean oscillation, equipped with its natural norm $\|.\|_{BMO}$ ([12], chapter 6) which we describe: let I be an arc on $\mathbf{T}$ equipped with its Haar measure m, and $f_I := \frac{1}{m(I)} \int_I f\, dm$. Then, $f \in BMO$ if

$$[f] := \sup_I \frac{1}{m(I)} \int_I |f - f_I|\, dm < \infty$$

where the supremum is taken over all subarcs I of $\mathbf{T}$, and

$$\|f\|_{BMO} := \|f\|_2 + [f].$$

Observe that, without this recentering by f_I, we would simply get the (smaller) space L^∞. To play at least once with the definition, note the following, in which f_α denotes the indicator function of the arc $(0,\alpha)$:

$$\text{If } 0<\alpha<\beta<1, \text{ then } \|f_\beta - f_\alpha\|_{BMO} \geq 1/2.$$

Indeed, set $\Delta = f_\beta - f_\alpha$ as well as $I=(\alpha-h,\alpha+h)$ with h small, $0<h<\min(\alpha,\beta-\alpha)$.
Then, $(f_\alpha)_I = 1/2,\ (f_\beta)_I = 1,\ \Delta_I = 1/2$ and

$$\begin{aligned}\|\Delta\|_{BMO} &\geq \frac{1}{m(I)}\int_I |\Delta-\Delta_I|dm \\ &= \frac{1}{2h}\int_{\alpha-h}^{\alpha} |-\frac{1}{2}|dt + \frac{1}{2h}\int_{\alpha}^{\alpha+h}|\frac{1}{2}|dt = \frac{1}{2}.\end{aligned}$$

As a consequence, BMO is a non-separable space.

One striking feature of this space (indeed the dual space of the real space H^1) is given by the John-Nirenberg inequality ([12] p. 223), which immediately implies the

Proposition 5.4. *One has the continuous inclusion*

$$BMO \subset L^{\psi_1}.$$

Propositions 5.2 and 5.4 together imply

$$f\in BMO \Rightarrow \|f\|_p = O(p) \quad \text{as} \quad p\to\infty. \tag{5.4}$$

For the proof of both theorems to come, the two following lemmas will be needed, one on diophantine approximation, one of harmonic analysis.

5.3 Two lemmas

Lemma 5.1. *For every integer $n\geq 0$ and every irrational α with convergents p_n/q_n, one has*

$$\sum_{0<|q|<q_{n+1}} \frac{1}{\|q\alpha\|^2} \leq \frac{C}{\|q_n\alpha\|^2} \tag{5.5}$$

with $C=\pi^2/3$. In particular :

$$\sum_{q_n\leq|q|<q_{n+1}} \frac{1}{\|q\alpha\|^2} \leq \frac{2C}{\|q_n\alpha\|^2}. \tag{5.6}$$

Proof. Set $\alpha_n = \|q_n\alpha\|$ and consider the half-open intervals $[j\alpha_n,(j+1)\alpha_n[$ where $j=1,2,\ldots$. Every number $\|q\alpha\|$, $1\leq q<q_{n+1}$, falls in one of those intervals (no one falls into $[0,\alpha_n[$) and at most two fall in the same interval, for if three of them, say $\|q\alpha\|$, $\|q'\alpha\|$, $\|q''\alpha\|$ belonged to $[j\alpha_n,(j+1)\alpha_n[$, at least two of them, say $\|q\alpha\|$ and $\|q'\alpha\|$ with $q'>q$, would be of the form $q\alpha-p$ and $q'\alpha-p'$ (otherwise, they would be of the form $p-q\alpha$ and $p'-q'\alpha$), so that

$$\begin{aligned}\Big|\|q\alpha\|-\|q'\alpha\|\Big| &= |(q'\alpha-p')-(q\alpha-p)| \\ &= |(q'-q)\alpha-(p'-p)| < \alpha_n.\end{aligned}$$

Setting $Q=q'-q$ with $0<Q<q_{n+1}$, this would imply $\|Q\alpha\|<\alpha_n$, a contradiction with (1.2). If follows immediately that

$$\sum_{0<q<q_{n+1}} \frac{1}{\|q\alpha\|^2} \leq 2\sum_{j=1}^{\infty}\frac{1}{j^2\alpha_n^2} = \frac{C}{\|q_n\alpha\|^2}.$$

This clearly ends the proof of the lemma, since $\|x\|$ is an even function. □

Our second lemma (the so-called embedding inequality) says the following (cf. [36] (Vol. II, page 132):

Lemma 5.2. *Let $\Lambda=(\lambda_n)_{n\geq1}\subset \mathbf{Z}$ be a Sidon set with constant K. Then, for every function $g\in L^{\varphi_2}$, with $\varphi_2(x) = x\Big(\log(1+x)\Big)^{1/2}$, we have:*

$$\Big(\sum_{n=1}^{\infty}|\widehat{g}(\lambda_n)|^2\Big)^{1/2} \leq CK\|g\|_{\varphi_2} \tag{5.7}$$

where C is an absolute constant.
If $g\in H^1$ (namely $g\in L^1$ and $\widehat{g}(k)=0$ for $k<0$), and if Λ is lacunary ($\lambda_{n+1}\geq q\lambda_n>1$ with $q>1$), then

$$\Big(\sum_{n=1}^{\infty}|\widehat{g}(\lambda_n)|^2\Big)^{1/2} \leq C_q\|g\|_1 \tag{5.8}$$

where C_q only depends on q.

