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Sign Patterns and Rigid Orders of Moduli

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Abstract

We consider the set of monic degree d real univariate polynomials $Q_d = x^d + \sum_{j=0}^{d-1} a_j x^j$ and their *hyperbolicity domains* Π_d, i.e. the subsets of values of the coefficients a_j for which the polynomial Q_d has all roots real. The subset $E_d \subset \Pi_d$ is the one on which a modulus of a negative root of Q_d is equal to a positive root of Q_d. At a point, where Q_d has d distinct roots with exactly s ($1 \leq s \leq [d/2]$) equalities between positive roots and moduli of negative roots, the set E_d is locally the transversal intersection of s smooth hypersurfaces. At a point, where Q_d has two double opposite roots and no other equalities between moduli of roots, the set E_d is locally the direct product of $\mathbb{R}^{d-3}$ and a hypersurface in $\mathbb{R}^3$ having a Whitney umbrella singularity. For $d \leq 4$, we describe the hyperbolicity domains in terms of sign patterns and (generalized) orders of moduli, and we draw pictures of the sets Π_d and E_d.

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1 Introduction

We consider *hyperbolic polynomials*, i.e. real univariate polynomials with all roots real. For degree ≤ 4, we discuss the question about the signs of the coefficients of such polynomials and the order of the moduli of their roots on the real positive half-line. We give a geometric illustration of this by representing the general families of monic hyperbolic polynomials of such degrees.

Recall that for a real (not necessarily hyperbolic) polynomial Q, Descartes' rule of signs states that the number pos of its positive roots is not more than c, the number of sign changes in the sequence of its coefficients; the difference $c - pos$ is even, see [6]. Hence the number of sign changes c' in the sequence of coefficients of $Q(-x)$ majorizes the number neg of negative roots of Q and $c' - neg \in 2\mathbb{Z}$. In the case when Q has no vanishing coefficient one has $c' = p$ (the number of sign preservations in the sequence of the coefficients of the polynomial Q).

If Q is hyperbolic and with no vanishing coefficient these conditions imply $pos = c$ and $neg = p$. A hyperbolic polynomial with non-zero constant term has no two consecutive vanishing coefficients, see Remark 2 in [13].

Definition 1.1. (1) A *sign pattern* is a finite sequence of $(+)$- and/or $(-)$-signs. We say that the polynomial $x^d + \sum_{j=0}^{d-1} a_j x^j$ defines (or realizes) the sign pattern $(+, \mathrm{sgn}(a_{d-1}), \ldots, \mathrm{sgn}(a_0))$. Sometimes, when we allow vanishing of some coefficients, the corresponding components of the sign pattern defined by the hyperbolic polynomial are zeros. In this case we speak about *generalized sign pattern*.

(2) We say that the roots of the hyperbolic polynomial Q define an *order of moduli* on the real positive half-line. Suppose that the moduli of the roots of Q are all distinct and nonzero. The order of moduli is obtained when the moduli are written in a string in the increasing order, with the sign $<$ between any two consecutive moduli; after this moduli of positive (resp. negative) roots are replaced by the letter P (resp. N). Example: suppose that the degree 6 hyperbolic polynomial has positive roots $\alpha_1 < \alpha_2$ and negative roots $-\gamma_i$, where

$$\gamma_1 < \alpha_1 < \gamma_2 < \gamma_3 < \alpha_2 < \gamma_4 \ .$$

Then the roots of the hyperbolic polynomial define the order of moduli $N < P < N < N < P < N$. Sometimes we consider hyperbolic polynomials with equal moduli of some roots and/or with roots at 0. In this case we speak about *generalized order of moduli.* If the degree 8 hyperbolic polynomial has a double root at 0, positive roots $\alpha_1 < \alpha_2$ and negative roots $-\gamma_i$, where

$$\gamma_1 = \alpha_1 < \gamma_2 = \gamma_3 < \alpha_2 < \gamma_4 \ ,$$

then we say that the roots define the generalized order of moduli $0 = 0 < N = P < N = N < P < N$.

There are two particular cases of orders of moduli which are of interest to us:

Definition 1.2. (1) Each sign pattern (not containing zeros) defines the corresponding *canonical order of moduli* as follows. The sign pattern is read from right to left and to each two consecutive equal (resp. opposite) signs one puts in correspondence the letter N (resp. the letter P) after which one inserts between the letters the signs $<$. For example, reading the sign pattern $(+,-,+,-,+,+,-,-,+)$ from the right yields $(+,-,-,+,+,-,+,-,+)$ and the canonical order of moduli is $P<N<P<N<P<P<P<P$. These sign pattern and order of moduli correspond to degree 8 hyperbolic polynomials. Suppose that a given sign pattern is realizable only by hyperbolic polynomials with roots defining the corresponding canonical order of moduli. Then we call such a sign pattern *canonical*.

(2) When all polynomials with roots defining one and the same order of moduli define one and the same sign pattern, we say that this order of moduli is *rigid*.

Remarks 1.3. (1) Some necessary and some sufficient conditions for a sign pattern to be canonical are formulated in [12]. In particular, the sign pattern

$$(+,-,+,-,+,-,\ldots)$$

and the all-units sign pattern are canonical.

(2) There are two kinds of rigid orders of moduli (see [13]):

(a) the ones in which all roots are of the same sign (this is the so-called trivial case, the sign pattern either equals $(+,-,+,-,+,-,\ldots)$ or is the all-units sign pattern) and

(b) the ones in which moduli of negative roots interlace with positive roots (hence half or about half of the roots are positive and the rest are negative). In this case the sign pattern is of one of the forms

$$\begin{array}{cc} & (+,+,-,-,+,+,-,-,+,\ldots) \\ \text{or} & (+,-,-,+,+,-,-,+,+,\ldots) \end{array}$$

Definition 1.4. For the general family of monic real degree d polynomials $Q_d := x^d + \sum_{j=0}^{d-1} a_j x^j$ we define the *hyperbolicity domain* Π_d as the set of values of the parameters a_j for which the polynomial Q_d is hyperbolic. We denote by E_d (resp. F_d or G_d) the subsets of $\mathbb{R}^d \cong Oa_0 \ldots a_{d-1}$ for which a negative and a positive root have equal moduli (resp. for which there is a complex conjugate pair of purely imaginary roots or a double root at 0). We set $\tilde{E}_d := E_d \cup F_d \cup G_d$. We denote by $\mathrm{Res}(U,V,x)$ the *resultant* of the polynomials $U, V \in \mathbb{R}[x]$, and by Δ_d the *discriminant set* $\mathrm{Res}(Q_d, Q_d', x) = 0$. This is the set of values of the coefficients a_j for which the polynomial Q_d has a multiple root.

Our first result is the following theorem (proved in Section 2).

Theorem 1.5. *(1) At a point of Π_d, where Q_d has d distinct real roots with exactly one equality between a positive root and a modulus of a negative root, the set E_d is locally a smooth hypersurface.*

(2) At a point of Π_d, where Q_d has d distinct real roots with exactly s ($2s \leq d$) equalities between positive roots and moduli of negative roots, the set E_d is locally the transversal intersection of s smooth hypersurfaces (transversality means that the s normal vectors are linearly independent).

(3) Suppose that the polynomial Q_d has d distinct roots, but is not necessarily hyperbolic. Suppose that at a point of $\mathbb{R}_d$ it has s_1 equalities between positive roots and moduli of negative roots and s_2 conjugate pairs of purely imaginary roots, $2(s_1+s_2) \leq d$ and no other equalities between moduli of roots. Then at this point the set $\tilde{E}_d$ is locally the transversal intersection of s_1+s_2 smooth hypersurfaces.

(4) At a point, where Q_d has two double opposite real roots or a double conjugate pair of purely imaginary roots (all other moduli of roots being distinct and different from these ones) the set $\tilde{E}_d$ is locally diffeomorphic to the direct product of $\mathbb{R}^{d-3}$ and a hypersurface in $\mathbb{R}^3$ having a Whitney umbrella singularity.

Our next aim is the description for $d \leq 4$ of the *stratification* of the closure of the set Π_d defined by the multiplicities of the roots of Q_d, the possible presence of roots equal to 0 and the eventual equalities between positive roots and moduli of negative roots. This is done in Sections 3, 4 and 5.

