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Zeeman's conjecture

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Abstract

Zeeman's conjecture says that for every contractible 2-polyhedron K the product $K \times I$ is collapsible. In this short survey, we discuss several examples and partial results, and explain that Zeeman's conjecture for restricted classes of K is equivalent to the Poincaré conjecture and Andrews–Curtis conjecture with stabilisations.

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This is an overview of Zeeman's conjecture. The conjecture first appeared in [30] and a good reference for it is [22].

1 Collapsibility

Before we state Zeeman's conjecture, we have to introduce some concepts from PL-topology. References for PL-topology include Zeeman's notes [29] and Hudson's book [16].

1.1 Polyhedra

Let us start with the basic definitions [16, Section I.1]. A *Euclidean cell* is a convex hull of a finite set of points of some $\mathbb{R}^n$. The prototypical example is the standard k-simplex, which is the convex hull of the $k+1$ standard basic vectors e_i in $\mathbb{R}^{k+1}$. A *Euclidean polyhedron* is a subset of a Euclidean space $\mathbb{R}^n$ that is a union of finitely many Euclidean cells. A continuous map of Euclidean polyhedra is *piecewise linear* (PL) if its graph is a Euclidean polyhedron. A *PL-structure* on a topological space X is a maximal collection of *polyhedral charts*, i.e. embeddings $f_i \colon P_i \to X$ with P_i a Euclidean polyhedron ("charts"), such that (i) the images of the maps f_i cover X, and (ii) for all i and j there exists a k such $f_i(P_i) \cap f_j(P_j) = f_k(P_k)$ with $f_i^{-1} f_k$ and $f_j^{-1} f_k$ piecewise linear. A continuous map between topological spaces with PL-structures is *PL* if it is so with respect to the polyhedral charts.

Every Euclidean polyhedron has a canonical PL-structure: those polyhedral charts compatible with the identity. If P is a Euclidean polyhedron and $P \cong X$ is a homeomorphism, then we can transfer the PL-structure on P along this homeomorphism to X. We refer to X with this PL-structure as a *polyhedron*. A *d-polyhedron* is a polyhedron that has no PL-charts whose domains are Euclidean polyhedra of dimension $> d$.

One can also describe polyhedra simplicially [16, Section I.2–4]. In that description one starts with the notion of a finite *simplicial complex*: a finite set V_0 of vertices and collections V_k of k-element subsets of V_0 such that if $\sigma \in V_k$, then any k'-element subset τ of σ (called a *face*) is in $V_{k'}$. These give rise to topological spaces by geometric realisation:

$$|V| := \left(\bigcup_{k \geq 0} V_k \times \Delta^k \right) / \sim$$

A homeomorphism $|V| \cong X$ is called a *triangulation*. Then equivalently a polyhedron is a topological space with an equivalence class of triangulations under subdivision. Here subdivision means either iterated stellar or iterated derived subdivision. Then a map $f \colon K \to L$ of polyhedra is *PL* if and only if there exist triangulations $|V| \cong K$, $|W| \cong L$ and a simplicial map $g \colon V \to W$ such that $|g| \colon |K| \to |L|$ is equal to f.

1.2 Collapsibility

The *standard PL-disk* D^n is given by $|\Delta^n|$ where Δ^n is a standard n-simplex. The *standard PL-sphere* S^{n-1} is given by $|\partial \Delta^n|$. These are examples of n- and $(n-1)$-polyhedra respectively. They are both examples of PL-manifolds, which by definition are polyhedra such that the link of each point is PL-homeomorphic to S^{n-1} (or D^{n-1} if it is a point in the boundary) [16, Section I.5]. Given a simplicial complex, the *star* $\mathrm{St}(v)$ of a vertex v consists of all simplices containing v, as well as their faces, and its *link* $\mathrm{Lk}(v)$ consists of all simplices in its star which do not contain v. For point v in a polyhedron, one defines these by choosing triangulation having v as a vertex; the result is independent of this choice up to PL-homeomorphism [16, Corollary 1.15].

Having established these definitions, we can proceed to the notion of collapsibility [16, Section II.1]

Definition 1.1.