Proof. Let (a_n) be a norm one sequence with compact support in ℓ^2. Let $h(t)=\sum_n a_n e(\lambda_n t)\in L^2$, with $\|h\|_2=1$. By Propositions 5.2 and 5.3, we have $\|h\|_{\psi_2}\leq (C/2)K$, where the function $\psi_2(x)=e^{x^2}-1$ is the Orlicz function conjugate to φ_2. Parseval's formula and the duality between the spaces L^{φ_2} and L^{ψ_2} now give us:

$$\Big|\sum_{n=1}^{\infty} a_n\widehat{g}(\lambda_n)\Big| = \Big|\sum_{k\in\mathbf{Z}}\widehat{h}(k)\widehat{g}(k)\Big| = \Big|\int_{\mathbf{T}} g(-t)h(t)dm(t)\Big|$$

$$\leq 2\|g\|_{\varphi_2}\|h\|_{\psi_2} \leq CK\|g\|_{\varphi_2}.$$

Taking the supremum on those test sequences, we obtain

$$\Big(\sum_{n=1}^{\infty}|\widehat{g}(\lambda_n)|^2\Big)^{1/2} \leq CK\|g\|_{\varphi_2}.$$

The second part of the lemma is due to Paley ([7] p. 104): by dividing Λ in at most J subsequences, where $q^J>2$, we can assume $q>2$; then, write $g=uv$ with $u,v\in H^2$ and $\|u\|_2\|v\|_2=\|g\|_1$, so that

$$\widehat{g}(\lambda_n) = \sum_{0\leq k\leq\lambda_{n-1}}\widehat{u}(k)\widehat{v}(\lambda_n-k) + \sum_{\lambda_{n-1}<k\leq\lambda_n}\widehat{u}(k)\widehat{v}(\lambda_n-k).$$

Now, apply Cauchy-Schwarz to each of the two sums on the RHS and add up, noting that $\lambda_n-\lambda_{n-1}>\lambda_{n-1}$. □

5.4 Two theorems

We begin by motivating the diophantine conditions to come (to come back!). Y. Meyer [23] proved the following, in which $W^{r,2}, r \geq 1$, denotes the Sobolev space of functions $f \in L^2$ whose r first derivatives (in the sense of distributions) are again in L^2, equipped with its natural norm

$$\|f\|_{W^{r,2}} = \left(|\widehat{f}(0)|^2 + \|f^{(r)}\|_2^2\right)^{1/2}.$$

Theorem 5.1. *The three following conditions are equivalent:*

1. *α belongs to* **Bad**.
2. *If $f \in W^{1,2}$, the equation $h(x+\alpha) - h(x) = f(x)$ has a square-integrable solution.*
3. *If $f \in W^{1,2}$, the equation $h(x+\alpha) - h(x) = f(x)$ has a solution in BMO.*

Proof. The implication $3 \Rightarrow 2$ is obvious thanks to (5.4).
The implication $2 \Rightarrow 1$ is nearly obvious: to each $f \in W^{1,2}$ is associated in a unique way a zero-mean solution $h \in L^2$ and then the map $f \mapsto h$ is linear. It is readily seen to have closed graph, so that, for some constant C:

$$\|h\|_2 \leq C\|f'\|_2 \text{ for all } f \in W^{1,2}.$$

We test this inequality on

$$f(x) = e(qx)\,(e(q\alpha) - 1),\ h(x) = e(qx)$$

to get

$$1 \lesssim q\|q\alpha\|, \text{ that is } \alpha \in \mathbf{Bad}.$$

The deep implication is $1 \Rightarrow 3$. A key point is a sufficient condition due to Y. Meyer [23] for membership in BMO (admitted here).

Proposition 5.5. *Let $h \in L^2$ with $\widehat{h}(0) = 0$, $h_k = \sum_{2^k \leq |j| < 2^{k+1}} \widehat{h}(k)e_k$, and*

$$H = \Big(\sum_{k \geq 0} |h_k|^2\Big)^{1/2}$$

the corresponding square function (the polynomial h_k is called the k-th block in the Littlewood-Paley decomposition of h). If $H \in L^\infty$, then $h \in BMO$ and

$$\|h\|_{BMO} \leq C\|H\|_\infty.$$

To check the assumptions of this proposition in the present case, we rely on a simple estimate:

$$S_k := \sum_{2^k \leq |j| < 2^{k+1}} \frac{1}{j^2\|j\alpha\|^2} \leq C. \tag{5.9}$$

Indeed, let $n+1$ be the smallest integer such that $q_{n+1} \geq 2^{k+1}$. We see that, thanks to Lemma 5.1 and the assumption $\alpha \in \mathbf{Bad}$:

$$\begin{aligned} S_k &\lesssim 2^{-2k} \sum_{2^k \leq |j| < 2^{k+1}} \frac{1}{\|j\alpha\|^2} \\ &\lesssim 2^{-2k} \sum_{1 \leq |j| < q_{n+1}} \|j\alpha\|^{-2} \\ &\lesssim 2^{-2k}\|q_n\alpha\|^{-2} \lesssim 2^{-2k}q_n^2 \lesssim 2^{-2k}2^{2k} = 1. \end{aligned}$$