The present paper is part of the study of real univariate polynomials in relationship with Descartes' rule of signs. The question for which sign patterns and pairs (pos, neg) compatible with this rule there exists such a polynomial Q_d is not a trivial one. It has been asked for the first time in [2] and the first non-trivial result has been obtained in [7]. The cases $d=5$ and 6 have been considered in [1]. The case $d=7$ (resp. $d=8$) has been treated in [4] (resp. in [4] and [8]). Other results in this direction are obtained in [3] and [9]. New facts about hyperbolic polynomials can be found in [10]. A tropical analog of Descartes' rule of signs is discussed in [5].

2 Proof of Theorem 1.5

Proof of part (1). One introduces the following local parametrization of Q_d:

$$W(x) := x^{d-2} + c_{d-3}x^{d-3} + \cdots + c_0 \ ,$$
$$Q_d = (x^2 - v^2)W(x) \ , \quad v, c_j \in \mathbb{R} \ , \ v \neq 0$$

The factor W is a degree $d-2$ hyperbolic polynomial and $W(\pm v) \neq 0$. Thus

$$\begin{array}{ll} a_{d-1} = c_{d-3} \ , & a_{d-2} = c_{d-4} - v^2 \ , \\ a_j = c_{j-2} - v^2 c_j \ , & 2 \leq j \leq d-3 \ , \\ a_1 = -v^2 c_1 \ , & a_0 = -v^2 c_0 \ . \end{array}$$

The mapping

$$\iota \ : \ \tilde{c} := (c_0, c_1, \ldots, c_{d-3}, v) \ \mapsto \ \tilde{a} := (a_0, a_1, \ldots, a_{d-1})$$

is of rank $d-1$, i. e. maximal possible. Indeed, the transposed of the Jacobian matrix $J := (\partial\tilde{a}/\partial\tilde{c})$, of size $(d-1)\times d$, equals

$$J^t = \begin{pmatrix} -v^2 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -v^2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -v^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -v^2 & 0 & 1 \\ -2vc_0 & -2vc_1 & -2vc_2 & \cdots & -2vc_{d-3} & -2v & 0 \end{pmatrix}$$

One can perform the following elementary operations which do not change the rank of J^t: for $j = d-2$, $d-1$, …, 1, add the $(j+2)$nd column multiplied by v^2 to the jth column. This makes disappear the terms $-v^2$ in the first $d-2$ rows. Only the units (see the matrix J^t) remain as nonzero entries in these rows. The two leftmost entries in the last row are now equal to

$$\begin{array}{rl} & U_0 := -2vc_0 - 2v^3c_2 - 2v^5c_4 - \cdots \\ \text{and} & U_1 := -2vc_1 - 2v^3c_3 - 2v^5c_5 - \cdots \end{array}$$

respectively (the sums are finite and their last terms depend on the parity of d). One can notice that $U_0 \pm vU_1 = -2vW(\pm v)$. As $v \neq 0$ and $W(\pm v) \neq 0$, at least one of the terms U_0 and U_1 is nonzero. Thus the $d-1$ rows of J^t after (and hence before) the elementary operations are linearly independent and rank(J) =rank(J^t) = $d-1$. □

Proof of part (2). If at some point of E_d the polynomial Q_d has d distinct real roots and exactly s equalities between positive roots and moduli of negative roots, then every such equality defines locally a smooth hypersurface. The proof of this repeats the proof of part (1) of this theorem. It remains to prove that the s normal vectors to these hypersurfaces are linearly independent.

Each of the s hypersurfaces can be given a local parametrization of the form $Q_d = (x^2 - v_j^2)W_j(x)$, $W_j = x^{d-2} + c_{d-3,j}x^{d-3} + \cdots + c_{0,j}$, see the proof of part (1). We define by analogy with the proof of part (1) the matrices J_j and J_j^t. Recall that each row of the matrix J_j^t is a vector tangent to the jth hypersurface. Hence a vector $\vec{w}_j$ normal to the jth hypersurface is orthogonal to all rows of J_j^t. Set $\vec{w}_j := (w_{1,j}, \ldots, w_{d,j})$. For $i = 1$, …, $d-2$, the condition that $\vec{w}_j$ is orthogonal to the ith row of J_j^t reads $w_{i+2,j} = v_j^2 w_{i,j}$.

Set $\vec{w}_j^\dagger := (w_{1,j}, w_{3,j}, w_{5,j}, \ldots, w_{2s-1,j})$. Hence up to a nonzero constant factor the vector $\vec{w}_j^\dagger$ is of the form $(1, v_j^2, v_j^4, \ldots, v_j^{2s-2})$. The vectors $\vec{w}_j^\dagger$ are the rows of a Vandermonde matrix whose determinant is nonzero, because the numbers v_j^2 are distinct. Hence the vectors $\vec{w}_j^\dagger$ are linearly independent and hence such are the vectors $\vec{w}_j$ as well. Part (2) of the theorem is proved. □

Proof of part (3). The proof of part (3) is performed by analogy with the proofs of parts (1) and (2). One uses the parametrization $Q_d = (x^2 + A)W(x)$ with W as in the proof of part (1). In the first $d-2$ rows of the matrix J^t the terms $-v^2$ are to be replaced by A and the last row becomes $(c_0, c_1, \ldots, c_{d-3}, 1, 0)$. □

Proof of part (4). For $d = 4$, the statement is part of Theorem 5.2. For $d > 4$, it results from Theorem 5.2 and from the following lemma about the product (see [14], p. 12):

Suppose that P, P_1, …, P_r are real monic polynomials, where for $i \neq j$, P_i and P_j have no root in common and $P = P_1 \cdots P_r$. Suppose that U_i (resp. U) is an open neighbourhood of P_i (resp. of P). Then the mapping

$$U_1 \times \cdots \times U_r \to U \ , \quad (R_1, \ldots, R_r) \mapsto R_1 \cdots R_r$$

is a diffeomorphism.

To use this result we represent the polynomial Q_d as a product Q^1Q^2, where Q^1 has two double opposite real roots or a double conjugate pair of purely imaginary roots, and Q^2 has the rest of the roots of Q_d. Thus one can apply the cited lemma with $r = 2$, R_1 close to Q^1 and R_2 close to Q^2. For any polynomial close to Q^2, the moduli of its roots are distinct and different from the ones of the roots of Q^2. □

3 The sets Π_1, Π_2 and $\tilde{E}_2$

For $d = 1$, the polynomial Q_d equals $x + a$, $a \in \mathbb{R}$. One has $\Pi_d = \mathbb{R}$ and the only root of Q_d equals $-a$. The sign of this root defines three strata: $\{a < 0\}$ and $\{a > 0\}$ (of dimension 1, they correspond to the sign patterns $(+,-)$ and $(+,+)$)and $\{a = 0\}$ (of dimension 0, it corresponds to the generalized sign pattern $(+,0)$). The set E_1 is not defined.

For $d = 2$, we set $Q_2 := x^2 + ax + b$. In Fig. 1 we show the sets Π_2, E_2 (in dashed line), F_2 (in dotted line) and G_2 in the coordinates (a, b), see Definition 1.4. One has

$$\begin{aligned} \Pi_2 &= \{ \ (a,b) \in \mathbb{R}^2 \mid b \leq a^2/4 \ \} \ , \\ E_2 &= \{ \ (a,b) \in \mathbb{R}^2 \mid a = 0, b < 0 \ \} \ , \\ F_2 &= \{ \ (a,b) \in \mathbb{R}^2 \mid a = 0, b > 0 \ \} \ , \\ G_2 &= \{(0,0) \in \mathbb{R}^2\} \ . \end{aligned}$$

The strata of dimension 2 and the sign patterns and orders of moduli corresponding to them are:

$$\begin{array}{rcl}
\{a>0, 0<b<a^2/4\} \ , & (+,+,+) \ , & N<N \\
\{a<0, 0<b<a^2/4\} \ , & (+,-,+) \ , & P<P \\
\{a>0, b<0\} \ , & (+,+,-) \ , & P<N \\
\{a<0, b<0\} \ , & (+,-,-) \ , & N<P
\end{array}$$

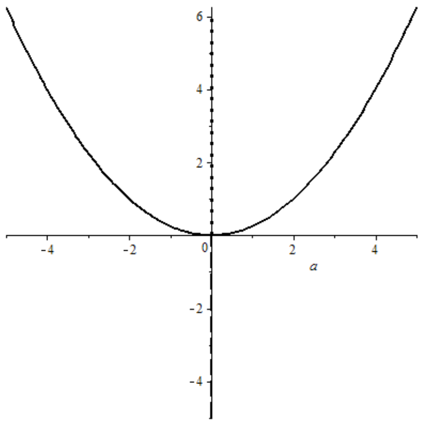

Fig. 1: The sets Π_2, E_2, F_2 and G_2.