(i) We say that there is an *elementary collapse* from K to L, denoted $K \searrow^e L$, if $L \subset K$ and there is a

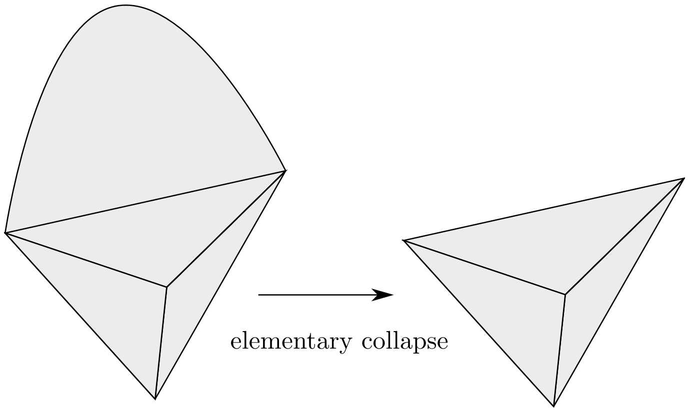


Fig. 1: An elementary collapse $D^2 \cup_{D^1} S^2 \to S^2$.

PL-ball $D^n \subset K$ such that $K = D^n \cup_{D^{n-1}} L$ where $D^{n-1} \subset \partial D^n$ and $D^{n-1} \subset L$ are subpolyhedra.

(ii) We say that K *collapses to* L if there is a finite sequence of elementary collapses starting with K and ending with L. We denote this $K \searrow L$.

(iii) If $K \searrow *$ we say that K is *collapsible.*

There is a related notion of a *simplicial collapse*, where one says that V collapses to W if $V = \Delta^n \cup_{\Lambda^n} W$ where Λ^n is $\partial\Delta^n$ with one face removed. A theorem of Whitehead says that if K is a simplicial complex and $|K|$ is collapsible, then there is an iterated stellar subdivision K' of K which is simplicially collapsible [27, Theorem 7]. Adiprasito and Benedetti proved that there is an integer $k \geq 0$ such that the k-fold iterated barycentric subdivision $\mathrm{Sd}^k(K)$ is simplicially collapsible [1, Theorem D]. You need these subdivisions, as the following example shows.

Example 1.2 (Rudin's triangulation of the 3-disk). In [25], Rudin constructed a triangulation of D^3 which is not simplicially collapsible, though of course it is collapsible and hence simplicially collapsible after some subdivisions.

There are several places in mathematics where collapses show up.

Example 1.3 (Simple homotopy theory, [6]). First of all, say that two polyhedra X, X' are *simply homotopy equivalent* if there exists a zigzag

$$X \searrow^e X_1 \searrow^e X_2 \searrow^e \ldots {}^e\swarrow X_n \searrow^e X'$$

of elementary collapses, each either leftwards or rightwards. A theorem of Whitehead says that any two simply-connected homotopy equivalent polyhedra are simply homotopy equivalent [27, Theorem 20]. Furthermore one can choose a zigzag such that none of the X_i's that appear in the sequence have dimension greater than $\max(\dim X, \dim X') + 1$ (provided that $\max(\dim X, \dim X')$ is not 2); this is [27, Addendum, p. 290] with an improvement due to Wall [26].[1] If they are not simply-connected there is an obstruction in the Whitehead group $\mathrm{Wh}(\pi_1)$, a quotient of the first algebraic K-theory group $K_1(\mathbb{Z}[\pi_1])$.

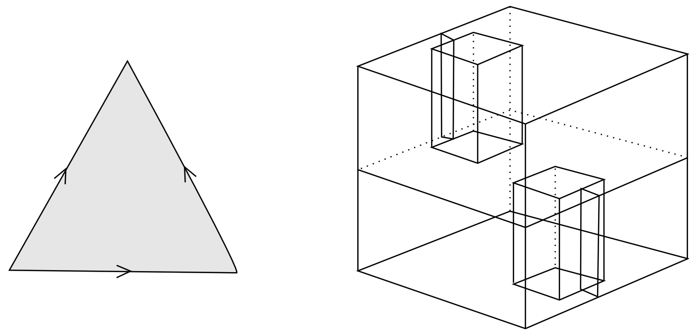
Example 1.4 (Regular neighborhoods in PL-manifolds, Section II.4 of [16]). If $K \subset M$ is a PL-embedding of a polyhedron in a PL-manifold, then there always exists a *regular neighborhood* N of K. This is by definition a closed neighborhood N of X that is submanifold of M and satisfies $N \searrow K$. These are essentially unique in the sense that if N and N' are two regular neighborhood, then there is a PL-homeomorphism of M fixing K and sending N to N'.