Now, write $f' = \sum c_j e_j$ with (c_j) square-summable. The zero-mean solution h of our coboundary equation $h(x+\alpha) - h(x) = f(x)$ is formally given by the Fourier series

$$h = \sum_{j \neq 0} \mu_j c_j e_j \quad \text{with} \quad \mu_j = \frac{1}{j(e(j\alpha) - 1)},\ |\mu_j| \approx \frac{1}{j\|j\alpha\|}.$$

If h_k (resp. f'_k) is the kth-block in the Littlewood-Paley decomposition of h (resp. f'), we see by Cauchy-Schwarz and (5.9) that

$$\begin{aligned} \|h_k\|_\infty^2 &\leq \Big(\sum_{2^k \leq |j| < 2^{k+1}} |\mu_j||c_j|\Big)^2 \\ &\leq \Big(\sum_{2^k \leq |j| < 2^{k+1}} |\mu_j|^2\Big)\Big(\sum_{2^k \leq |j| < 2^{k+1}} |c_j|^2\Big) \\ &\lesssim \sum_{2^k \leq |j| < 2^{k+1}} |c_j|^2 = \|f'_k\|_2^2. \end{aligned}$$

Summing up, we obtain that

$$\sum_{k \geq 0} \|h_k\|_\infty^2 \lesssim \sum_{k \geq 0} \|f'_k\|_2^2 = \|f'\|_2^2,$$

implying $H \in L^\infty$ and $h \in BMO$ by Proposition 5.5. □

Remarks. 1. Similarly, if the equation has an L^2-solution for each $f \in W^{r,2}$, then α is **r-diophantine**, namely verifies

$$1 \lesssim q^r\|q\alpha\|.$$

This is why we consider this class of numbers in this section.

2. It follows from Theorem 5.1 and (5.4) that if $f' \in L^2$, the zero-mean solution h will satisfy

$$\|h\|_p = O(p).$$

We will now see that relaxing $f' \in L^2$ to $f' \in L^{\varphi_2}$ preserves some control on h, namely

$$\|h\|_p = O(p^{3/2}).$$

Throughout the rest of this subsection, we fix the r-diophantine number α, i.e. α satisfying an inequality of the form:

$$q^r\|q\alpha\| \geq \delta, \quad q = 1, 2, \dots \tag{5.10}$$

and we assume without lost of generality that $r \in \mathbf{N}$. Having both preceding lemmas at our disposal, we will give a simpler proof of a theorem due to Rozhdestvenskii [31]. In this theorem, one significantly improves the Orlicz class of functions to which the solution h given there belongs (here, H^1 denotes once more the Hardy space of those $f \in L^1$ with vanishing Fourier coefficients of negative index). Here is now our second theorem.

Theorem 5.2. 1. *Suppose that f is absolutely continuous and that the r^{th} derivative $f^{(r)} \in L^{\varphi_2}$ where φ_2 is the Orlicz function defined by $\varphi_2(x) = x\sqrt{\log(1+x)}$. Then (1.1) has a solution $h \in L^2$.*

2. *Indeed, the solution h of (1.1) is not only in L^2 but in $L^{\psi_{2/3}}$, where $\psi_{2/3}(x) = e^{x^{2/3}} - 1$ for $x \geq 0$. In other terms, one has $h \in \cap_{q<\infty} L^q$ with*

$$\|h\|_q \leq M q^{3/2} \text{ for every } q \geq 2. \tag{5.11}$$

3. *If $f \in H^1$, then $f^{(r)} \in L^1$ is enough to imply that any integrable solution h of the coboundary equation is actually in L^2.*

Before proving Theorem 5.2, we would first like to (re)-consider an instructive example:

Example. A function f to which Theorem 5.2 to come, with $\alpha \in \mathbf{Bad}$, *will not apply* is

$$f(x) = \sum_{q\geq 1} \frac{\sin 2\pi q x}{q\sqrt{\log(q+1)}}$$

Indeed, f is absolutely continuous (with the poor modulus of continuity $O((\log 1/h)^{-1/2})$ and

$$f' = 2\pi \sum_{q\geq 1} \frac{\cos 2\pi q x}{\sqrt{\log(q+1)}} \in L^1$$

by the usual properties of the cosine series, since the sequence $\frac{1}{\sqrt{\log(q+1)}}$ is convex and tends to zero ([19] p. 24). But if $f' \in L^{\varphi_2}$, Lemma 5.2 will give us:

$$\sum_{n\geq 0} |\widehat{f'}(2^n)|^2 = c \sum_{n\geq 0} \left(\frac{1}{\sqrt{\log(2^n+1)}}\right)^2 < \infty,$$

contradicting the divergence of the harmonic series. We can even say that f is not a L^2-coboundary. Indeed, since $\alpha \in \mathbf{Bad}$, we have $q_n \leq C^n$ for some $C > 1$, so that:

$$\begin{aligned}\sum_{q\geq 1} \frac{|\widehat{f}(q)|^2}{\|q\alpha\|^2} &\geq \sum_{n\geq 0} \frac{|\widehat{f}(q_n)|^2}{\|q_n\alpha\|^2} \gtrsim \sum_{n\geq 0} q_n^2 |\widehat{f}(q_n)|^2 \\ &\gtrsim \sum_{n\geq 0} \frac{1}{\log(q_n+1)} = \infty.\end{aligned}$$

This example was considered in more detail in a previous section, where the sharpness of $f' \notin L^{\varphi_2}$ was observed: $f' \in L^{\varphi_r}$ for all $r > 2$.