The set E_2 coincides with one of the strata of Π_2 of dimension 1 to which corresponds the generalized sign pattern $(+,0,-)$ and the generalized order of moduli $P=N$. We list the four remaining such strata and their corresponding sign patterns or generalized sign patterns and their generalized orders of moduli:

$$\begin{array}{ll}
\{a>0, b=a^2/4\} & (+,+,+) \ , \\
 & N=N \\
 & \\
\{a<0, b=a^2/4\} & (+,-,+) \ , \\
 & P=P \\
 & \\
\{a>0, b=0\} & (+,+,0), \\
 & 0<N \\
 & \\
\text{and } \{a<0, b=0\} & (+,-,0) \\
 & 0<P.
\end{array}$$

Finally, the origin is the only stratum of Π_2 of dimension 0. Its corresponding generalized sign pattern is $(+,0,0)$ and the generalized order of moduli is $0=0$.

Remarks 3.1. (1) Each order of moduli is present only in one of the strata of dimension 2, i.e. to each order of moduli corresponds exactly one sign pattern. Geometrically this can be interpreted as all orders of moduli being rigid.

(2) To each sign pattern corresponds exactly one order of moduli. Hence this is the canonical order of moduli and all sign patterns are canonical.

(3) The set E_2 coinciding with one of the strata of dimension 1 means that one obtains exactly the same stratification if one considers only the question whether roots are equal or not and whether there is a root at 0, but not the question whether a positive root is equal to a modulus of a negative root. Indeed, sign patterns define orders of moduli and vice versa.

(4) For $b>a^2/4$, the polynomial Q_2 has two complex conjugate roots.

4 The sets Π_3 and $\tilde{E}_3$

We use the notation $Q_3 := x^3+ax^2+bx+c$, and the fact that the sets Π_3, E_3, F_3 and G_3 are invariant w.r.t. the one-parameter group of quasi-homogeneous dilatations $a\mapsto ua$, $b\mapsto u^2b$, and $c\mapsto u^3c$, $u\in\mathbb{R}^*$. This allows to limit oneself to drawing only the pictures of $\Pi_3|_{a=1}$ and $\Pi_3|_{a=0}$. (Thus, for instance, one obtains the set $\Pi_3|_{a=-1}$ from the set $\Pi_3|_{a=1}$ by the symmetry $a\mapsto -a$, $b\mapsto b$, $c\mapsto -c$.)

The set $\Pi_3|_{a=1}$ is the interior of a real algebraic curve $\mathcal{C}$ with a cusp point at $(1/3, 1/27)$ (denoted by T in Fig. 2) and the curve itself; "interior" means the part of the plane which forms a narrow tongue approaching the cusp point. For $a=0$, this curve is a semi-cubic parabola. The set $E_3|_{a=1}$ is a half-line with endpoint at the origin which for $a=1$ is tangent to the curve $\mathcal{C}$ at the point M; the set $E_3|_{a=0}$ is the open negative half-axis b. To obtain the set $E_3|_{a=1}$ one can use the parametrization

$$Q_3 = (x^2-v^2)(x+1) = x^3+x^2-v^2x-v^2 \ , \quad v>0 \quad (1)$$

hence one has $b=c<0$. We represent in Fig. 2 the set E_3 by dashed and the set F_3 by dotted line.

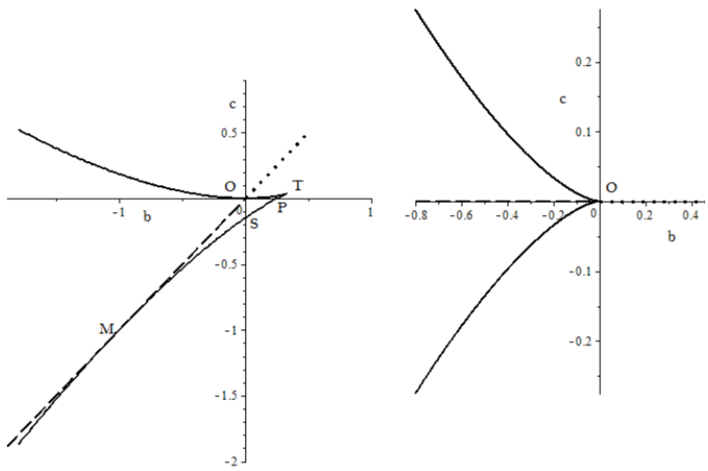

Fig. 2: The sets Π_3, E_3, F_3 and G_3 for $a=1$ (left) and $a=0$ (right).

There are six strata of $\Pi_3|_{a=1}$ of dimension 2. We list them together with the corresponding sign patterns and orders of moduli and we justify the orders of moduli in Remarks 4.1 below:

1) the curvilinear triangle OPT, the sign pattern is $(+,+,+,+)$, the order of moduli is $N < N < N$;

2) the curvilinear triangle OPS, the sign pattern is $(+,+,+,-)$, the order of moduli is $P < N < N$;

3) the curvilinear triangle OSM, the sign pattern is $(+,+,-,-)$, the order of moduli is $P < N < N$;

4) the open sector defined by the negative b-half-axis and the half-line OM, the sign pattern is $(+,+,-,-)$, the order of moduli is $N < P < N$;

5) the open curvilinear sector defined by the part of the half-line OM, which is below the point M, and the infinite arc of the curve $\mathcal{C}$ belonging to the third quadrant, the sign pattern is $(+,+,-,-)$, the order of moduli is $N < N < P$;

6) the open curvilinear sector defined by the negative b-half-axis and the part of the curve $\mathcal{C}$ contained in the second quadrant, the sign pattern is $(+,+,-,+)$, the order of moduli is $P < P < N$.

Remarks 4.1. (1) In OPT all coefficients are positive, so all roots are negative. When crossing the segment OP a negative root becomes positive, so this root is of smallest modulus. By continuity the order of moduli is $P < N < N$ in OPS and OSM, because one does not cross the set E_3. On the segment OM the only possible generalized order of moduli is $P = N < N$, so when passing from OSM into the sector from 4) one passes from the order of moduli $P < N < N$ to the order of moduli $N < P < N$. When one crosses then the negative b-half-axis it is the root of smallest modulus that changes sign, so the order of moduli becomes $P < P < N$ in 6). At the point P (see 20) below) one has a double negative root. By continuity this is the case along the arc PM and its continuation, so inside the domain 5) close to this continuation only the orders of moduli $N < N < P$ or $P < N < N$ are possible. One can choose a point of the domain 5) to show by computation that the order of moduli is $N < N < P$.

(2) By analogy with the presentation (1) one can parametrize the set $F_3|_{a=1}$ of cubic polynomials whose first two coefficients are equal to 1 and which have a purely imaginary pair of roots as follows: $(x^2+v^2)(x+1) = x^3+x^2+v^2x+v^2$, hence $b = c > 0$. The set $\tilde{E}_3|_{a=1}$ (see Definition 1.4) is the whole line $b = c$. For $v = 0$, there is a double root at 0 and a simple root at -1.

(3) Polynomials from the exterior of the curves drawn in solid line on Fig. 2 have one real and two complex conjugate roots.

Further when we speak about arcs, these are open arcs of the curve $\mathcal{C}$. The eleven strata of $\Pi_3|_{a=1}$ of dimension 1 are:

7) the arc OT, the sign pattern is $(+,+,+,+)$, the generalized order of moduli is $N = N < N$;

8) the arc TP, the sign pattern is $(+,+,+,+)$, the generalized order of moduli is $N < N = N$;

9) the arc PS, the sign pattern is $(+,+,+,-)$, the generalized order of moduli is $P < N = N$;

10) the arc SM, the sign pattern is $(+,+,-,-)$, the generalized order of moduli is $P < N = N$;

11) the infinite arc contained in the third quadrant, the sign pattern is $(+,+,-,-)$, the generalized order of moduli is $N = N < P$;

12) the infinite arc contained in the second quadrant, the sign pattern is $(+,+,-,+)$, the generalized order of moduli is $P = P < N$;

13) the open segment OM, the sign pattern is $(+,+,-,-)$, the generalized order of moduli is $P = N < N$;

14) the half-line OM without the segment OM and the point M, the sign pattern is $(+,+,-,-)$, the generalized order of moduli is $N < P = N$;

15) the open segment OP, the generalized sign pattern is $(+,+,+,0)$, the generalized order of moduli is $0 < N < N$;

16) the open negative b-half-axis, the generalized sign pattern is $(+,+,-,0)$, the generalized order of moduli is $0 < P < N$;

17) the open segment OS, the generalized sign pattern is $(+,+,0,-)$, the order of moduli is $P < N < N$.

