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Asymptotic for a semilinear hyperbolic equation with asymptotically vanishing damping term, convex potential and integrable source

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Abstract

We investigate the long time behavior of solutions to semilinear hyperbolic equations of the form:

$$u''(t) + \gamma(t)u'(t) + Au(t) + f(u(t)) = g(t),\ t \geq 0, \qquad (\mathrm{E}_\alpha)$$

where A is a self-adjoint nonnegative operator, f a function which is the gradient of a regular convex function, and γ a nonnegative function which behaves, for t large enough, as $\frac{K}{t^\alpha}$ with $K > 0$ and $\alpha \in [0, 1[$. We obtain sufficient conditions on the source term $g(t)$, that ensure the weak and strong convergence of any solution $u(t)$ of (E_α), as $t \to +\infty$, to a solution of the stationary equation $Av + f(v) = 0$, if one exists.

Keywords: Dissipative hyperbolic equation, asymptotically small dissipation, asymptotic behavior, energy function, convex function.

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1 Introduction and statement of the main results

Let H be a real Hilbert space with inner product and associated norm respectively denoted by $\langle .,. \rangle$ and $|.|$. Let V be another real Hilbert space continuously and densely embedded in H. Let V' be the dual space of V. We denote by $\langle .,. \rangle_{V',V}$ the dual product between V and its dual space V' which means that for every $u \in V'$ and $v \in V$,

$$\langle u, v \rangle_{V',V} = u(v).$$

Let us notice that by identifying every element $u \in H$ with the associated linear continuous form $T_u : V \to \mathbb{R}$ defined by:

$$T_u(v) = \langle u, v \rangle,$$

we get $H \hookrightarrow V'$ and the useful identity

$$\langle u, v \rangle_{V',V} = \langle v, w \rangle\ \forall (v, w) \in H \times V. \qquad (1.1)$$

Throughout this paper, $A : V \to V'$ is linear and continuous operator such that the associated bilinear form $a : V \times V \to \mathbb{R}$ defined by

$$a(v, w) = \langle Av, w \rangle_{V',V}$$

is symmetric, positive and satisfies the semi-coercivity property:

$$\exists \lambda \geq 0, \mu > 0 :\ a(v, v) + \lambda |v|^2 \geq \mu \|v\|_V^2\ \ \forall v \in V.$$

A typical example of the operator A is the Laplacian operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ where Ω is an open set of $\mathbb{R}^n$. Let $f : V \to V'$ be a continuous function that derives from a C^1-convex function $F : V \to \mathbb{R}$ in the sense:

$$\forall u, v \in V, F'(u)(v) = \langle f(u), v \rangle_{V',V}, \qquad (1.2)$$

which is in turn equivalent to

$$\forall u \in V, \nabla F(u) = f(u). \qquad (1.3)$$

We consider the problem

$$\begin{cases} v \in V, \\ Av + f(v) = 0 \end{cases} \qquad (1.4)$$

It is clear that the solutions of this problem are exactly the global minimizers of the convex function $\Phi : V \to \mathbb{R}$ defined by:

$$\Phi(v) = \frac{1}{2}a(v, v) + F(v).$$

Hereafter, we assume that the set

$$\arg\min \Phi = \{v \in V : \Phi(v) = \min_V \Phi := \Phi^*\},$$

of minimizers of Φ is nonempty. We aim to approximate numerically the elements of this set. For this reason we investigate, the long time behavior as $t \to +\infty$ of solutions $u(t)$ to the following second order semi-linear hyperbolic equation:

$$u''(t)+\gamma(t)u'(t)+Au(t)+f(u(t))=g(t),\ t\geq 0,\ (\mathrm{E}_\alpha)$$

where the damping term $\gamma:[0,\infty[\to]0,\infty[$ is absolutely continuous which behaves like $\frac{K}{t^\alpha}$ for some $K>0,\ \alpha\in[0,1[$, and t large enough. More precisely, we assume that there exist $K>0,\ t_0\geq 0$ and $\alpha\in[0,1[$ such that:

$$\gamma(t)\geq\frac{K}{(1+t)^\alpha}\ \forall t\geq t_0, \tag{1.5}$$

$$((1+t)^\alpha\gamma(t))'\leq 0 \text{ a.e. } t\geq t_0. \tag{1.6}$$

The function $g:[0,\infty[\to H$, which represents the source or the error term, is assumed to be integrable i.e. $g\in L^1(\mathbb{R}^+,H)$. For the problem of the existence of solution to the equation Eq. (E_α) we refer the reader to the reference [10]. In this paper, we assume the existence of a global solution u to Eq. (E_α) in the class

$$W^{2,1}_{loc}(\mathbb{R}^+,H)\cap W^{1,1}_{loc}(\mathbb{R}^+,V), \tag{1.7}$$

where

$$W^{2,1}_{loc}(\mathbb{R}^+,H) = \{v\in L^1_{loc}(\mathbb{R}^+,H): v''\in L^1_{loc}(\mathbb{R}^+,H)\}$$
$$W^{1,1}_{loc}(\mathbb{R}^+,V) = \{v\in L^1_{loc}(\mathbb{R}^+,V): v'\in L^1_{loc}(\mathbb{R}^+,V)\},$$

and we focus our attention on the study of the asymptotic behavior of $u(t)$ as t goes to infinity. We aim to establish that under suitable conditions on the source term g, the solution $u(t)$ converges at infinity a minimizer of the convex function Φ.