5.5 Proof of Theorem 5.2

By examining the Fourier coefficients, we can only have:

$$h(x) = \sum_{k\neq 0} \frac{\widehat{f}(k)}{e(k\alpha) - 1} e(kx)$$

and we *first* show that the right-hand side is in L^2. Denote by q_n' and q_n'' integers of $[q_n, q_{n+1}[$ and $]-q_{n+1}, -q_n]$ respectively where $|\widehat{f}(q)|$ is maximum on this block. We can write, using Lemma 5.1 in the wake, as well as the diophantine hypothesis $||q\alpha|| \gtrsim q^{-r}$ on α:

$$\begin{aligned}&\sum_{k\neq 0} \left|\frac{\widehat{f}(k)}{e(k\alpha) - 1}\right|^2 \\ \lesssim\ &\sum_{n\geq 0} \sum_{q_n \leq |q| < q_{n+1}} \frac{|\widehat{f}(q)|^2}{\|q\alpha\|^2} \\ \lesssim\ &\sum_{n\geq 0} \left(\frac{|\widehat{f}(q_n')|^2}{\|q_n\alpha\|^2} + \frac{|\widehat{f}(q_n'')|^2}{\|q_n\alpha\|^2}\right) \\ \lesssim\ &\sum_{n\geq 0} q_n^{2r} \left(|\widehat{f}(q_n')|^2 + |\widehat{f}(q_n'')|^2\right) \\ \lesssim\ &\sum_{n\geq 0} (q_n')^{2r} |\widehat{f}(q_n')|^2 + \sum_{n\geq 0} (q_n'')^{2r} |\widehat{f}(q_n'')|^2 \\ \lesssim\ &\sum_{n\geq 0} \left(|\widehat{g}(q_n')|^2 + |\widehat{g}(q_n'')|^2\right)\end{aligned}$$

where $g = f^{(r)}$. Now, since $q_{n+2} \geq 2q_n$ for continued fractions, taking the points of even and odd index separately, we get that the set of integers $E = \{q_n', q_n'',\ n \geq 1\}$ is the union of at most four Hadamard lacunary sets with ratio ≥ 2, and is therefore a Sidon set with an absolute constant. An appeal to Lemma 5.2 ends the proof, since we assumed that $g = f^{(r)} \in L^{\varphi_2}$.

The second part is proved similarly, using now the second item of Lemma 5.2. It remains to prove the more ambitious conclusion $h \in L^{\psi_{2/3}}$. To that effect, we will first make a more careful study of the function h. We need the additional:

Lemma 5.3. *One can write*

$$h = \sum_{j\geq 1} G_j \quad \textit{with} \quad \|G_j\|_2 \leq C/j \tag{5.12}$$

where the functions G_j have disjoint spectra E_j, each E_j being Sidon with a constant less than an absolute constant. As a consequence, for each integer $N \geq 1$, one has a decomposition $h = u_N + v_N$ with

$$\|u_N\|_{\psi_2} \lesssim \log N \quad \textit{and} \quad \|v_N\|_2 \lesssim \frac{1}{\sqrt{N}}.$$

Proof. The proof of Lemma 5.3 is just a matter of rearrangement; to save notation, we will assume that $f \in H^2$, i.e. $\widehat{f}(k) = 0$ if $k < 0$. From Lemma 5.1, we can write *by increasing rearrangement* (recall that $\alpha_n=\|q_n\alpha\|$):

$$[q_n,q_{n+1}[=\{l_{n,j},\ 1\le j\le q_{n+1}-q_n\}$$

with $\|l_{n,j}\alpha\|\gtrsim j\alpha_n$. We have seen that

$$h(x)=\sum_n\sum_{1\le j\le q_{n+1}-q_n}\frac{\widehat{f}(l_{n,j})}{e(l_{n,j}\alpha)-1}e(l_{n,j}x)$$

$$=\sum_{j\ge1}\Big[\sum_{\substack{n,\\ q_{n+1}-q_n\ge j}}\frac{\widehat{f}(l_{n,j})}{e(l_{n,j}\alpha)-1}e(l_{n,j}x)\Big]=:\sum_{j\ge1}G_j.$$

Now, for fixed j, the spectrum $E_j=\{l_{n,j}\}$ of G_j is as before the union of two Hadamard sets with ratio ≥ 2 and is Sidon with a uniform constant. The E_j are clearly disjoint, and finally, using once more that α is r-diophantine and recalling that $g=f^{(r)}$:

$$\begin{aligned}\|G_j\|_2^2 &\lesssim \sum_n\frac{|\widehat{f}(l_{n,j})|^2}{\|l_{n,j}\alpha\|^2}\lesssim\sum_n\frac{1}{j^2\alpha_n^2}|\widehat{f}(l_{n,j})|^2\\ &\lesssim \frac{1}{j^2}\sum_n q_n^{2r}|\widehat{f}(l_{n,j})|^2\lesssim\frac{1}{j^2}\sum_n l_{n,j}^{2r}|\widehat{f}(l_{n,j})|^2\\ &\lesssim \frac{1}{j^2}\sum_n|\widehat{g}(l_{n,j})|^2\lesssim\frac{1}{j^2},\end{aligned}$$

the last inequality coming from Lemma 5.2 applied to $g\in L^{\varphi_2}$ and E_j. Finally, take

$$u_N=\sum_{j=1}^N G_j,\ v_N=\sum_{j=N+1}^{\infty}G_j$$

and use the triangle inequality in L^{ψ_2}, as well as the Rudin-Khintchine inequalities of Lemma 5.2, which give $\|G_j\|_{\psi_2}\lesssim\|G_j\|_2\lesssim 1/j$, to get the estimate for u_N. Besides, since the G_j's have disjoint spectra:

$$\int_{\mathbf{T}}|v_N|^2dm=\sum_{j>N}\int_{\mathbf{T}}|G_j|^2dm\lesssim\sum_{j>N}j^{-2}\lesssim N^{-1}.$$