The five strata of $\Pi_3|_{a=1}$ of dimension 0 are the following points listed with sign patterns or generalized sign patterns and with the hyperbolic polynomials which they define (the coordinates of the points are the last two coefficients of the polynomials):

18) O, the generalized sign pattern is $(+,+,0,0)$, $x^2(x+1) = x^3+x^2$;

19) T, the sign pattern is $(+,+,+,+)$, $(x+1/3)^3 = x^3+x^2+x/3+1/27$;

20) P, the generalized sign pattern is $(+,+,+,0)$, $x(x+1/2)^2 = x^3+x^2+x/4$;

21) S, the generalized sign pattern is $(+,+,0,-)$, $(x+2/3)^2(x-1/3) = x^3+x^2-4/27$;

22) M, the sign pattern is $(+,+,-,-)$, $(x-1)(x+1)^2 = x^3+x^2-x-1$.

Only some, but not all, of these strata have analogs for $a = 0$:

5a) the curvilinear sector defined by the negative b-half-axis and the infinite branch of the semi-cubic parabola belonging to the third quadrant, the generalized sign pattern is $(+,0,-,-)$, the order of moduli is $N < N < P$;

6a) the curvilinear sector defined by the negative b-half-axis and the infinite branch of the semi-cubic parabola belonging to the second quadrant, the eneralized sign pattern is $(+,0,-,+)$, the order of moduli is $P < P < N$;

11a) the infinite open arc of the semi-cubic parabola contained in the third quadrant, the generalized sign pattern is $(+,0,-,-)$, the generalized order of moduli is $N=N<P$;

12a) the infinite open arc of the semi-cubic parabola contained in the second quadrant, the generalized sign pattern is $(+,0,-,+)$, the generalized order of moduli is $P=P<N$;

14a and 16a) the open negative b-half-axis, the generalized sign pattern is $(+,0,-,0)$, the generalized order of moduli is $0<P=N$;

18a) O, the generalized sign pattern is $(+,0,0,0)$, the generalized order of moduli is $0=0=0$.

Remarks 4.2. (1) For $d=3$, one can parametrize the closure of the set E_3 as follows: one sets

$$\begin{aligned} Q_3 &:= (x^2-v^2)(x+A) \\ &= x^3+Ax^2-v^2x-Av^2\ , \quad v\geq 0\ , \quad A\in\mathbb{R} \end{aligned}$$

Thus setting $B:=-v^2$ one can view the closure of the set E_3 as the graph of the function AB defined for $(A,B)\in\mathbb{R}\times\mathbb{R}_-$. Hence the coefficient c is expessed as a function in (a,b): $c=ba$. The latter equation is satisfied by polynomials Q_3 parametrized as follows:

$$Q_3:=(x^2+B)(x+A)=x^3+Ax^2+Bx+AB\ .$$

For $B<0$, $B>0$ and $B=0$, one obtains the sets E_3, F_3 and G_3 respectively, see Definition 1.4.

(2) The sign patterns $(+,+,+,+)$, $(+,+,-,+)$ and $(+,+,+,-)$ are canonical (but the sign pattern $(+,+,-,-)$ is not). Geometrically this is expressed by the fact that to each of them corresponds a single dimension 2 stratum of $\Pi_3|_{a=1}$ (and three such strata to $(+,+,-,-)$). In the same way, when considering the set $\Pi_3|_{a=-1}$, one sees that the sign patterns $(+,-,+,-)$, $(+,-,-,-)$ and $(+,-,+,+)$ are canonical and $(+,-,-,+)$ is not.

(3) The orders of moduli $N<N<N$, $N<P<N$, $P<N<P$ and $P<P<P$ are rigid, see Remarks 1.3 – each of them is present in only one stratum of Π_3 of maximal dimension. The order of moduli $P<N<N$ is present in the strata OPS and OSM, i.e. 2) and 3), so it is not rigid. In the same way the order of moduli $N<P<P$ is not rigid. The order of moduli $P<P<N$ is present in the strata 6) and 6a) hence it is present in a maximal-dimension stratum of $\Pi_3|_{a=-1}$ as well, so it is not rigid. In the same way the order of moduli $N<N<P$ is not rigid. Thus there are eight orders of moduli of which four are present each in one and the remaining four are present each in two strata of Π_3 of maximal dimension.

5 The sets Π_4 and $\tilde{E}_4$

5.1 General properties of the sets Π_4 and $\tilde{E}_4$

We consider the family of polynomials $Q_4:=x^4+ax^3+bx^2+cx+h$, a, b, c, $h\in\mathbb{R}$.

Remark 5.1. For $d=4$, the set $\tilde{E}_4$ intersects also the domain of $\mathbb{R}^4\cong Oabch$ in which the polynomial Q_4 has two real roots and one complex conjugate pair, and the domain in which it has two complex conjugate pairs.

Theorem 5.2. *(1) The hypersurface $\tilde{E}_4$ is defined by the condition $\Phi(a,b,c,h)=0$, where $\Phi:=a^2h+(c-ab)c$. This hypersurface is irreducible.*

(2) The set of singular points of $\tilde{E}_4$ is the plane $a=c=0$. This is the set of even polynomials.

(3) For $a=c=0$, the Hessian matrix of Φ is of rank 2 for $4h-b^2\neq 0$ and of rank 1 for $4h-b^2=0$ (that is, when the polynomial Q_4 has either two double opposite real roots or a double complex conjugate purely imaginary pair or a quadruple root at 0).

(4) At a point of the set $\{a=c=0,\ 4h-b^2\neq 0\}$ the hypersurface $\Phi=0$ is locally the transversal intersection of two smooth hypersurfaces.

(5) At a point of the set $\{a=c=0,\ 4h-b^2=0\}$ the hypersurface $\Phi=0$ is locally diffeomorphic to the direct product of $\mathbb{R}$ with a hypersurface in $\mathbb{R}^3$ having a Whitney umbrella singularity.

Proof. Part (1). The hypersurface $\tilde{E}_4$ can be defined with the help of the parametrization

$$\begin{aligned} Q_4 &:= (x^2+A)(x^2+ux+v) \\ &= x^4+ux^3+(A+v)x^2+Aux+Av \end{aligned}$$

Hence $a^2h=Au^2v$, $c-ab=-uv$ and $(c-ab)c=-Au^2v$. The polynomial Φ is irreducible. Indeed, it is linear in h. Should it be reducible, it should be of the form $(a^2+\cdots)(h+\cdots)$. However a^2h is its only term containing h, so this factorization should be of the form $a^2(h+\cdots)$. This is impossible, because Φ is not divisible by a^2.

Part (2). We set $\Phi_a:=\partial\Phi/\partial a$ and similarly for Φ_b, Φ_c and Φ_h. Hence

$$\Phi_a=2ah-bc,\quad \Phi_b=-ac,\quad \Phi_c=2c-ab\ \text{ and }\ \Phi_h=a^2$$

The singular points of $\tilde{E}_4$ are defined by the condition $\Phi=\Phi_a=\Phi_b=\Phi_c=\Phi_h=0$ which is equivalent to $a=c=0$. This is the set of even polynomials.

Part (3). The Hessian matrix of Φ equals

$$\begin{pmatrix} 2h & -c & -b & 2a \\ -c & 0 & -a & 0 \\ -b & -a & 2 & 0 \\ 2a & 0 & 0 & 0 \end{pmatrix}.$$

For $a=c=0$, only its first and third rows contain non-zero entries and the third row contains the entry 2, so the rank equals 1 or 2. It equals 1 exactly when $4h-b^2=0$ (to be checked directly). In this case $Q_4=(x^2+b/2)^2$ from which part (3) follows.

Part (4). Any even degree 4 polynomial has two pairs of opposite roots each of which might be real or purely imaginary. Therefore part (4) results from part (3) of Theorem 1.5.