Before stating our main theorems, let us first recall some previous results related to this subject. In the pioneer paper [1], Alvarez considered the case where $V=H$, the damping term γ is a non negative constant and the source g is equal to 0. He proved that $u(t)$ converges weakly to a minimizer of the function Φ. Moreover, he showed that the convergence is strong if the function Φ is even or the interior of $\arg\min\Phi$ is not empty. In [6], Haraux and Jendoubi extended the weak convergence result of Alvarez to the case where the source term is in the space $L^1(\mathbb{R}^+,H)$. Cabot and Frankel [5] studied Eq. (E_α) where $g=0$ and $\gamma(t)$ behaviors at infinity like $\frac{K}{t^\alpha}$ with $K>0$ and $\alpha\in]0,1[$. They proved that every *bounded* solution converges weakly toward a critical point of Φ. In the paper [8], the second author of the present paper improved the result of Cabot and Frankel by getting rid of the supplementary hypothesis on the boundedness of the solution. In [7], it was proved that the main convergence result of Cabot and Frankel remains true if the source term g satisfies the condition $\int_0^{+\infty}(1+t)\,|g(t)|\,dt<\infty$.

The first purpose of the present paper is to improve this last result of [7]. We prove that the convergence holds under the weaker and optimal condition

$$\int_0^{+\infty}(1+t)^\alpha\,|g(t)|\,dt<\infty. \tag{1.8}$$

More precisely, we establish the following result.

Theorem 1.1. *Assume that* $\int_0^{+\infty}(1+t)^\alpha\,|g(t)|\,dt<\infty$. *Let u be a solution to Eq. (E_α) in the class (1.7). If $u\in L^\infty(\mathbb{R}^+,H)$, then $u(t)$ converges weakly in V as $t\to+\infty$ toward some element of* $\arg\min\Phi$. *Moreover, the energy function*

$$\mathcal{E}(t):=\frac{1}{2}\,|u'(t)|^2+\Phi(u(t))-\Phi^* \tag{1.9}$$

satisfies $\mathcal{E}(t)=\circ(t^{-2\alpha})$ as $t\to+\infty$.

Our next result asserts that we can get rid of the hypothesis on the boundedness of the solution by adding a second condition on the source term. This theorem generalizes the main result of [8].

Theorem 1.2. *Assume that* $\int_0^{+\infty}(1+t)^\alpha\,|g(t)|\,dt<\infty$ *and* $\int_0^{+\infty}(1+t)^{3\alpha}\,|g(t)|^2\,dt<\infty$. *Let u be a solution to Eq. (E_α) in the class (1.7). Then $u\in L^\infty(\mathbb{R}^+,H)$ and, therefore, we have the same conclusion as in Theorem 1.1.*

Our two last main results concern the strong convergence of the solution when the potential function Φ is even or the interior of the set $\arg\min\Phi$ is nonempty.

Theorem 1.3. *Assume that the function Φ is even,* $\int_0^{+\infty}(1+t)^\alpha\,|g(t)|\,dt<\infty$, *and*

$$\int_0^{+\infty}(1+t)^{2\alpha+1}\,|g(t)|^2\,dt<\infty.$$

Let u be a solution to Eq. (E_α) in the class (1.7). Then there exists $u_\infty\in\arg\min\Phi$ such that $u(t)\to u_\infty$ strongly in V as $t\to+\infty$.

Theorem 1.4. *Assume that the interior of the set* $\arg\min\Phi$ *with respect of the strong topology of V is not empty. Let u be a solution to Eq. (E_α) in the class (1.7). If $\int_0^{+\infty}(1+t)^\alpha\,|g(t)|\,dt<\infty$ and $u\in L^\infty(\mathbb{R}^+,H)$, then $u(t)$ converges strongly in V as $t\to+\infty$ to some element of* $\arg\min\Phi$.

Remark 1.5. A typical example of Eq. (E_α) is the following nonlinear damped wave equation:

$$u_{tt}+\gamma(t)u_t-\Delta u+f(u)=g \text{ on } \Omega\times]0,+\infty[,$$

with the Dirchlet boundary condition:

$$u=0 \text{ on } \partial\Omega\times]0,+\infty[,$$

where Ω is a bounded open subset of

$$\mathbb{R}^N,\ g\in L^1([0,+\infty[,L^2(\Omega)),$$

and $f:\mathbb{R}\to\mathbb{R}$ is a continuous and nondecreasing function which satisfies

$$|f(s)|\leq C(1+|s|)^m\ \forall s\in\mathbb{R},$$

where C and m are nonnegative constants with $m \leq \frac{N}{N-2}$ if $N \geq 3$. Here $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$, $a(v,w) = \int_\Omega \nabla v \nabla w dx$, and F is the function defined on $H_0^1(\Omega)$ by:

$$F(v) = \int_\Omega \int_0^{v(x)} f(s) ds dx.$$

Using Sobolev's inequalities, one can easily verify that the function $v \mapsto f(v)$ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and F is a C^1 convex function which satisfies the property (1.2), in fact

$$\forall v, w \in H_0^1(\Omega), F'(v)(w) = \int_\Omega f(v(x))w(x)dx.$$

2 Preliminary results

In this section, we prove some important preliminary results which will be very useful in the next section to prove the main theorems.