A polyhedron X in M is called a *spine* of X if M is a regular neighborhood of X. If M is path-connected with non-empty boundary, you can always find a spine with all cells of dimension $\leq \dim M - 1$. The idea is to "push in" top-dimensional cells from the boundary until they are all gone.

A collapsible polyhedron must be contractible, but there are many contractible polyhedra which are not collapsible. We will describe two classical examples.

Example 1.5 (Dunce cap and house with two rooms). The *dunce cap* D is the space obtained from the simplex Δ^2 by identifying its three faces as part (i) of Fig. 2; it inherits a canonical PL-structure from Δ^2.

It is homeomorphic to the CW-complex obtained by glueing a 2-cell to a circle x via the attaching map xxx^{-1}. This attaching map shows that it is homotopy equivalent to the CW-complex obtained by glueing a 2-cell to a circle via the attaching x, i.e. a 2-disk, and hence it contractible. On the other hand, the dunce cap is *not* collapsible, as there is no free face to get started. To make this precise, one proves that the link of every point in D (with respect to its canonical PL-structure) is ○ (for a point in the interior of Δ^2), ⦶ (for a point in the interior of a 1-simplex of Δ^2), or ○-○ (for a 0-simplex of Δ^2). An elementary collapse requires a free face and hence a point whose link is a disk.



(i) dunce cap (ii) house with two rooms

Fig. 2: Two examples of non-collapsible polyhedra.

[1] Look at [26, p. 351] to see why the case $\max(\dim X, \dim X') = 2$ is different.

A similar example is the *house with two rooms*, given by taking a lump of clay, poking in a hole from both the top and the bottom, and from these holes hollowing out U-shaped rooms; it is shown in part (ii) of Fig. 2. The definition makes clear it is a deformation retract of I^3 and hence contractible, but it is again not collapsible because there is no place to get started. Indeed, one can prove that this lack of free faces is the only obstruction for collapsing contractible 2-polyhedra; if it is not collapsible, then whatever order one starts collapsing things, one will always end up with something that has no free faces.

Other examples with interesting names are the *house with one room* and the *abalone* [22, p. 336].

We end with a characterisation of collapsible PL-manifolds due to Whitehead.

Theorem 1.6 (Corollary 1_n of [27]). *A PL-manifold is collapsible if and only if it is PL-homeomorphic to a disk.*

Sketch of proof. We prove the stronger statement that a regular neighborhood N of a collapsible polyhedron K in a PL-manifold M is always a PL-disk [16, p. 57]. This is done by induction over the dimension using uniqueness of regular neighborhoods. In particular, assuming regular neighborhoods in n-dimensional PL-manifold are unique we give the following argument. If $k_0 \in K$ is such that $K \searrow k_0$ and $N \subset M$ is a regular neighborhood of K, it is also a regular neighborhood of k_0, since $N \searrow K \searrow k_0$. In a PL-manifold the star $\mathrm{St}(k_0) \subset M$ of k_0 with respect to some triangulation is a regular neighborhood of k_0, and is easily shown to be PL-homeomorphic to a disk. By the assumed uniqueness of regular neighborhoods we conclude that $N \cong \mathrm{St}(k_0) \cong D^n$ and we are done. □

2 Zeeman's conjecture and some basic observations

Zeeman's conjecture is the following deceivingly elementary statement [30]. It is still open, though we will later see that it is partially settled by Perelman's proof of the Poincaré conjecture.

Conjecture 2.1 (Zeeman's conjecture).
If K is a contractible 2-polyhedron, then $K \times I \searrow *$.

We start with a result due to Dierker [9] that makes the conjecture sound more plausible.