Part (5). The equation $\Phi = 0$ can be given the equivalent form

$$(c - ab/2)^2 + (h - b^2/4)a^2 = 0 \ .$$

The change of coordinates

$$(a, b, c, h) \mapsto (a, b, \omega := c - ab/2, \varrho := -h + b^2/4)$$

is a global diffeomorphism of $\mathbb{R}^4$ onto itself. The conditions $a = c = h - b^2/2 = 0$ and $a = \omega = \varrho = 0$ are equivalent. In the new coordinates the above equation becomes $\omega^2 = \varrho a^2$ $(*)$ which is the equation of the direct product of the Whitney umbrella and the b-axis. One can observe that in the space $Oa\varrho\omega$, the equations $a = \omega = 0$ define the ϱ-axis which satisfies the equation $(*)$. However, if one parametrizes the Whitney umbrella as $a = u$, $\varrho = v^2$, $\omega = uv$, then this parametrization does not cover the negative ϱ-half-axis. □

We remind that the one-parameter group of diffeomorphisms

$$x \mapsto tx, \quad a \mapsto ta, \quad b \mapsto t^2 b, \quad c \mapsto t^3 c, \quad h \mapsto t^4 h, \quad t \neq 0,$$

preserves the set $\tilde{E}_4$ and the set of zeros of the polynomial Q_4. Therefore for $a \neq 0$, the set $\tilde{E}_4|_{a=1}$ gives an adequate idea about the set $\tilde{E}_4$. The set $\tilde{E}_4|_{a=0}$ is discussed in Subsection 5.3.

The set $\Delta_4|_{a=1}$ is defined by the equation

$$\mathrm{Res}(Q_4|_{a=1}, Q_4'|_{a=1}, x) = 0,$$

i. e.

$$\begin{aligned} \Delta_4|_{a=1} \quad : \quad & 16b^4h - 4b^3c^2 - 4b^3h + b^2c^2 \\ & -80b^2ch - 128b^2h^2 + 18bc^3 + 144bc^2h \\ & -27c^4 + 18bch + 144bh^2 - 4c^3 \\ & -6c^2h - 192ch^2 + 256h^3 - 27h^2 = 0 \end{aligned}$$

We represent the sets $\tilde{E}|_{a=1}$ and $\Pi_4|_{a=1}$ by means of some pictures. One can notice that each set $\tilde{E}_4|_{a=1,b=b_0}$ is a parabola of the form $h = -c^2 + b_0 c$, see part (1) of Theorem 5.2. The intersections of the set $\Pi_4|_{a=1}$ with the planes $\{b = \text{const}\}$ are either empty or curvilinear triangles or (for $b = 3/8$) a point. When this is a curvilinear triangle, its border has a transversal self-intersection point (see the point I in Fig. 12) and two cusp points (see the points L and R in Fig. 12). At the point L the polynomial Q_4 has a triple real root to the left and a simple root to the right; and vice versa at the point R. Along the arc LI (resp. IR or RL) there are two simple and one double real roots; the double root is to the right (resp. to the left or in the middle). At the point I the polynomial Q_4 has two double real roots.

In the curvilinear sector above the point I the polynomials have two pairs of complex conjugate roots and in the domain to the left, below and to the right of the curvilinear triangle LRI they have two real and two complex conjugate roots.

5.2 The sets $\Pi_4|_{a=1}$ and $\tilde{E}_4|_{a=1}$

In Figures 3 and 4 we show the projections on the plane (b, c) of some strata of the set $\Pi_4|_{a=1}$ and the intersection $\Pi_4|_{a=1} \cap \{h = 0\}$. The curve drawn in solid line has a cusp at $(3/8, 1/16)$; we denote it by $\mathcal{S}$. This is the projection in the plane (b, c) of the set of polynomials $Q_4|_{a=1}$ having a root of multiplicity ≥ 3:

$$\begin{aligned} Q_4 &= (x-u)^3(x-v) \\ &= x^4 - (3u+v)x^3 + (3u^2 + 3uv)x^2 \\ &\quad -(u^3 + 3u^2 v)x + u^3 v \end{aligned}$$

Since $3u + v = -1$,

$$Q_4 = x^4 + x^3 - (3u + 6u^2)x^2 + (3u^2 + 8u^3)x - u^3 - 3u^4 \ .$$

Thus $\mathcal{S} : b = -3u - 6u^2$, $c = 3u^2 + 8u^3$ hence

$$\mathcal{S} \ : \ 32b^3 - 9b^2 - 108bc + 108c^2 + 27c = 0 \ .$$

The cusp point corresponds to a polynomial with a quadruple root:

$$(x + 1/4)^4 = x^4 + x^3 + (3/8)x^2 + x/16 + 1/256 \ .$$

The upper (resp. lower) branch of this curve consists of polynomials with $u > v$ (resp. $u < v$). The half-line in dashed line (we denote it by $\mathcal{L}_R$) represents the projection on the plane (b, c) of the stratum consisting of two double real roots:

$$\begin{aligned} & (x - \zeta)^2(x - \theta)^2 \\ = \ & x^4 - 2(\zeta + \theta)x^3 + (\zeta^2 + \theta^2 + 4\zeta\theta)x^2 \\ & -2\zeta\theta(\zeta + \theta)x + \zeta^2\theta^2 \end{aligned}$$

hence $\zeta + \theta = -1/2$, so $b = 1/4 + 2\zeta\theta$, $c = \zeta\theta$ and $c = b/2 - 1/8$. Thus

$$\mathcal{L}_R \ : \ c = b/2 - 1/8 \ , \ c \leq 3/8 \ .$$

The continuation of this half-line is drawn in dotted line; we denote it by

$$\mathcal{L}_I \ : \ c = b/2 - 1/8 \ , \ c \geq 3/8 \ .$$

It corresponds to the point B in Fig. 23 and represents polynomials having a double complex conjugate pair. In the space (b, c, h) the union of the two strata projecting on $\mathcal{L}_R \cup \mathcal{L}_I$ on the plane (b, c) is a parabola. Indeed, one has $h = c^2$.

The union of the curve represented in dash-dotted line (we denote it by $\mathcal{H}$) and the segment $[0, 1/4]$ of the b-axis is the intersection $\Delta_4|_{a=1} \cap \{h = 0\}$. The equation of this intersection is

$$\begin{aligned}&\mathrm{Res}(Q_4|_{a=1}, Q_4'|_{a=1}, x)|_{h=0}\\ &= -c^2(4b^3 - b^2 - 18bc + 27c^2 + 4c) = 0 \ ,\end{aligned}$$

so $\mathcal{H} : 4b^3 - b^2 - 18bc + 27c^2 + 4c = 0$. The curve $\mathcal{H}$ has a cusp at$(1/3, 1/27)$ (this point corresponds to the polynomial $(x + 1/3)^3x)$ which is situated on the lower branch of the curve $\mathcal{S}$. It is tangent to the latter curve at the origin (this is the polynomial $(x + 1)x^3$) and its lower branch is tangent at the point $(1/4, 0)$ to the half-line $\mathcal{L}_R$ drawn in dashed line (this is the polynomial $(x + 1/2)^2x^2$). For $a = 1$, we show in

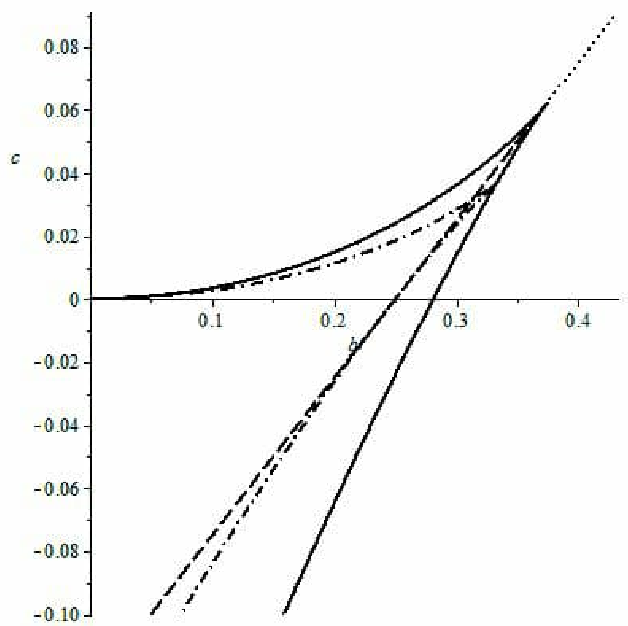

Fig. 3: The projection of $\Pi_4|_{a=1}$ on the plane of parameters (b, c) .