Proposition 2.1. *let u be a solution to Eq. (E_α) in the class (1.7). Assume that there exists $\nu \in [0, 1+\alpha[$ such that: $\int_0^{+\infty} (1+t)^{\frac{\nu}{2}} |g(t)| dt < \infty$. Assume moreover that $u \in L^\infty(\mathbb{R}^+, H)$ or $\int_0^{+\infty} (1+t)^{\nu+\alpha} |g(t)|^2 dt < \infty$. Then*

$$\int_0^{+\infty} (1+t)^{\nu-\alpha} |u'(t)|^2 dt < \infty,$$

and the energy function $\mathcal{E}$, given by (1.9), satisfies $\mathcal{E}(t) = \circ(t^{-\nu})$ as $t \to +\infty$.

Proof. The proof of this proposition makes use of a modified version of a method introduced by Cabot et Frankel in [5] and developed in [8]. Let $\bar{u} \in \arg\min \Phi$ and define the function $p : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$p(t) = \frac{1}{2} |u(t) - \bar{u}|^2.$$

Since u is in the class (1.7), the function p belongs to the space $W^{2,1}_{loc}(\mathbb{R}^+, \mathbb{R}^+)$ and satisfies almost everywhere on $\mathbb{R}^+$

$$\begin{aligned} & p''(t) + \gamma(t)p'(t) && (2.1) \\ &= |u'(t)|^2 + \langle \nabla\Phi(u(t)), \bar{u} - u(t)\rangle + \langle g(t), u(t) - \bar{u}\rangle \\ &= |u'(t)|^2 + \langle \nabla\Phi(u(t)), \bar{u} - u(t)\rangle_{V',V} + \langle g(t), u(t) - \bar{u}\rangle \\ &\leq |u'(t)|^2 + \Phi(\bar{u}) - \Phi(u(t)) + |g(t)| \sqrt{2p(t)} \\ &= \frac{3}{2} |u'(t)|^2 - \mathcal{E}(t) + |g(t)| \sqrt{2p(t)}, && (2.2) \end{aligned}$$

where we have used (1.1) and the convexity inequality $\Phi(\bar{u}) \geq \Phi(u(t)) + \langle \nabla\Phi(u(t)), \bar{u} - u(t)\rangle_{V',V}$. On the other hand the energy function $\mathcal{E}$ belongs to $W^{1,1}_{loc}(\mathbb{R}^+, \mathbb{R})$ and satisfies for almost every $t \geq 0$,

$$\begin{aligned} \mathcal{E}'(t) &= \langle u''(t), u'(t)\rangle + \langle \nabla\Phi(u(t)), u'(t)\rangle_{V',V} \\ &= \langle u''(t), u'(t)\rangle + \langle \nabla\Phi(u(t)), u'(t)\rangle \\ &= -\gamma(t) |u'|^2 + \langle g(t), u'(t)\rangle. \end{aligned} \tag{2.3}$$

For every $r \in \mathbb{R}$, we define the function λ_r on $\mathbb{R}^+$ by $\lambda_r(t) = (1+t)^r$. In view of (2.3),

$$(\lambda_\nu \mathcal{E})' = \lambda'_\nu \mathcal{E} - \lambda_\nu \gamma |u'|^2 + \lambda_\nu \langle g, u'\rangle. \tag{2.4}$$

Hence,

$$\lambda_\nu \gamma |u'|^2 \leq \lambda'_\nu \mathcal{E} - (\lambda_\nu \mathcal{E})' + \lambda_{\frac{\nu}{2}} |g| \sqrt{2\lambda_\nu \mathcal{E}}. \tag{2.5}$$

Since γ satisfies (1.5) with $\alpha < 1$,

$$\lambda'_\nu(t) |u'(t)|^2 = \circ(\lambda_\nu(t)\gamma(t) |u'(t)|^2)$$

as $t \to +\infty$. Then there exists $t_1 \geq t_0$ such that

$$\frac{3}{2}\lambda'_\nu(t) |u'(t)|^2 \leq \frac{1}{2}\lambda_\nu(t)\gamma(t) |u'(t)|^2 \text{ a.e. } t \geq t_1. \tag{2.6}$$

Thus, by multiplying the inequality (2.2) by $\lambda'_\nu(t)$ and using (2.5)-(2.6), we obtain

$$\begin{aligned} & \frac{1}{2}\lambda'_\nu \mathcal{E} + \frac{1}{2}(\lambda_\nu \mathcal{E})' \\ & \leq -\lambda'_\nu p'' - \lambda'_\nu \gamma p' + \lambda'_\nu |g| \sqrt{2p} + \frac{1}{2}\lambda_{\frac{\nu}{2}} |g| \sqrt{2\lambda_\nu \mathcal{E}}, \end{aligned}$$

almost everywhere on $[t_1, \infty[$.
Integrating this last inequality between t_1 and $t \geq t_1$, we get after integrations by parts

$$\frac{1}{2}\int_{t_1}^t \lambda'_\nu \mathcal{E} ds + \frac{1}{2}(\lambda_\nu \mathcal{E})(t) \leq C_0 + A(t) + B(t) + C(t), \tag{2.7}$$

where

$$\begin{aligned} C_0 &= \frac{1}{2}(\lambda_\nu \mathcal{E})(t_1) + (\lambda'_\nu p')(t_1) - (\lambda''_\nu p)(t_1) + (\lambda'_\nu \gamma p)(t_1), \\ A(t) &= -(\lambda'_\nu p')(t) + (\lambda''_\nu p)(t) - (\lambda'_\nu \gamma p)(t), \\ B(t) &= \int_{t_1}^t (-\lambda_\nu^{(3)} + (\lambda'_\nu \gamma)')p + \lambda'_\nu |g| \sqrt{2p} ds, \\ C(t) &= \int_{t_1}^t \lambda_{\frac{\nu}{2}} |g| \sqrt{\lambda_\nu \mathcal{E}} ds. \end{aligned}$$