Lemma 2.2. *For any contractible 2-polyhedron K there exists an integer $k \geq 0$ such that $K \times I^k \searrow *$.*

Proof. We claim that if $L \searrow^e K$ and $L \searrow *$, then $K \times I \searrow *$. Suppose that K is obtained by an elementary collapse of D^n in L. Then we note that $K \times I \searrow L$ by taking a triangulation of K compatible with inclusion $D^{n-1} \to K$ and collapsing for each of the simplices σ of this triangulation the subset $\sigma \times I \subset K \times I$ to $\sigma \times 0$ *except* those σ that are contained in D^{n-1}. The end result is $K \cup_{D^{n-1}} (D^{n-1} \times I)$, which is PL-homeomorphic to L.

To complete the proof we note that since K is contractible, it is simply homotopy equivalent to a point by an aforementioned result of Whitehead. This means that there exists some sequence of collapses and expansions relating it to the point. The number k will then be the number of expansions in this sequence, by repeated application of our observation and the fact that if $K \searrow *$ then $K \times I \searrow *$ as well. To prove the latter, one notes that crossing with I simply replaces D^n and D^{n-1} in every elementary collapse with D^{n+1} and D^n. Thus using a sequence of elementary collapses from K to a point gives a sequence of elementary collapses of $K \times I$ to I and I collapses to a point. □

The dunce cap and house with two rooms are not counterexamples to Zeeman's conjecture. Indeed, Zeeman made his conjecture after proving that dunce cap becomes collapsible after taking the product with an interval.

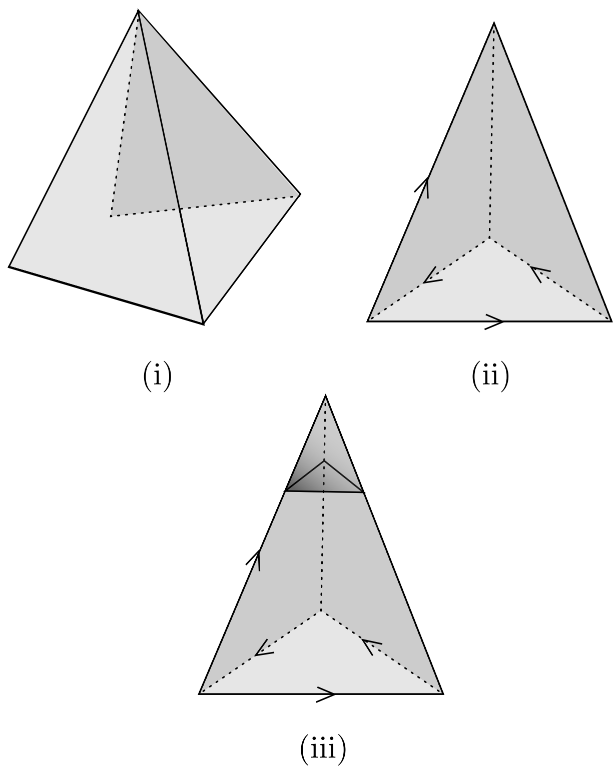


Fig. 3: Part (i) and (ii) hopefully make clear our way of thinking of the dunce cap; you start folding and end with "the top of a pyramid", its bottom circle identified with one edge. Part (iii) displays D', obtained by glueing a tetrahedron to D.

Example 2.3 (Dunce cap and house with two rooms). We claim that the dunce cap and house with two rooms both become collapsible after taking the product with an interval. This will use the observation in the previous proof that if $L \searrow^e K$ and $L \searrow *$ then $K \times I \searrow *$.

In other words, it suffices to find an elementary expansion of the dunce cap or the house with two rooms so that the result is collapsible. We will give a complete description for the dunce cap, and only tell you that the required expansion for the house with two rooms is thickening one of the internal walls linking a chimney to the side [12].

Let D denote the dunce cap and think of it as being obtained by taking a 2-simplex, folding it along two diagonals lines from the top, glueing the two vertical edges and glueing the horizontal edge to that, see Fig. 3 (i) and (ii). In Fig. 3 (ii), we see that D consists of three triangles glued along their edges. In particular, there is *no* bottom triangle.

We glue on a tetrahedron in the top, as in Fig. 3 (iii) and denote the result by D'. We claim that this is collapsible. We first get rid of the tetrahedron by pushing in from the right side, Fig. 4 (i), and next collapse the two remaining vertical faces of the tetrahedron, Fig. 4 (ii). At this point, we might as well forget about the vertical line segment from v to w as it is identified with the corresponding line segment on the bottom-right. If we do so, the line segment from v to w on the bottom-right becomes free, and we can do a collapse to get rid of the face to the right, Fig. 4 (iii). Then we collapse the back and top faces, after which we end up with a disk, Fig. 4 (iv).