Fig. 5 (in long-dashed line) the projection on the plane (b, c) of the intersection $\Pi_4 \cap \tilde{E}_4$. This is the parabola $\mathcal{P} : (b - 2c)^2 + c = 0$. Indeed, one can parametrize a polynomial from the set $\tilde{E}_4|_{a=1}$ as follows:

$$\begin{aligned} Q_4 &:= (x^2 - u^2)(x^2 + x + w \qquad (2)\\ &= x^4 + x^3 + (w - u^2)x^2 - u^2x - wu^2 \ , \ u, w \in \mathbb{R} \ )\end{aligned}$$

The condition $\mathrm{Res}(Q_4, Q_4', x) = 0$ reads:

$$-4u^2(u^2 + u + w)^2(u^2 - u + w)^2(4w - 1) = 0 \ .$$

For $w = \pm u - u^2$, one obtains $b = -u(2u \mp 1)$ and $c = -u^2$ from which results the equation of $\mathcal{P}$. For $u = 0$, the polynomial $Q_4 = x^4+x^3+wx^2$ has a double root at 0 hence the corresponding half-line $c = h = 0$, $b \leq 1/4$, belongs to the closure of the set $\tilde{E}_4|_{a=1}$. Its projection on the plane (b, c) is drawn in long-dashed line on Fig. 5.

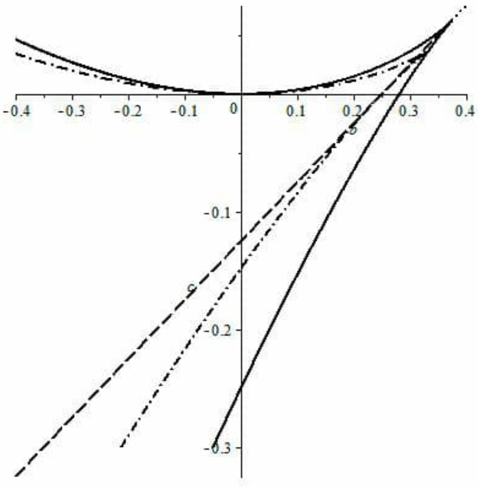

Fig. 4: The projection of $\Pi_4|_{a=1}$ on the plane of parameters (b, c) (global view).

The projection of its analytic continuation is drawn in dotted line. For $w = 1/4$, the polynomial Q_4 has a double root at $-1/2$. The corresponding half-line $c = b-1/4 = 4h$, $c \leq 0$, belongs to the closure of $\tilde{E}_4|_{a=1}$. Its projection on the plane $(b, \ c)$ is drawn in long-dashed line and the projection of its analytic continuation in dotted line.

The symmetry axis of the parabola $\mathcal{P}$ is represented in dotted line, its equation is $c = b/2 - 1/10$. It is parallel to the projection $\mathcal{L}_R$ on the plane $(b, \ c)$ of the stratum of $\Pi_4|_{a=1}$ of polynomials having two double roots. At the intersection point of the projection of this stratum with the parabola $\mathcal{P}$ the tangent line to the parabola is parallel to the c-axis.

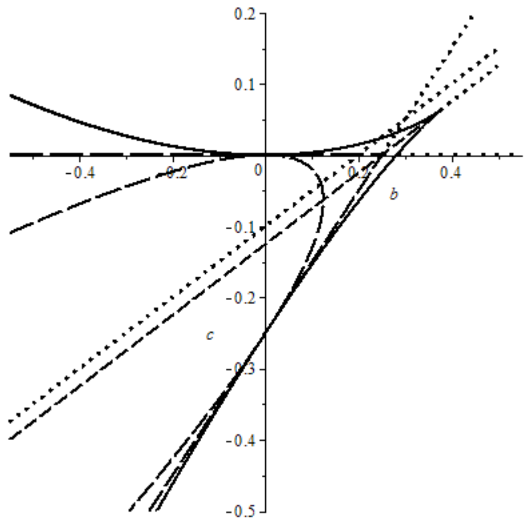

Fig. 5: The projection of the intersection of $\tilde{E}_4$ and Π_4.

In Fig. 6 we show the intersection of the set $\tilde{E}_4$ with the plane (b, c) and the curve $\mathcal{H}$. The intersection $\tilde{E}_4|_{a=1} \cap \{h = 0\}$ consists of the two half-lines $b = c$, $c \leq 0$, and $c = 0$, $b \leq 1/4$ (drawn in dashed line).

Their continuations are drawn in dotted line. One can observe that (see (2)) the set $\tilde{E}_4 \cap \{a = 1\}$ is defined by the equation $h = c(b - c)$.

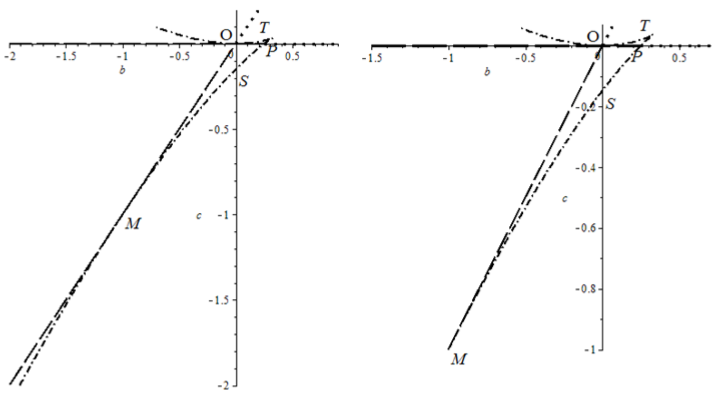

Fig. 6: The intersections of the sets $\tilde{E}_4$ and Π_4 with the plane (b, c).

Remarks 5.3. (1) For $d = 4$ and $a > 0$, there are five canonical sign patterns. We list them together with the corresponding orders of moduli:

$$\begin{array}{ll} (+,+,+,+,+) & N < N < N < N \\ (+,+,+,+,-) & P < N < N < N \\ (+,+,-,+,+) & N < P < P < N \\ (+,+,+,-,+) & P < P < N < N \\ (+,+,-,+,-) & P < P < P < N \; . \end{array}$$

The first of them is trivially canonical. For the other ones being canonical follows from the results in [11]; for the last one it is to be observed that if the polynomial $Q_4(x)$ realizes the sign pattern $(+,+,+,+,-)$, then $-x^4 Q_4(-1/x)$ realizes the sign pattern $(+,+,-,+,-)$. The remaining sign patterns beginning with $(+,+)$, namely

$$(+,+,+,-,-), (+,+,-,-,-) \text{ and } (+,+,-,-,+)$$

are not canonical, see [11]. Geometrically this is illustrated by the fact that the set E_4 divides the intersections of Π_4 with the corresponding orthants in two or more parts, see Fig. 7-21.

(2) For $d = 4$, the only rigid orders of moduli for which one has $a > 0$ are $P < N < P < N$ and $N < N < N < N$. The corresponding sign patterns are $(+,+,-,-,+)$ and $(+,+,+,+,+)$. The order of moduli $P < N < P < N$ concerns the part of $\Pi_4|_{a=1,b=-0.5}$ (see Fig. 10) between the negative c-half-axis and the set $E_4|_{a=1,b=-0.5}$.

Remark 5.4. We indicate how the roots of the polynomial Q_4 change when it runs along the arc $OUVW$, see Fig. 7. Greek letters indicate positive quantities:

$$\begin{array}{lll} \text{At } O & : & x^2(x-\xi)(x+\eta) \; , \\ & & -\xi+\eta = 1 \; , \xi < \eta \; . \\ \text{Along } OU & : & (x-\alpha)(x+\alpha)(x-\xi)(x+\eta) \; , \\ & & -\xi+\eta = 1 \; , \alpha < \xi < \eta \; . \\ \text{At } U & : & (x-\xi)^2(x+\xi)(x+\eta) \; , \\ & & -\xi+\eta = 1 \; , \xi < \eta \; . \\ \text{Along } UV & : & (x-\alpha)(x+\alpha)(x-\xi)(x+\eta) \; , \\ & & -\xi+\eta = 1 \; , \xi < \alpha < \eta \; . \\ \text{At } V & : & (x-\alpha)(x+\alpha)^2(x-\xi) \; , \\ & & -\xi+\alpha = 1 \; , \xi < \alpha \; . \\ \text{Along } VW & : & (x-\alpha)(x+\alpha)(x\mp\xi)(x+\eta) \; , \\ & & \mp\xi+\eta = 1 \; , \xi < \eta < \alpha \; . \\ \text{At } W & : & (x-\alpha)(x+\alpha)(x+\xi)^2 \; , \\ & & 2\xi = 1 \; , \xi < \alpha \; . \end{array}$$

When a point of the set $\tilde{E}_4|_{a=1}$ is outside the set $\Pi_4|_{a=1}$ and close to the point O, the polynomial Q_4 has a pair of conjugate purely imaginary roots close to 0. The sign $\mp$ of ξ above is explained by the fact that when crossing the c-axis the corresponding root changes sign.