Let us estimate separately $A(t), B(t)$, and $C(t)$. Firstly, by using the fact that $\sqrt{\lambda_\nu \mathcal{E}} \leq 1 + \lambda_\nu \mathcal{E}$, we get

$$C(t) \leq \int_0^{+\infty} (1+s)^{\frac{\nu}{2}} |g(s)| ds + \int_{t_1}^t \lambda_{\frac{\nu}{2}} |g| \lambda_\nu \mathcal{E} ds. \tag{2.8}$$

On the other hand, in view of (1.5)

$$\begin{aligned} A(t) &\leq \lambda'_\nu(t) |\langle u'(t), u(t) - \bar{u}\rangle| \\ &\quad - \nu[K - (\nu-1)(1+t)^{\alpha-1}](1+t)^{\nu-\alpha-1} p(t) \\ &\leq 2\lambda'_\nu(t)\sqrt{\mathcal{E}(t)}\sqrt{p(t)} \\ &\quad - \nu[K - (\nu-1)(1+t)^{\alpha-1}](1+t)^{\nu-\alpha-1} p(t). \end{aligned}$$

Therefore, since $\alpha < 1$, there exists $t_2 \geq t_1$ such that for every $t \geq t_2$,

$$A(t) \leq 2\lambda'_\nu(t)\sqrt{\mathcal{E}(t)}\sqrt{p(t)} - \frac{\nu K}{2}(1+t)^{\nu-\alpha-1} p(t).$$

Using now the elementary inequality

$$\forall a>0\ \forall b,x\in\mathbb{R},\ -ax^2+bx\le\frac{b^2}{4a} \tag{2.9}$$

with $x=\sqrt{p(t)}$, we get

$$A(t)\le\frac{2\nu}{K}(1+t)^{\nu+\alpha-1}\mathcal{E}(t)\ \forall t\ge t_2.$$

Using once again the fact that $\alpha<1$, we infer the existence of $t_3\ge t_2$ such that

$$A(t)\le\frac{1}{4}\lambda_\nu(t)\mathcal{E}(t)\ \forall t\ge t_3. \tag{2.10}$$

Let us now prove that the function B is bounded. To this end we first notice that, thanks to (1.5) and (1.6), we have for almost every $t\ge t_1$

$$\begin{aligned}&-\lambda_\nu^{(3)}(t)+(\lambda_\nu'\gamma)'(t)\\&\le-\lambda_\nu^{(3)}(t)+\lambda_\nu''\gamma-\alpha\lambda_\nu'(t)\frac{\gamma(t)}{(1+t)}\\&\le-\lambda_\nu^{(3)}(t)-\nu K(1+\alpha-\nu)(1+t)^{\nu-2-a}.\end{aligned}$$

Since $\nu<1+\alpha$ and $\alpha<1$, there exists $t_4\ge t_3$ such that for almost every $t\ge t_4$,

$$-\lambda_\nu^{(3)}(t)+(\lambda_\nu'\gamma)'(t)\le-\mu(1+t)^{\nu-2-a}, \tag{2.11}$$

where $\mu=\dfrac{\nu K(1+\alpha-\nu)}{2}>0$. Therefore, if $u\in L^\infty(\mathbb{R}^+,H)$ then for every $t\ge t_4$ we have

$$\begin{aligned}B(t)&\le B(t_4)+\sqrt{\sup_{t\ge0}2p(t)}\int_0^{+\infty}\lambda_\nu'\,|g|\,dt\\&\le B(t_4)+\nu\sqrt{\sup_{t\ge0}2p(t)}\int_0^{+\infty}(1+t)^{\frac{\nu}{2}}\,|g|\,dt\end{aligned}$$

Let us now examine the boundedness of the function B under the other hypothesis $\int_0^{+\infty}(1+t)^{\nu+\alpha}\,|g(t)|^2\,dt<\infty$. By using (2.11) and the inequality (2.9) with $x=\sqrt{p(t)}$ we easily get that for every $t\ge t_4$

$$\begin{aligned}B(t)&\le B(t_4)+\frac{2\nu^2}{\mu}\int_{t_4}^{t}(1+s)^{\nu+\alpha}\,|g(s)|^2\,dt\\&\le B(t_4)+\frac{2\nu^2}{\mu}\int_0^{+\infty}(1+s)^{\nu+\alpha}\,|g(s)|^2\,dt.\end{aligned}$$

Coming back to (2.7) and using the estimates (2.8)-(2.10) and the boundedness of the function B, we infer the existence of a constant $C_1\ge0$ such that for every $t\ge t_4$,

$$\frac{1}{2}\int_{t_1}^{t}\lambda_\nu'\mathcal{E}ds+\frac{1}{4}(\lambda_\nu\mathcal{E})(t)\le C_1+\int_{t_1}^{t}\lambda_{\frac{\nu}{2}}\,|g|\,\lambda_\nu\mathcal{E}dt.$$