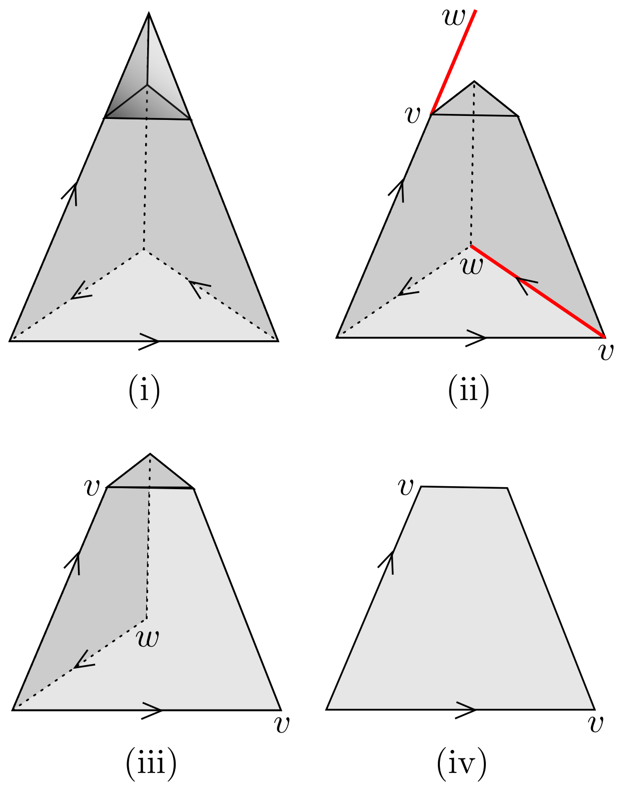


Fig. 4: Collapsing D'.

The closest result to a proof of Zeeman's conjecture is the following result of M. Cohen.

Theorem 2.4 (Corollary 4 of [7])**.** *If K is a contractible 2-polyhedron, then $K \times I^6$ is collapsible. Similarly, if K is a contractible n-polyhedron for $n \geq 3$, then $K \times I^{2n}$ is collapsible.*

Strategy of proof. One first shows that if X and Y are spines of M then $\mathrm{Cone}(\partial M) \times X \searrow Y$, and then uses that any 2-dimensional polyhedron embeds in $M = I^6$ and any n-dimensional polyhedron embeds in $M = I^{2n}$. The latter uses engulfing [10] and hence relies implicitly on an analogue of the Whitney trick in the PL-setting, explaining why the case $n = 2$ is different. □

The closest result to a counterexample is the following result.

Theorem 2.5 (Theorem 3 of [8])**.** *For all $n \geq 3$ there exists a contractible n-polyhedron K such that $K \times I$ is not collapsible.*

Strategy of proof. One considers the "dimension of the Whitehead torsion": this invariant $\dim \tau$ of an element $\tau \in \mathrm{Wh}(\pi_1)$ is the integer given by $\min\{\dim(P \backslash L) \mid L \subset P,\ P$ deformation retracts on L, and $\tau(P, L) = \tau\}$. It turns out that there exists a finitely-presented group π_1 and $\tau \in \mathrm{Wh}(\pi_1)$ with $\dim(\tau) = 3$. These can be used to construct counterexamples of the higher-dimensional Zeeman conjecture of the form $K = \mathrm{Cone}(L \to P)$ where L is a $(n-1)$-dimensional polyhedron with $\pi_1(L) = \pi_1$ and P obtained from L by attaching I^n to L, such that $\tau(P, L) = \tau$. □

3 The Poincaré conjecture

We will show how Zeeman's conjecture implies the Poincaré conjecture. The proof is relatively straightforward, except for the following fact about 3-dimensional manifolds [20] (see also [21]), which we will use to give a triangulation on a topological 3-manifold homotopy equivalent to S^3. It is false in higher dimensions.

Theorem 3.1 (Moise)**.** *All 3-dimensional topological manifolds admit unique PL- and smooth structures.*

For completeness we state the now proven Poincaré conjecture [23], using the previous theorem to remark that the same is true for PL-manifolds and smooth manifolds.