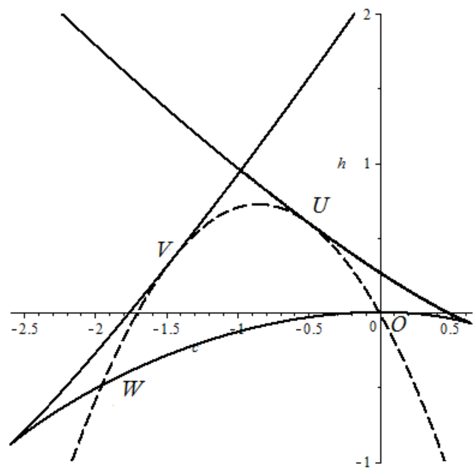

Fig. 7: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $(b = -1.7)$ (global view).

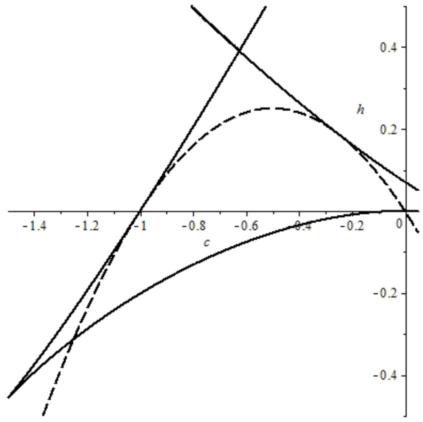

Fig. 8: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $(b = -1)$ (global view).

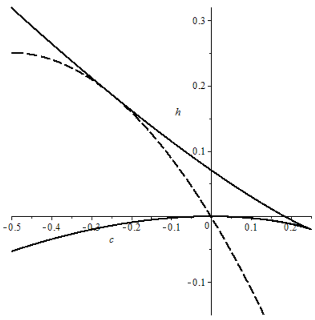

Fig. 9: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = -1$ (local view).

Remark 5.5. For $b = -1$ (see Fig. 8) , the point V is on the c-axis. It is above the c-axis for $b < -1$ and below it for $b \in (-1, 0)$. For $b = 0$, it coincides with the cusp point L, see Fig. 12.

For $b = 0$, the point U coincides with the cusp point R, where the tangent line is horizontal, see Fig. 12 and Fig. 13. For $b = b_0 \in (0, 1/8)$, both points of tangency between the parabola $E^* := \tilde{E}_4|_{a=1,b=b_0}$ and the set $\Pi^* := \Pi_4|_{a=1,b=b_0}$ belong to the lower arc of Π^*, see Fig. 14 and Fig. 15. For $b = 1/8$ (see Fig. 15), these points coincide. This gives the polynomial

$$x^4+x^3+x^2/8-x/16-3/256 = (x+1/4)^2(x-1/4)(x+3/4) .$$

For $b_0 \in (0, 1/4)$, the parabola E^* intersects transversally the two side arcs of the border of the set Π^*, see Fig. 16 and Fig. 17. For $b_0 = 1/4$, the parabola passes at the self-intersection point of the border of Π^* which is the point $c = h = 0$, see Fig. 18; this is the polynomial

$$x^4 + x^3 + x^2/4 = x^2(x + 1/2)^2 .$$

For $b_0 > 1/4$, the parabola does not intersect the set Π^* (see Fig.Fig. 19–21), and for $b_0 > 3/8$, the latter set is void, see Fig. 23. For $a = 1$, $b = 3/8$, the only hyperbolic polynomial is

$$(x + 1/4)^4 = x^4 + x^3 + 3x^2/8 + x/16 + 1/256 .$$

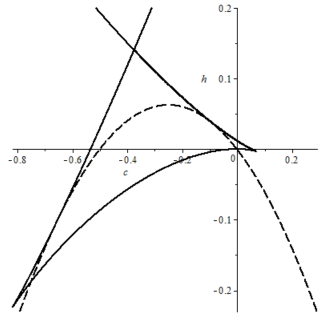

Fig. 10: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $(b = -0.5)$ (global view).

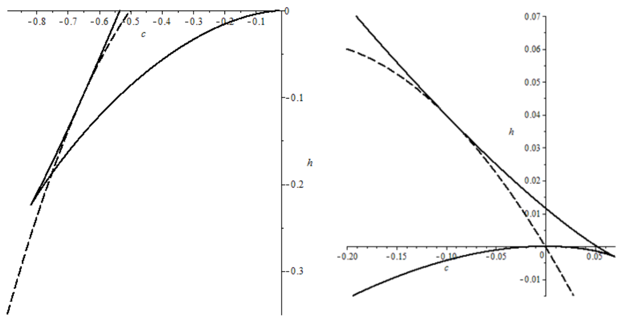

Fig. 11: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = -0.5$ (local views).

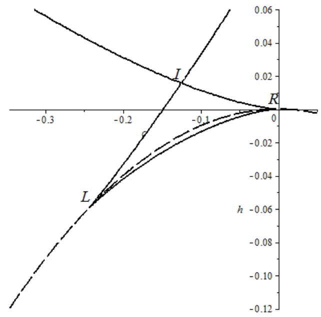

Fig. 12: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0$ (global view).

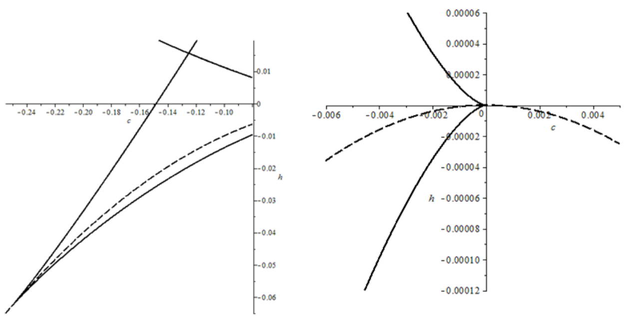

Fig. 13: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0$ (local views).

In Fig. 14 we show the sets $\Pi_4|_{a=1}$ and $\tilde{E}_4|_{a=1}$ for $b = 1/8 = 0.125$. The sets $\Pi_4|_{a=1}$ and $\tilde{E}_4|_{a=1}$ have a fourth order tangency at the point $(c, h) = (-1/16, -3/256) = (-0.0625, -0.01171875)$. The corresponding polynomial equals $(x - 1/4)(x + 1/4)^2(x + 3/4)$. For $b \in (0, 1/8)$, these two sets have two points of second order tangency and they appear very close to one another.

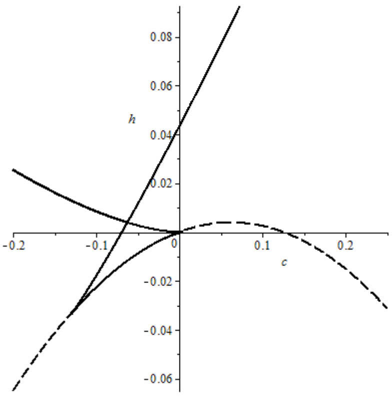

Fig. 14: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.125$ (global view).

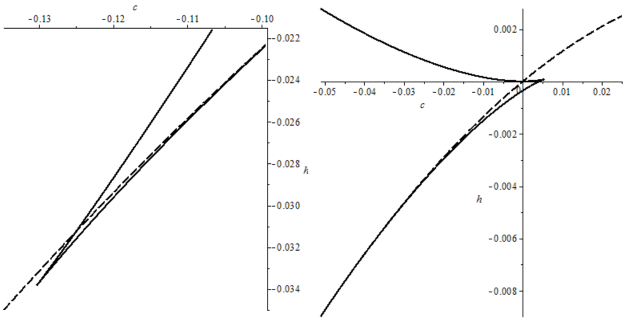

Fig. 15: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.125$ (local views).

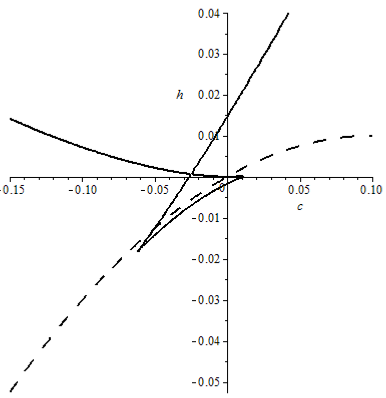

Fig. 16: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.2$ (global view).