Therefore, by applying Gronwall's inequality we first get that $\sup_{t\ge t_1}\lambda_\nu(t)\mathcal{E}(t)<+\infty$ and then we deduce that $\int_{t_1}^{+\infty}\lambda_\nu'(t)\mathcal{E}(t)dt<+\infty$. Recalling that the energy function $\mathcal{E}$ is continuous and hence locally bounded on $\mathbb{R}^+$, we infer that

$$\int_0^{+\infty}\lambda_\nu'(t)\mathcal{E}(t)dt<+\infty \tag{2.12}$$

and

$$\sup_{t\ge0}\lambda_\nu(t)\mathcal{E}(t)<+\infty. \tag{2.13}$$

Hence by using the equality (2.4) we obtain

$$\begin{aligned}&\int_0^{+\infty}[(\lambda_\nu\mathcal{E})']_+dt\\&\le\int_0^{+\infty}\lambda_\nu'\mathcal{E}dt+\sqrt{\sup_{t\ge0}2\lambda_\nu(t)\mathcal{E}(t)}\int_0^{+\infty}\lambda_{\frac{\nu}{2}}\,|g|\,dt\\&<+\infty,\end{aligned}$$

where $[(\lambda_\nu\mathcal{E})']_+$ is the positive part of $(\lambda_\nu\mathcal{E})'$. The last inequality implies that $\lambda_\nu(t)\mathcal{E}(t)$ converges as t goes to $+\infty$ to some real number m. If $m\ne0$ then

$$\lambda_\nu'(t)\mathcal{E}(t)=\frac{\lambda_\nu(t)\mathcal{E}(t)}{\nu(1+t)}\sim\frac{m}{\nu(1+t)}\quad\text{as}\quad t\to+\infty$$

which contradicts the result (2.12). Thus $m=0$ and therefore $\mathcal{E}(t)=\circ(t^{-\nu})$ as $t\to+\infty$. Finally, by using the inequality (2.5), we obtain

$$\begin{aligned}&\int_0^{+\infty}\lambda_\nu\gamma\,|u'|^2\,dt\\&\le\int_0^{+\infty}\lambda_\nu'\mathcal{E}dt+\mathcal{E}(0)+\sqrt{\sup_{t\ge0}2\lambda_\nu(t)\mathcal{E}(t)}\int_0^{+\infty}\lambda_{\frac{\nu}{2}}\,|g|\,dt.\end{aligned}$$

In view of (2.12) and (2.13), the right hand side of the previous inequality is finite, then thanks to the hypothesis (1.5) we conclude that

$$\int_0^{+\infty}(1+t)^{\nu-\alpha}\,|u'(t)|^2\,dt<+\infty$$

as desired. □

Proposition 2.2. *Let u be a solution to Eq. (E_α). Assume that the integrals*

$$\int_0^{+\infty}(1+t)^\alpha\,|g(t)|\,dt\quad\text{and}\quad\int_0^{+\infty}(1+t)^\alpha\,|u'(t)|^2\,dt$$

are finite and $\Phi(u(t))\to\Phi^*$ as $t\to+\infty$. Then $u(t)$ converges weakly in V as $t\to+\infty$ toward some element u_∞ of $\arg\min\Phi$.

The proof of this proposition relies on the classical Opial's lemma [9] (see [3] for a simple proof) and an elementary lemma which will be also used to prove Theorem 1.3 and Theorem 1.4. Let us first recall Opial's lemma.

Lemma 2.3 (Opial's lemma)**.** *Let $x:[t_0,+\infty[\to\mathcal{H}$. Assume that there exists a nonempty subset S of $\mathcal{H}$ such that:*

(i) If $t_n \to +\infty$ and $x(t_n) \rightharpoonup x$ weakly in $\mathcal{H}$, then $x \in S$.

(ii) For every $z \in S$, $\lim_{t\to+\infty} \|x(t) - z\|$ exists.

Then there exists $z_\infty \in S$ such that $x(t) \rightharpoonup z_\infty$ weakly in $\mathcal{H}$ as $t \to +\infty$.

Lemma 2.4. *There exists $\tau_0 \geq 0$ such that for every $\tau \geq \tau_0$*

$$\int_\tau^{+\infty} e^{-\Gamma(t,\tau)} dt \leq \frac{2}{K}(1+\tau)^\alpha$$

where $\Gamma(t,\tau) = \int_\tau^t \gamma(s)ds$.

Proof. Let $\tau \geq t_0$. In view of (1.5),

$$\begin{aligned}
&\int_\tau^{+\infty} e^{-\Gamma(t,\tau)} dt \\
&\leq \frac{1}{K}\int_\tau^{+\infty} (1+t)^\alpha \gamma(t) e^{-\Gamma(t,\tau)} dt \\
&= -\frac{1}{K}\int_\tau^{+\infty} (1+t)^\alpha \left(e^{-\Gamma(t,\tau)}\right)' dt \\
&= \frac{1}{K}(1+\tau)^\alpha + \frac{\alpha}{K}\int_\tau^{+\infty} (1+t)^{\alpha-1} e^{-\Gamma(t,\tau)} dt \\
&\leq \frac{1}{K}(1+\tau)^\alpha + \frac{\alpha}{K(1+\tau)^{1-\alpha}}\int_\tau^{+\infty} e^{-\Gamma(t,\tau)} dt.
\end{aligned}$$

It is then enough to choose τ_0 large enough such that $\frac{\alpha}{K(1+\tau_0)^{1-\alpha}} \leq \frac{1}{2}$. □