Theorem 3.2 (Perelman)**.** *If a 3-dimensional topological manifold is homotopy equivalent to S^3, then it is homeomorphic to S^3.*

We need the following version of the topological Schoenflies theorem, due to Brown [4] and Mazur [19]. Note that it is easier than the smooth Schoenflies theorem, which requires the h-cobordism theorem in dimensions ≥ 5, is open in dimension 4, and follows from topological version in dimension 1, 2 and 3.

Theorem 3.3 (Topological Schoenflies theorem)**.** *For any n, any locally flat topological embedding of S^{n-1} into S^n bounds two standard n-balls.*

Sketch of proof. Locally flat embeddings as in the statement locally look like $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, so in particular have closed locally contractible image. By Alexander duality we have $\tilde{H}^0(S^n \backslash S^{n-1}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, and so there are two path-components. Using once more that the embedding is locally flat, the closures of these path-components are topological n-manifolds with boundary. Let X and Y be the closed oriented topological n-dimensional manifolds obtained by glueing a D^n to the boundaries of these closures. Now consider the infinite connected sum

$$X\#Y\#X\#Y\#\cdots$$

Bracketing this as $(X\#Y)\#(X\#Y)\#\cdots$ gives S^n and bracketing as $X\#(Y\#X)\#\cdots$ gives X, so that X is homeomorphic to S^n. This is an example of an Eilenberg swindle. □

Zeeman already observed that his conjecture implies the Poincaré conjecture, using the following argument.

Proposition 3.4 (Theorem (2) of [30])**.** *Zeeman's conjecture implies the Poincaré conjecture.*

Proof. Let S be a topological 3-manifold homotopy equivalent to S^3. By Theorem 3.1 it admits a PL-structure so a triangulation, and we can remove a open 3-simplex to obtained a 3-manifold M with boundary S^2. Mayer–Vietoris implies that $\tilde{H}_*(M) = 0$ and Seifert–van Kampen that $\pi_1(M) = 0$. So M is contractible.

Let K be a spine of M, which is a contractible 2-polyhedron. If Zeeman's conjecture is true, then $K \times I \searrow *$. Since $M \searrow K$, we also get $M \times I \searrow *$. By Theorem 1.6, thus $M \times I$ is PL-homeomorphic to D^4. The boundary of $M \times I$ is S^3 and M sits in it as $M \times \{0\}$ by a locally flat embedding. Next $S^2 \cong \partial M \times \{0\}$ is a locally flat embedding of S^2 into S^3. The Schoenflies theorem implies that M is homeomorphic to D^3. Thus S is obtained from glueing D^3 to D^3 along S^2 via some homeomorphism, which we may assume is orientation-preserving. Since $\text{Homeo}^+(S^2)$ is path-connected [17] (see also [11, Corollary 3]), the result is homeomorphic to glueing D^3 to D^3 along S^2 via the identity homeomorphism, and S is homeomorphic to S^3. □

The following definition appears in [13, §2], and is due to Casler [5]:

Definition 3.5. A *standard 2-polyhedron* is one that locally looks like one of the situations in Fig. 5.

Remark 3.6. A standard 2-polyhedron, by a twist of fate, is sometimes also called a special 2-polyhedron (e.g. in [22]) or a fake closed surface (e.g. in [28]).

Lemma 3.7. *Every connected 3-dimensional PL-manifold with boundary has a standard spine.*

Proof. I will quote Gillman and Rolfsen quoting Zeeman: *...choose a spine in the interior; expand each edge like a banana and collapse from one side; then expand each vertex like a pineapple and collapse from one face".* □

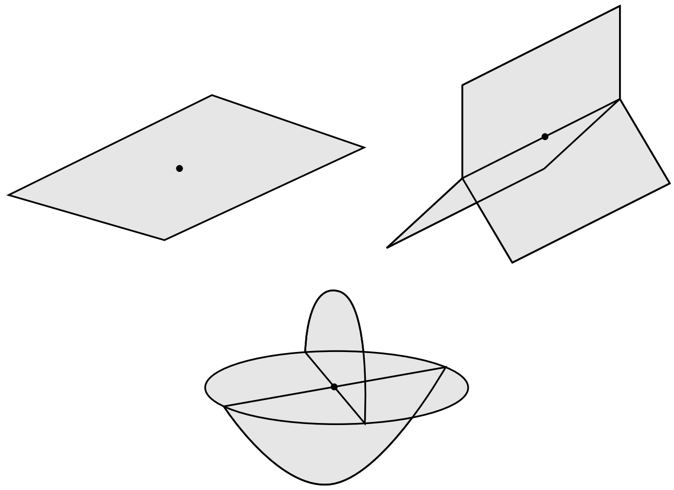


Fig. 5: The three allowed local situations for a standard 2-polyhedron.