This ends the proof of Lemma 5.3. □

We can now end the proof of Theorem 5.2 by a *Marcinkiewicz interpolation type argument,* which lets $L^{\psi_{2/3}}$ appear as a real interpolate space between L^2 and L^{ψ_2}. We will denote by $\delta>0$ some numerical constant. Let t be a real number $\ge t_0$ where the constant $t_0\ge1$ will be fixed, depending on δ. We write $h=:u_N+v_N$, where $N=N(t)$ will be adjusted later. Clearly

$$m(|h|>2t)\le m(|u_N|>t)+m(|v_N|>t).\qquad(5.13)$$

We will find a good upper bound for the RHS of (5.13) in two steps:

Step 1. We have

$$m(|u_N|)>t)\lesssim e^{-\delta t^2/(\log N)^2}.\qquad(5.14)$$

Indeed, if $\lambda\ge\|u_N\|_{\psi_2}$, Markov's inequality gives us

$$\begin{aligned}m(|u_N|>t)&=m(e^{|u_N|^2/\lambda^2}>e^{t^2/\lambda^2})\qquad(5.15)\\ &\le e^{-t^2/\lambda^2}\int e^{|u|^2/\lambda^2}dm\le 2e^{-t^2/\lambda^2}.\end{aligned}$$

Now, we know from Lemma 5.3 that we can take $\lambda=C\log N$. This gives the relation (5.14).

Step 2. It holds

$$m(|v_N|>t)\lesssim\frac{1}{Nt^2}\le\frac{1}{N}.\qquad(5.16)$$

This is Chebyshev's inequality and the estimate $\|v_N\|_2\lesssim\frac{1}{\sqrt{N}}$ of Lemma 5.3.

Now, it is easy to conclude. Adjust N in order to balance the contributions of (5.14) and (5.16); namely aiming for

$$N=e^{\delta t^2/(\log N)^2}\quad\text{or}\quad\log N=(\delta t^2)^{1/3},\quad N=e^{(\delta t^2)^{1/3}},$$

we take

$$N=\left[e^{(\delta t^2)^{1/3}}\right]+1\qquad(5.17)$$

where [ ] denotes the integer part. By adjusting now t_0 large enough in terms of δ, we obtain

$$e^{(\delta t^2)^{1/3}}\le N\le e^{(2\delta t^2)^{1/3}},$$

so that

$$e^{-\delta t^2/(\log N)^2}\le e^{-\delta t^2/(2\delta t^2)^{2/3}}\le e^{-\delta' t^{2/3}}$$

where $\delta'>0$ only depends on δ. Inserting this value in (5.14) and (5.16) gives, taking (5.13) into account and changing the value of δ if this is necessary: $m(|h|>2t)\lesssim e^{-\delta t^{2/3}}$ for $t\ge t_0$. Equivalently (changing δ and t_0 again):

$$m(|h|>t)\lesssim e^{-\delta t^{2/3}}\ \text{for}\ t\ge t_0.$$

Using the classical "integration by parts" formula

$$\int_{\mathbf{T}}\rho(|h|)dm=\int_0^\infty\rho'(t)m(|h|>t)dt$$

now gives

$$\begin{aligned}\int_{\mathbf{T}}e^{\varepsilon|h|^{2/3}}dm&=\int_0^\infty(2\varepsilon/3)t^{-1/3}e^{\varepsilon t^{2/3}}m(|h|>t)dt\\ &<\infty\quad\text{for }\varepsilon<\delta.\end{aligned}$$

This ends the proof of Theorem 5.2. ♢

6 The equation $g(2x)-g(x)=f(x)$

More general coboundary equations, associated to arbitrary measure-preserving transformations, appear as well in ergodic theory. The case of the 2-shift $\sigma:x\mapsto 2x\ (\mathrm{mod}\ 1)$ is of specific interest, since its very rich dynamics (chaos) is the opposite of the rotation's one; in return, the condition $f\in L^2_\Lambda\implies f\circ\sigma\in L^2_\Lambda$ is much more restrictive for Λ.

Raikov proved the following theorem for $f\in L^1(\mathbf{T})$:

$\lim_N \frac{1}{N} \sum_{n<N} f(2^n x)$ exists almost everywhere and is equal to $\int_{\mathbf{T}} f dm$.