Proof of Proposition 2.2 . Let us first prove that $u \in L^\infty(\mathbb{R}^+, V)$. Let $\bar{u} \in \arg\min \Phi$ and define, as in the proof of Proposition 2.1, the function $p : \mathbb{R}^+ \to \mathbb{R}^+$ by $p(t) = \frac{1}{2}|u(t) - \bar{u}|^2$. This function belongs to the space $W^{2,1}_{loc}(\mathbb{R}^+, \mathbb{R}^+)$ and satisfies almost everywhere on $\mathbb{R}^+$,

$$\begin{aligned}
&p'' + \gamma p' \\
&= |u'|^2 - \langle \nabla\Phi(u), u - \bar{u}\rangle + \langle g, u - \bar{u}\rangle \\
&= |u'|^2 - \langle \nabla\Phi(u) - \nabla\Phi(\bar{u}), u - \bar{u}\rangle_{V',V} + \langle g, u - \bar{u}\rangle \\
&\leq |u'|^2 + |g|\sqrt{2p},
\end{aligned}$$

where we have used the monotonicity of the operator $\nabla\Phi$. Therefore, for almost every $t \geq \tau_0$,

$$p'(t) \leq e^{-\Gamma(t,\tau_0)} p'(\tau_0) + \int_{\tau_0}^t e^{-\Gamma(t,s)}\rho(s)ds$$

where $\rho := |u'|^2 + |g|\sqrt{2p}$.

Thus, by using the previous lemma and Fubini's theorem, we get for every $t \geq \tau_0$

$$\begin{aligned}
&\int_{\tau_0}^t [p'(\tau)]_+ d\tau && (2.14)\\
&\leq \frac{2\,(1+\tau_0)^\alpha}{K}|p'(\tau_0)| + \frac{2}{K}\int_{\tau_0}^t (1+s)^\alpha \rho(s)ds \\
&\leq c_0 + \frac{2}{K}\int_{\tau_0}^t (1+s)^\alpha |g(s)|\sqrt{2p(s)}ds && (2.15)
\end{aligned}$$

where $c_0 = \frac{2(1+\tau_0)^\alpha}{K}|p'(\tau_0)| + \frac{2}{K}\int_0^{+\infty}(1+s)^\alpha |u'(s)|^2 ds$ and $[p'(\tau)]_+$ is the positive part of $p'(\tau)$.

Using now the inequalities $\sqrt{2p} \leq 1 + 2p$ and $p(t) \leq p(\tau_0) + \int_{\tau_0}^t [p'(\tau)]_+ d\tau$, we obtain

$$p(t) \leq c_1 + \frac{4}{K}\int_{\tau_0}^t (1+s)^\alpha |g(s)|\, p(s)ds, \ \forall t \geq \tau_0,$$

with

$$c_1 = c_0 + p(\tau_0) + \frac{2}{K}\int_0^{+\infty}(1+s)^\alpha |g(s)|\, ds.$$

Hence, by applying Gronwall's inequality, we deduce that the function p is bounded, which is equivalent to $u \in L^\infty(\mathbb{R}^+, H)$. Using now [5, Remark 3.4], we obtain that $u \in L^\infty(\mathbb{R}^+, V)$. Coming back to the estimate (2.15), we infer that

$$\begin{aligned}
&\int_{\tau_0}^{+\infty} [p'(\tau)]_+ d\tau \\
&\leq c_0 + \frac{2}{K}\int_0^{+\infty}(1+s)^\alpha |g(s)|\, ds \sqrt{\sup_{t\geq 0} 2p(t)} \\
&< +\infty
\end{aligned}$$

which implies that $\lim_{t\to+\infty} p(t)$ and therefore $\lim_{t\to+\infty} |u(t) - \bar{u}|$ exist. Now, let $\bar{x} \in H$ such that there exists a sequence $(t_n)_n$ of positive real numbers tending to $+\infty$ such that $u(t_n)$ converges weakly in H to $\bar{x}$. Since $u \in L^\infty(\mathbb{R}^+, V)$, $u(t_n)$ converges weakly also in the space V to the same element $\bar{x}$. Using now the weak lower semi-continuity of the continuous and convex function Φ, we deduce that $\Phi^* = \liminf \Phi(u(t_n)) \leq \Phi(\bar{x})$. Thus $\bar{x} \in \arg\min \Phi$. Therefore, applying Opial's lemma with $S = \arg\min \Phi$ ensures that $u(t)$ converges weakly in H as $t \to +\infty$ to some element of $\arg\min \Phi$. Recalling that $u \in L^\infty(\mathbb{R}^+, V)$, we conclude that this weak convergence holds also in the space V. □

We close this section by proving the following simple lemma that will be used in the proof of Theorem 1.4.

Lemma 2.5. *For every $v \in V$,*

$$|v| \leq \|v\|_{V'}^{\frac{1}{2}} \|v\|_V^{\frac{1}{2}}. \tag{2.16}$$

Proof. Let $v \in V$. From (1.1),

$$|v|^2 = \langle v, v\rangle = \langle v, v\rangle_{V',V} \leq \|v\|_{V'}\|v\|_V .$$

which gives the result. □

3 Proof of the main results

This section is devoted to the proof of our main theorems. Let us first notice that Theorem 1.1 and Theorem 1.2 follow immediately from Proposition 2.1 (with $\nu = 2\alpha$) and Proposition 2.2. Hence it remains to prove Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. The proof is based on the adaptation of a method introduced by Bruck [4] for the steepest descent method and used by Alvarez [1] for the heavy ball with friction system.
Since $2\alpha+1 \geq 3\alpha$, then in view of Theorem 1.2, Proposition 2.1, and Proposition 2.2, $u(t)$ converges weakly in V to some $u_\infty \in \arg\min \Phi$, and $\int_0^{+\infty}(1+t)^\alpha \left|u'(t)\right|^2 dt < \infty$. Let $\tau \geq \tau_0$ where τ_0 is the real defined in Lemma 2.4. We define the function q on the interval $[\tau_0, \tau]$ by:

$$q(t) = |u(t)|^2 - |u(\tau)|^2 - \frac{1}{2}\left|u(t) - u(\tau)\right|^2.$$

The function q belongs to the space $W^{2,1}([\tau_0,\tau],\mathbb{R})$ and satisfies almost everywhere

$$q'(t) = \langle u'(t), u(t) + u(\tau)\rangle \tag{3.1}$$

$$q''(t) = \left|u'(t)\right|^2 + \langle u''(t), u(t) + u(\tau)\rangle. \tag{3.2}$$