Proposition 3.8 (Theorem 3 of [13])**.** *Zeeman's conjecture for standard 2-polyhedra that are spines of 3-dimensional PL-manifolds is equivalent to the Poincaré conjecture.*

Sketch of proof. The implication $\Rightarrow$ is a direct consequence of Lemma 3.7 and the proof of Proposition 3.4, in which we only applied Zeeman's conjecture to the spine of a 3-dimensional PL-manifold. For the implication $\Leftarrow$, Gillman and Rolfsen prove that if M is a homotopy ball (or more generally homology ball) and K is a standard spine, then $K \times I$ collapses to something homeomorphic to M. The Poincaré conjecture implies that there are no other homotopy 3-balls than the standard one. But the standard one is obviously collapsible, so we get $K \times I \searrow M \searrow *$. □

Since the Poincaré conjecture is nowadays known to be true, so is Zeeman's conjecture for standard 2-polyhedra that are spines of 3-dimensional PL-manifolds.

4 The Andrews–Curtis conjecture

The Andrews–Curtis conjecture is a conjecture about recognising balanced presentations of the trivial group [2] [3] (see [15] for a survey of recent work).

Let $\langle x_1, \ldots, x_n \rangle$ denote the free group on $x_1, \ldots, x_n$ and let $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ denote the group generated by $x_1, \ldots, x_n$ satisfying the relations $r_1, \ldots, r_m \in \langle x_1, \ldots, x_n \rangle$ (that is, take the quotient by the normal subgroup generated by $r_1, \ldots, r_m$). We say such a presentation is *balanced* if $n = m$.

There are certain moves we can perform on the relations to obtain a different presentation of the same group, called *extended Nielson operations*:

(i) Replace an r_i with r_i^{-1}.

(ii) Permute the r_i.

(iii) Replace r_i with gr_ig^{-1} for $g \in \langle x_1, \ldots, x_n \rangle$.

(iv) Replace r_i with r_ir_j for $i \neq j$.

The *Andrews–Curtis conjecture* says that these moves suffice to reduce a balanced presentation of the trivial group to the trivial balanced presentation.

Conjecture 4.1 (Andrews–Curtis conjecture)**.** *If $\langle x_1, \ldots, x_n | r_1, \ldots, r_n \rangle$ is a presentation of the trivial group, then we can transform $r_1, \ldots, r_n$ into $x_1, \ldots, x_n$ with a finite number of extended Nielsen operations.*

Remark 4.2. Suppose that f is an automorphism of the free group $\langle x_1, \ldots, x_n \rangle$ then we claim it suffices to transform $r_1, \ldots, r_n$ into $f(x_1), \ldots, f(x_n)$. This is because the automorphism group of a free group is generated by the *elementary Nielsen transformations* [24]. These are the automorphisms given by:

(I) Send x_i to x_i^{-1}.

(II) Permute the x_i's.

(III) Send x_i to x_ix_j for $i \neq j$.

These can visibly be implemented by the moves (i), (ii), and (iv) respectively.

To understand the relationship between the Andrews–Curtis conjecture and Zeeman's conjecture, we will explain how a generalisation of the former is a consequence of the latter. To do so, we will interpret the operations (i)–(iv) geometrically. For a path-connected polyhedron K with a given triangulation, one can obtain a presentation of its fundamental group $\pi_1(K)$ by making a few additional choices.

First, we make some choices for the 1-skeleton: we pick a basepoint in the 0-skeleton and a spanning tree of the 1-skeleton. We then further pick an order $a_1, \ldots, a_n$ of the 1-simplices not in the spanning tree, as well as an orientation of each of these. From this we obtain an identification of the fundamental group of the 1-skeleton as the free group $\langle x_1, \ldots, x_n \rangle$ with the homotopy class of loop x_i given by the 1-simplex a_i and the paths from its endpoints to the basepoints along the spanning tree (unique up to homotopy).