This is actually just a version of the "ergodic theorem" as observed by F. Riesz a few years later, and the convergence rate aroused the interest of M. Kac who established a central limit theorem for those sums [17]. The following is a striking analog (for the special transformation $x \mapsto 2x$) of Lemma 4.1:

Theorem 6.1. *Let f be an α-hölderian function with $\alpha > 1/2$. Then $\lim_{N\to\infty} \|S_N f\|_2^2/N = 0 \iff f$ is a L^2-coboundary for the shift i.e. $f(x) = g(2x) - g(x)$ for some L^2-function g.*

Proof. The function f is assumed to satisfy some Hölder's condition and, under the condition

$$\tau^2 := \lim_{N\to\infty} \frac{1}{N} \Big\| \sum_{n=0}^{N} f(2^n t) \Big\|_2^2 = 0, \qquad (6.1)$$

we shall prove that f is a coboundary. We make use of the spectral local correspondence by defining the spectral measure of $f \in L^2$, σ_f, through its Fourier coefficients: $\hat{\sigma}_f(k) = \langle f \circ \sigma^k, f \rangle$ if $k \geq 0$, and $\hat{\sigma}_f(-k) = \overline{\hat{\sigma}_f(k)}$. If U is the isometry of L^2 associated to σ, we have for every trigonometric polynomial R:

$$||R(U)f||_{L^2} = ||R||_{L^2(\sigma_f)}.$$

Thus, hypothesis (6.1) is nothing but $\lim_{N\to\infty} K_N * \sigma_f(0) = 0$, where K_N is the Fejér kernel (the square of a Dirichlet kernel), and this means (thanks to the Fejér convergence theorem) that $\sigma_f\{0\} = 0$, i.e. σ_f has no mass at 0. Now, the following lemma gives a description of a coboundary in terms of its spectral measure ([27] page 288).

Lemma 6.1. *The function f is a coboundary for the shift in L^2 if and only if the function $1/\sin^2 \pi t \in L^1(\sigma_f)$.*

To pursue, more information on f and σ_f is needed, and here the hölderian regularity of f is an unbreakable assumption as we shall see ([17]).

Lemma 6.2. *Let f be an α-hölderian real function* with $\alpha > 1/2$. *Then,*

$$\Big| \int_{\mathbf{T}} f(2^k t) f(2^j t) dt \Big| = O(2^{-|j-k|\alpha}),$$

the constant depending on f and α. In particular,

$$\widehat{\sigma}_f(k) = O(2^{-k\alpha}),\ k \geq 0.$$

It follows from Lemma 6.2 that σ_f is an absolutely continuous measure with a C^∞ density, say F; by hypothesis (6.1), $F(0) = \sigma_f(\{0\}) = 0$, and $F'(0) = 0$ since F is even. Thus, clearly, $1/\sin^2 \pi t \in L^1(\sigma_f)$ and f is a coboundary in L^2. □

Kàc ([17]) asked for completing his result when $0 < \alpha \leq 1/2$ and gave an example of an $f \in A(\mathbf{T})$ (the Wiener algebra) without any integrable solution g: consider

$$f(t) = \sum_{j\geq 1} a_j \cos \pi 2^j t, \quad \text{with} \quad a_j = \frac{1}{\sqrt{j}} - \frac{1}{\sqrt{j-1}} \quad \text{for} \quad j \geq 2,\ a_1 = 1;$$

we shall prove that $\tau = 0$ though no solution $g \in L^2$ exists to the coboundary equation $f(x) = g(2x) - g(x)$. Indeed we can compute the partial sum

$$S_N f(t) = \sum_{r=1}^{N} \frac{\cos \pi 2^r t}{\sqrt{r}} + \sum_{r=N+1}^{\infty} \Big(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r-N}}\Big) \cos \pi 2^r t.$$

Parseval identity then gives

$$2\|S_N f\|_2^2 = \sum_{r=1}^{N} \frac{1}{r} + \sum_{r=N+1}^{\infty} \Big(\frac{1}{\sqrt{r-N}} - \frac{1}{\sqrt{r}} \Big)^2$$

$$\lesssim \sum_{r=1}^{N} \frac{1}{r} + \sum_{r=N+1}^{2N} \frac{1}{r-N} + \sum_{r>2N} N^2 r^{-3} = 2 \log N + O(1)$$

because $\left|(r-N)^{-1/2} - r^{-1/2}\right| \lesssim N r^{-3/2}$ for $r > 2N$ by the mean-value theorem. This implies $\tau = 0$.
Actually, there exists no $g \in L^1$ satisfying $f(t) = g(2t) - g(t)$. In the opposite case, the Fourier (cosine) coefficients of such a g would satisfy

$$b_j = 0 \quad \text{if j is not a power of 2 and} \quad b_{2^k} = \frac{1}{\sqrt{k}}\cdot$$

But this is impossible since $\sum_k b_{2^k}^2 = \infty$, and $\sum_{k=1}^{\infty} \frac{e(2^k t)}{\sqrt{k}}$ is not even the Fourier series of a bounded measure on $\mathbf{T}$. Indeed, if $\Lambda = \{\lambda_n\}$ is a Sidon set (here $\Lambda = \{2^n\}$), and μ a complex measure with spectrum inside Λ (here $\mu = g(t)dt$), then

$$\sum_{n=1}^{\infty} |\widehat{\mu}(\lambda_n)|^2 < \infty.$$

Because if K_N is the Fejér kernel, it holds

$$\|\mu * K_N\|_2 \leq C\|\mu * K_N\|_1 \leq C\|\mu\|$$

the spectrum of the trigonometric polynomial $\mu * K_N$ being contained in the Sidon set Λ, on which the L^1 and L^2-norms are equivalent. That is

$$\sum_{n=1}^{\infty} |\widehat{\mu}(\lambda_n)|^2 (K_N(\lambda_n))^2 \leq C^2 \|\mu\|^2,$$

which implies $\sum_{n=1}^{\infty} |\widehat{\mu}(\lambda_n)|^2 \leq C^2 \|\mu\|^2$ by letting N tend to infinity.