Combining this two equalities, we obtain

$$\begin{aligned}
&q''(t) + \gamma(t)q'(t) &(3.3)\\
&= \left|u'(t)\right|^2 + \langle \nabla\Phi(u), -u(\tau) - u(t)\rangle_{V',V} &(3.4)\\
&\quad + \langle g(t), u(t) + u(\tau)\rangle\\
&\leq \left|u'(t)\right|^2 + \Phi(-u(\tau)) - \Phi(u(t)) + 2M\left|g(t)\right|\\
&= \left|u'(t)\right|^2 + \Phi(u(\tau)) - \Phi(u(t)) + 2M\left|g(t)\right|\\
&= \frac{3}{2}\left|u'(t)\right|^2 + \tilde{\mathcal{E}}(\tau) - \tilde{\mathcal{E}}(t) + 2M\left|g(t)\right| + \int_t^\tau \frac{|g(s)|^2}{4\gamma(s)}ds &(3.5)
\end{aligned}$$

where $M = \sup_{t\geq 0}|u(t)|$ and $\tilde{\mathcal{E}}$ is the modified energy function defined by:

$$\tilde{\mathcal{E}}(t) = \mathcal{E}(t) + \int_t^{+\infty} \frac{|g(s)|^2}{4\gamma(s)}ds,$$

where $\mathcal{E}$ is the energy function given by (1.9). Using (2.3), we get

$$\begin{aligned}
\tilde{\mathcal{E}}'(t) &= -\gamma(t)\left|u'(t)\right|^2 + \langle g(t), u'(t)\rangle - \frac{|g(t)|^2}{4\gamma(t)}\\
&\leq -\left(\sqrt{\gamma(t)}\left|u'(t)\right| - \frac{|g(t)|}{2\sqrt{\gamma(t)}}\right)^2.
\end{aligned}$$

Therefore the function $\tilde{\mathcal{E}}$ is non increasing. Hence (3.5) and (1.5) yield

$$q''(t) + \gamma(t)q'(t) \leq \omega(t),$$

where

$$\omega(t) = \frac{3}{2}\left|u'(t)\right|^2 + 2M\left|g(t)\right| + \frac{1}{4K}\int_t^{+\infty}(1+s)^\alpha \left|g(s)\right|^2 ds.$$

Therefore, for almost every $t \in [\tau_0, \tau]$,

$$q'(t) \leq e^{-\Gamma(t,\tau_0)}\left|q'(\tau_0)\right| + \int_{\tau_0}^t e^{-\Gamma(t,s)}\omega(s)ds \equiv \kappa(t). \tag{3.6}$$

A simple calculation, using Fubini's theorem and Lemma 2.4, gives

$$\int_{\tau_0}^{+\infty}\kappa(t)dt \leq c_0 + \frac{2}{K}\int_{\tau_0}^{+\infty}(1+s)^\alpha\omega(s)ds$$

where $c_0 = \frac{2}{K}(1+\tau_0)^\alpha\left|q'(\tau_0)\right|$.

Using once again Fubini's theorem, we get

$$\begin{aligned}
&\int_{\tau_0}^{+\infty}(1+t)^\alpha\int_t^{+\infty}(1+s)^\alpha\left|g(s)\right|^2 dsdt\\
&\leq \frac{1}{\alpha+1}\int_{\tau_0}^{+\infty}(1+s)^{2\alpha+1}\left|g(s)\right|^2 ds.
\end{aligned}$$

Then we deduce that the integral $\int_{\tau_0}^{+\infty}(1+s)^\alpha\omega(s)ds$ is finite which implies

$$\int_{\tau_0}^{+\infty}\kappa(t)dt < +\infty. \tag{3.7}$$

Integrating now (3.6) between t and τ, with $\tau_0 \leq t \leq \tau$, we get

$$\frac{1}{2}\left|u(t) - u(\tau)\right|^2 \leq |u(t)|^2 - |u(\tau)|^2 + \int_t^\tau \kappa(s)ds. \tag{3.8}$$

In the proof of Proposition 2.2, we showed that

$$\lim_{t\to+\infty}\left|u(t) - \bar{u}\right|^2$$

exists for all $\bar{u}$ in $\arg\min\Phi$. But $0 \in \arg\min\Phi$ since Φ is convex and even, then $\lim_{t\to+\infty}|u(t)|^2$ exists. Therefore, (3.8) and (3.7) imply

$$|u(\tau) - u(t)| \to 0 \text{ as } t, \tau \to +\infty.$$

Thus, in view of Cauchy criteria, $u(t)$ converges strongly in H as $t \to +\infty$. Therefore, by using [5, Corollary 3.6], we deduce that $u(t)$ converges strongly in V as $t \to +\infty$. Finally, since $u(t) \rightharpoonup u_\infty$ weakly in V, we conclude that $u(t) \to u_\infty$ strongly in V. □