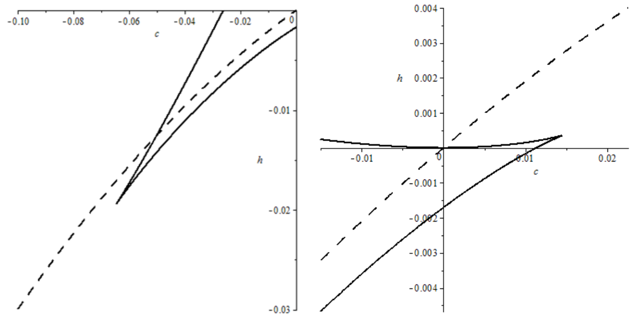

Fig. 17: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.2$ (local views).

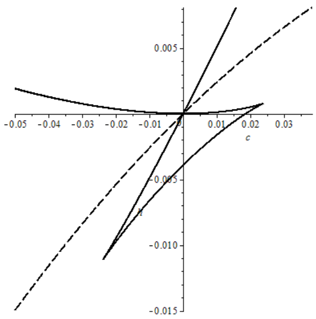

Fig. 18: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.25$ (global view).

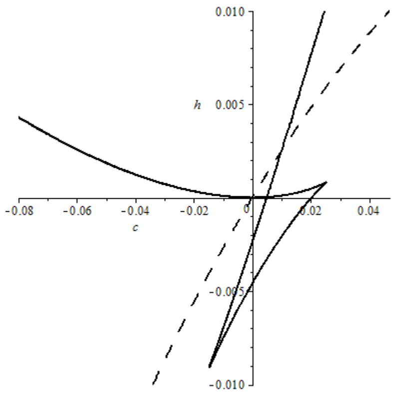

Fig. 19: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.26$ (global view).

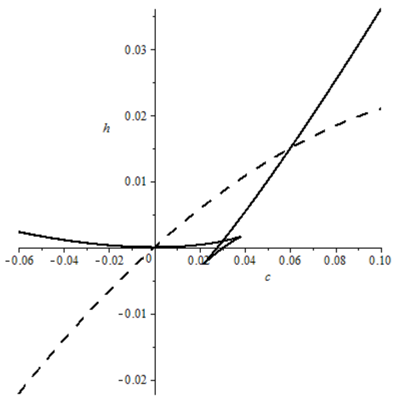

Fig. 20: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.31$ (global view).

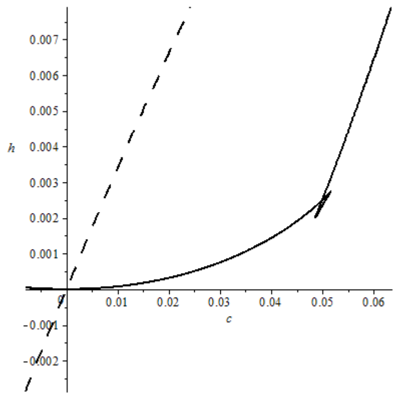

Fig. 21: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.35$ (global view).

Set $Q_4^{\ddagger} := Q_4|_{a=1,b=b_0}$. Although Figures 22 and 23 are much alike, the curve defined by the equation $\mathrm{Res}(Q_4^{\ddagger}, (Q_4^{\ddagger})', x) = 0$ (drawn in solid line) is smooth for $b_0 = 0.4$ whereas for $b_0 = 3/8$ it has a $(4/3)$-singularity at $(c, h) = (1/16, 1/256)$. For $a = 1$, the hypersurface $\mathrm{Res}(Q_4, Q_4', x) = 0$ has a swallowtail singularity at $(3/8, 1/16, 1/256)$, see about swallowtail catastrophe in [15]. For $b_0 = 0.4$, the set $\mathrm{Res}(Q_4^{\ddagger}, (Q_4^{\ddagger})', x) = 0$ contains also the isolated point $(2/5, 3/40, 9/1600)$ (see the point B in Fig. 23) at which the polynomial $Q_4^{\ddagger}$ has a double complex conjugate pair:

$$x^4 + x^3 + 2x^2/5 + 3x/40 + 9/1600 = (x^2 + x/2 + 3/40)^2 .$$

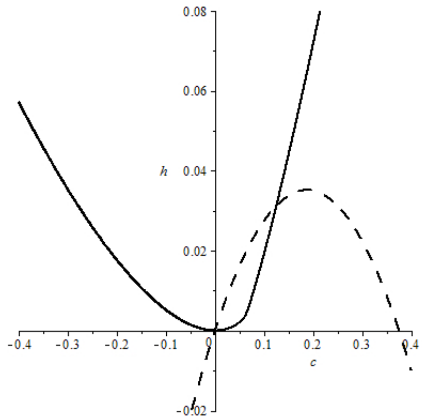

Fig. 22: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 3/8 = 0.375$ (global view).

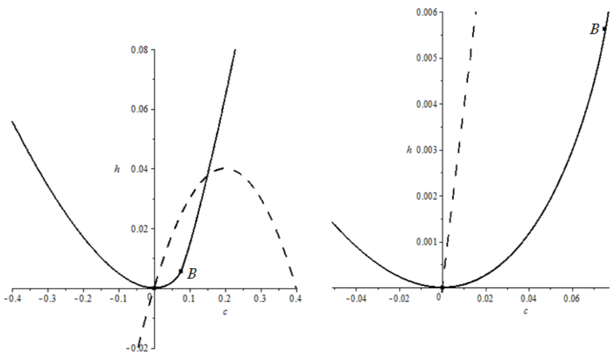

Fig. 23: The sets $\tilde{E}_4|_{a=1}$ and $\Pi_4|_{a=1}$ for $b = 0.4$ (global and local view).

5.3 The sets $\Pi_4|_{a=0}$ and $\tilde{E}_4|_{a=0}$

We consider the polynomial $Q_4^0 := x^4 + bx^2 + cx + h$. Set $Q_4^{\triangle} := Q_4^0|_{b=b_0}$. We show the sets $\mathrm{Res}(Q_4^{\triangle}, (Q_4^{\triangle})', x) = 0$ for $b_0 = -1$, 0 and 1, see Fig. 24, 25 and 26. The hypersurface $\mathrm{Res}(Q_4^0, (Q_4^0)', x) = 0$ has a swallowtail singularity at the origin, see about swallowtail catastrophe in [15]. The curve in solid line in Fig. 25 has a 4/3-singularity at the origin. The point $(b, c, h) = (1, 0, 1/4)$ (this is the point N in Fig. 26) represents a polynomial having two conjugate double imaginary roots. We denote the set $E_4|_{a=0}$ in dashed line (see Fig. 24) and the set $F_4|_{a=0}$ in dotted line (see Fig. 24, 25 and 26). The set $\{a = c = 0,\ h = b^2/4\}$ is represented in Fig. 24 by the self-intersection point of the discriminant set $\Delta_4|_{a=0}$, in Fig. 25 by the origin and in Fig. 26 by the point N.

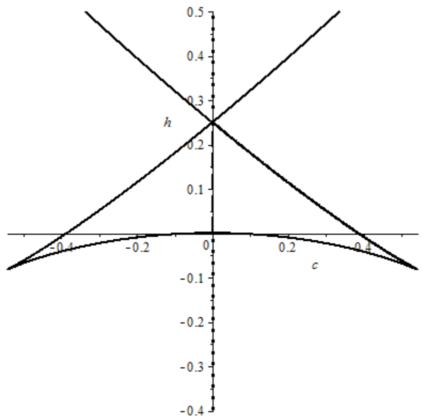

Fig. 24: The set $\Pi_4|_{a=0}$ for $b = -1$ (global view).

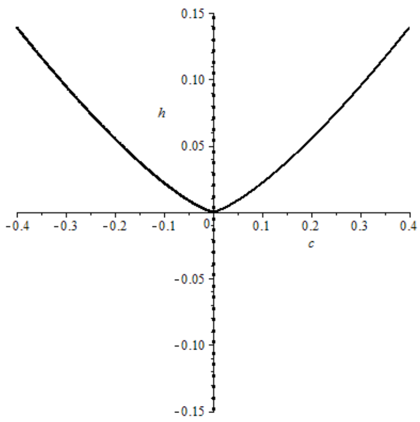

Fig. 25: The set $\Pi_4|_{a=0}$ for $b = 0$ (global view).

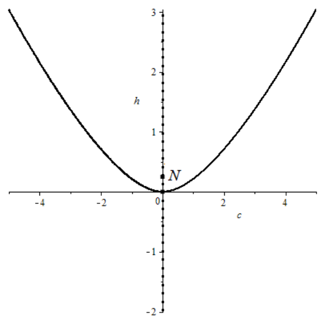

Fig. 26: The set $\Pi_4|_{a=0}$ for $b = 1$ (global view).

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