Whence the question of M. Kàc: anyway, could f be a coboundary but now with some measurable g ? Fukuyama [8] proved that this equation actually has no measurable solution by establishing

Theorem 6.2. *Suppose that $f \in A(\mathbf{T})$ and $\hat{f}(n) = 0$ if $n \neq \pm 2^k$, $(k \geq 0)$; if the coboundary equation*

$$f(x) = g(2x) - g(x) \text{ a.e.} \tag{6.2}$$

has a measurable solution g, it has also a solution in $L^2(\mathbf{T})$.

We focus in turn on this coboundary equation and look for conditions ensuring the existence of L^2-solutions. The Walsh decomposition of course is very well appropriate to the 2-shift. By mimicking Herman's approach, we are able to prove:

Theorem 6.3. *Let $f \in L^2$ have its Fourier-Walsh spectrum in the set Λ_s where $\Lambda_s = \{w_k : [k] = s\}$, s is fixed and $[k]$ is the sum of digits of the integer k in base 2. Then if the coboundary equation*

$$f(x) = g(2x) - g(x) \text{ a.e.} \tag{6.3}$$

has a measurable solution g, it has also a solution in $L^2(\mathbf{T})$.

Remarks. 1. Observe that the hypothesis can also be expressed in terms of the already encountered set $\Lambda_s = \{\pm 2^{k_1} \pm \cdots \pm 2^{k_s}\}$, $k_i \geq 1$.
2. The set Λ_s being sub-lacunary, this result provides an improvement to Fukuyama's theorem.

Proof. Our proof makes use of three arguments:
- The previous Lemma 4.1.
- The property 3.1 since Λ_s is a $\Lambda(p)$-set with $p > 2$.
- The strong mixing property of the 2-shift.

We now use the precise version of Theorem 3.3 for the $\Lambda(p)$-character of the sets Λ_s. We write $\Lambda := \Lambda_s$ and we suppose that g is a measurable solution of (6.2) with $f \in L^2_\Lambda$ i.e. $g(x) - g(\sigma x) = f(x)$. The function g is "almost bounded" :

$$\forall \varepsilon > 0,\ \exists K \subset [0,1);\ m(K) \geq 1-\varepsilon \text{ and } g_{|K} \text{ is bounded;}$$

we just have to observe that $\mathbf{T} = \cup_n\{|g| \leq n\} = \lim \uparrow_n \{|g| \leq n\}$.

As $S_n f(x) = f(x) + f(\sigma x) + \cdots + f(\sigma^{(n-1)}x)$, we can write

$$\begin{aligned} \textstyle\int_{K\cap\sigma^{-n}K} |S_n f(x)|^2\, dx &= \textstyle\int_{K\cap\sigma^{-n}K} |g(x) - g(\sigma^n x)|^2\, dx \\ &\leq \sup_{K\cap\sigma^{-n}K} |g(x) - g(\sigma^n x)|^2 \\ &\leq (2\sup_K |g(x)|)^2 < +\infty. \end{aligned}$$

We check that $S_n f \in L^2_\Lambda$ for every n; indeed, observe that, for any $h \in L^2$, (with the Walsh notations of section 3),

$$\begin{aligned} \langle h(\sigma^j x), w_k\rangle &= 0 && \text{if } 2^j \not| k \\ &= \langle h, w_{k/2^j}\rangle && \text{otherwise.} \end{aligned}$$

Now, $w_k \in \Lambda \Longrightarrow w_{2k} \in \Lambda$ (because $[2k] = [k] = s$) so that, if $k \notin \Lambda$ and $k = 2^j \ell$, then $\ell \notin \Lambda$ in turn; this implies $< S_n f, w_k >= 0$ for every $k \notin \Lambda$.

Let us then choose ε so that $(1-\varepsilon)^2 - \varepsilon \geq b$ where b is given by Lemma 3.1. Since the 2-shift is strongly mixing with respect to the Lebesgue measure,

$$\lim_{n\to\infty} m(K \cap \sigma^{-n}K) = m(K)^2$$

and there exists $N > 0$ such that, for $n \geq N$,

$$m(K \cap \sigma^{-n}K) \geq m(K)^2 - \varepsilon.$$

We may apply Lemma 3.1 to the function $\varphi = S_n f \in L^2_\Lambda$ and the set $E = K \cap \sigma^{-n}K$, since $m(E) \geq b$ by our choice of N and ε. Finally,

$$\begin{aligned} \|S_n f\|_2 &\leq C\Big(\textstyle\int_{K\cap\sigma^{-n}K} |S_n f(x)|^2 dx\Big)^{1/2} \\ &\leq 2C\|g\|_K. \end{aligned}$$

It follows that $\sup_n \|S_n f\|_2 < +\infty$ and f is a coboundary in L^2 by Lemma 4.1. □

Remark. The asymptotic behaviour of general Raikov sums $S_n f(x) := \sum_{k\leq n} f(\omega_k x)$ has been investigated for less lacunary sequences (ω_k): an emblematic example consists in the so-called Fürstenberg sequence of integers (s_n), which is the semigroup $\langle 2, 3\rangle$ rearranged as an increasing sequence; more general semigroups $\langle p_1, \ldots, p_s\rangle$ have been studied, where the p_j are coprime numbers. The following result deserves to be recalled. [9].