Next, we make some choices for the 2-simplices: we pick an order $b_1, \ldots, b_m$, as well as an orientation of each of these together with a path in the 1-skeleton from the basepoint to a 0-simplex in its boundary. Combining these paths with the attaching map of each 2-simplex gives a well-defined word in $\langle x_1, \ldots, x_n \rangle$, and we obtain a collection of relations $r_1, \ldots, r_m$ in $\langle x_1, \ldots, x_n \rangle$. By an application of Seifert–van Kampen, we conclude that $\pi_1(K) \cong \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$.

This presentation depends on some choices, and when we make other ones the resulting presentation will differ by repeated applications of the following moves [14, p. 20]:

(i') Replace an r_i with r_i^{-1}.

(ii') Permute the r_i.

(iii') Replace r_i with gr_ig^{-1} for $g \in \langle x_1, \ldots, x_n \rangle$.

(iv') Take an automorphism of $\langle x_1, \ldots, x_n \rangle$, the free group, and apply it to each of the r_i.

The move (iv') is a consequence of making different choices for the 1-skeleton. Indeed, these yield compositions of *elementary Nielsen transformations* as in Remark 4.2. The remaining moves (i')–(iii') are consequences of making different choices for the 2-simplices: move (i') corresponds to reading an attaching map of a 2-simplex in opposite direction, move (ii') to reordering the 2-simplices, and move (iii') to picking other paths.

Now further suppose that K is a 2-polyhedron and we allow ourselves to change K by a zigzag of elementary collapses $K \searrow^e K_1 \searrow^e \cdots {}^e\swarrow K_n \searrow^e K'$, so that the dimension of the K_i does not exceed 3. This adds two additional moves [14, p. 21]:

(v') Replace r_i with r_ir_j for $i \neq j$.

(vi') Add a new generator x_{n+1} and a new relation r_{n+1} given by x_{n+1}; or do the opposite, that is, delete a generator x_{n+1} and a relation r_{n+1} given by x_{n+1} (when x_{n+1} does not appear in the other relations).

Note that the moves (i')–(iii') and (v') are exactly the extended Nielsen operations (i)–(iii) and (iv) of the Andrews–Curtis conjecture. Since the moves (i')–(iii') and (v') are preserved by applying an automorphism of $\langle x_1, \ldots, x_n \rangle$, by Remark 4.2 we may as well also allow move (iv') in Conjecture 4.1. The *Andrews–Curtis conjecture with stabilisations* is then obtained by further allowing the move (vi') in Conjecture 4.1.

Proposition 4.3. *Zeeman's conjecture implies the Andrews–Curtis conjecture with stabilisations.*

Proof. Consider the 2-polyhedron K given by a single 0-cell, n 1-cells corresponding to the x_i and n 2-cells attached via the r_i. If $\langle x_1, \ldots, x_n | r_1, \ldots, r_n \rangle$ is a presentation of the trivial group, K is contractible and under the assumption of Zeeman's conjecture there is a zigzag $K \swarrow K \times I \searrow *$. Since $K \times I$ is at most 3-dimensional, the result follows. □

Remark 4.4. The Andrews–Curtis conjecture with stabilisations can be rephrased geometrically: it is equivalent to the statement that if K is a contractible 2-polyhedron then there exists a zigzag $K \searrow^e K_1 \searrow^e \cdots {}^e\swarrow K_n \searrow^e *$ such that none of the K_i's have dimension exceeding $3 \geq \max(\dim K, \dim *) + 1$ [28]. That is, it removes the exception for dimension 2 from Example 1.3.

The Poincaré conjecture involved standard 2-polyhedra that are spines of 3-manifolds. Matveev considered what happens when one restricts Zeeman's conjecture to those standard 2-polyhedra that are *not* spines of 3-manifolds.

Proposition 4.5 (Corollary of the main theorem of [18]). *Zeeman's conjecture for standard polyhedra that are not spines of 3-manifolds is equivalent to the Andrews–Curtis conjecture with stabilisations.*

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