Theorem 6.4 (Fukuyama-Petit)**.** *Assume f satisfies a "strong" Hölder type condition. Then*

$$\lim_N \|S_N f\|^2/N = 0 \Longleftrightarrow f(x) = \sum_{j=1}^{s} (f_j(p_j x) - f_j(x))$$

for some L^2-functions f_j.

7 Acknowledgments

The authors thank the anonymous referee, whose careful reading helped them to significantly improve the initial version of this work.

References

[1] Anosov, D. V., *The additive functional homology equation that is connected with an ergodic rotation of the circles*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 1259–1274.

[2] A. Bonami, *Étude des coefficients de Fourier des fonctions de $L^p(G)$*, Ann. Inst. Fourier (Grenoble) **20** (1971), 335–402.

[3] J. Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$-set problem,* Acta Mathematica **162** (1989) 227–246.

[4] C. Borell, *On the integrability of Banach space valued Walsh polynomials,* Séminaire de Probabilités XIII, Lecture Notes in Math. **721** (1979), 1–3.

[5] D. Cohn, *Measure Theory,* Second Edition, Birkhäuser, 2013.

[6] S. Drury, *Sur les ensembles de Sidon,* C.R.A.S. **271** (1970), no. 2, 162–163.

[7] P. Duren, *Theory of H^p spaces,* Dover Publications (2000).

[8] K. Fukuyama, *On a gap series of Mark Kac*, Coll. Math. **81** (1999), 157–160.

[9] K. Fukuyama, B. Petit, *Le théorème limite central pour les suites de R. C. Baker*, Erg. Th. Dyn. Systems **21** (2001), 479–492.

[10] B. Fayad, A. Windsor, *A dichotomy between discrete and continuous spectrum for a class of special flows over rotations*, J. Mod. Dyn. **1** (2007), 107–122.

[11] R. Gundy, N. Varopoulos, *A martingale that occurs in harmonic analysis,* Arkiv. Math **14** (1976), no.2, 179-187.

[12] J. Garnett, *Bounded analytic functions,* Revised First Edition, Springer 2007.

[13] G. H. Hardy, E. M. Wright *An introduction to the theory of numbers*, Fifth edition, Oxford Science Publications, 1995.

[14] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Études Sci. Publ. Math. Systems **49** (1979), 5–233.

[15] M. Herman, *L^2-regularity of measurable solutions of a finite-difference equation of the circle*, Erg. Th. Dyn. Systems **24** (2004), 1277–1281.

[16] A. Iwanik, M. Lemanczyk, C. Mauduit, *Piecewise absolutely continuous cocycles over irrational rotations*, J. London Math. Soc. (2) **59** (1999), no. 1, 171–187.

[17] M. Kac, *On the distribution of sums of the type $\sum f(2^k t)$,* Annals of Mathematics **47** (1)(1946), 33–49.

[18] J. P. Kahane, *Some random series of functions*, Second edition, Cambridge, 1985.

[19] Y. Katznelson, *An introduction to harmonic analysis*, Third edition, Cambridge, 2004.

[20] M. A. Krasnosel'skii, J. B. Ruticki, *Convex functions and Orlicz spaces*, P. Noordhoff, 1961.

[21] D. Li, H. Queffélec, L. Rodríguez-Piazza, *Some new thin sets of integers in harmonic analysis,* J. Anal. Math **86**, 105-138, 2002.

[22] D. Li, H. Queffélec, *Introduction to Banach spaces, Analysis and Probability,* Cambridge Studies in Advanced mathematics, Vol. 1, 2, 2018.

[23] Y. Meyer, *Sur un problème de Michael Herman*, Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 726–731, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983.

[24] F. Nazarov, A. Nishry, M. Sodin, *Log-integrability of Rademacher Fourier Series with applications to random analytic functions,*, arXiv:1301.0529v1, 3 Jan 2013.

[25] K. Petersen, *Ergodic theory*, Cambridge Studies in Advanced Mathematics, 2. Cambridge University Press, Cambridge, 1983.

[26] G. Pisier, *Sur l'espace de Banach des séries de Fourier aléatoires presque sûrement convergentes,* Séminaire sur la Géométrie des espaces de Banach 1977-78, Exposé 14, École polytechnique, Paris.

[27] M. Queffélec, *Substitution dynamical systems–spectral analysis*, Lecture Notes in Mathematics 1294, Second edition, Springer-Verlag, Berlin, 2010.

[28] H. Queffélec, M. Queffélec, *Diophantine approximation and Dirichlet series*, HRI Lecture Notes Series–2, 2013.

[29] H. Queffélec, *Topologie*, Dunod (2016).

[30] M. M. Rao, Z. D. Ren, *Theory of Orlicz spaces*, Pure and Applied Mathematics **146**, Marcel Dekker, 1991.

[31] A. V. Rozhdestvenskii, *On absolutely continuous weakly mixing cocycles over irrational rotations,* Sbornik Mathematics **194** (5) (2003), 775-792.

[32] W. Rudin, *Fourier Analysis on groups,* Wiley Interscience, 1962.

[33] D. Volny, *Constructions of smooth and analytic cocycles over irrational circle rotations*, Comment. Math. Univ. Carolin. **36** (1995), 745–764.

[34] A. Wintner, *The linear difference equation of first order for angular variables*, Duke Math. J. **12**, (1945). 445–449.

[35] J.C. Yoccoz, *Petits diviseurs en dimension 1*, Astérisque **231** (1995).

[36] A. Zygmund, *Trigonometric series,* Second Edition, Cambridge Mathematical Library, 1993.

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