Proof of Theorem 1.4. By assumption, there exists $x^* \in \arg\min\Phi$ and $r > 0$ such that for all v in the unit Ball $B_V(0,1)$ of V we have $\nabla\Phi(x^* + rv) = 0$. Therefore the monotonicity of $\nabla\Phi$ implies that for every $x \in V$, $\langle\nabla\Phi(x), x - x^* - rv\rangle_{V',V} \geq 0$ which yields that $\langle\nabla\Phi(x), v\rangle_{V',V} \leq \frac{1}{r}\langle\nabla\Phi(x), x - x^*\rangle_{V',V}$. Hence by taking the supremum on $v \in B_V(0,1)$, we get

$$\|\nabla\Phi(x)\|_{V'} \leq \frac{1}{r}\langle\nabla\Phi(x), x - x^*\rangle_{V',V}. \tag{3.9}$$

Let us now define the function $p(t) = \frac{1}{2}\left|u(t) - x^*\right|^2$. We already know that p satisfies the differential inequality

$$\begin{aligned}
&p''(t) + \gamma(t)p'(t)\\
&\leq \left|u'(t)\right|^2 - \langle\nabla\Phi(u(t)), u(t) - x^*\rangle_{V',V} + \langle g(t), u(t) - x^*\rangle.
\end{aligned}$$

Hence by using (3.9), we obtain

$$r\left\|\nabla\Phi(u(t))\right\|_{V'}\leq -p''(t)-\gamma(t)p'(t)+\sigma(t), \quad (3.10)$$

where $\sigma(t)=\left|u'(t)\right|^2+\left|g(t)\right|\sup_{t\geq 0}\left|u(t)-x^*\right|$.
Recalling that in view Proposition 2.1,

$$\int_0^{+\infty}\lambda_\alpha(t)\sigma(t)dt<\infty$$

where $\lambda_\alpha(t)=(1+t)^\alpha$. Hence, by multiplying (3.10) by $\lambda_\alpha(t)$ and integrating between t_0 and $\tau\geq t_0$, we get after integration by parts and simplification

$$\begin{aligned} & r\int_{t_0}^{\tau}\lambda_\alpha(t)\left\|\nabla\Phi(u(t))\right\|_{V'}dt \\ & \leq C-\lambda_\alpha(\tau)p'(\tau)+\lambda_\alpha'(\tau)p(\tau) \\ & -\underbrace{(\lambda_\alpha\gamma)}_{\geq 0}(\tau)p(\tau)+\int_{t_0}^{\tau}[\underbrace{(\lambda_\alpha\gamma)'}_{\leq 0}-\lambda_\alpha''](t)p(t)dt \end{aligned}$$

where C is a constant independent of τ. Since $\alpha<1$ and $u\in L^\infty(\mathbb{R}^+,H)$, the integral $\int_{t_0}^{+\infty}\left|\lambda_\alpha''(t)\right|p(t)dt$ and the supremum $\sup_{\tau\geq t_0}\lambda_\alpha'(\tau)p(\tau)$ are finite. Moreover, from Proposition 2.1, $|u'(\tau)|=\circ(\tau^{-\alpha})$ as $\tau\to+\infty$, then

$$\sup_{\tau\geq t_0}\lambda_\alpha(\tau)\left|p'(\tau)\right|\leq\sup_{\tau\geq t_0}\lambda_\alpha(\tau)\left|u'(\tau)\right|\left|u(\tau)-x^*\right|<\infty.$$

Therefore, we conclude that

$$\int_{t_0}^{+\infty}\lambda_\alpha(t)\left\|\nabla\Phi(u(t))\right\|_{V'}dt<+\infty. \quad (3.11)$$

From Eq. (E_α), we have

$$u''(t)+\gamma(t)u'(t)=g(t)-\nabla\Phi(u(t))$$

Hence, by integrating this equation we get

$$u'(t)=e^{-\Gamma(t,\tau_0)}u'(\tau_0)+\int_{\tau_0}^{t}e^{-\Gamma(t,s)}[g(s)-\nabla\Phi(u(s))]ds, \quad (3.12)$$

for almost every $t\geq\tau_0$ where τ_0 is the real defined by Lemma 2.4. Up to replace τ_0 by $\tau_0'>\tau_0$, we can assume that $u'(\tau_0)\in H$. Thus by applying Lemma 2.4 and Fubini's theorem to the equality (3.12), we obtain

$$\begin{aligned} & \int_{\tau_0}^{+\infty}\left\|u'(t)\right\|_{V'}dt \\ \leq\ & \frac{2}{K}(1+\tau_0)^\alpha\left\|u'(\tau_0)\right\|_{V'}+\frac{2}{K}\int_{\tau_0}^{+\infty}(1+s)^\alpha\left\|g(s)\right\|_{V'} \\ & +\frac{2}{K}\int_{\tau_0}^{+\infty}(1+s)^\alpha\left\|\nabla\Phi(u(s))\right\|_{V'}ds. \end{aligned}$$

Hence $\int_{\tau_0}^{+\infty}\left\|u'(t)\right\|_{V'}dt<+\infty$ thanks to the continuous injection $H\hookrightarrow V'$, the hypothesis on g, and the estimate (3.11). Thus we deduce that $u(t)$ converges strongly in V' as $t\to+\infty$ to some u_∞. Recalling that, in view of Theorem 1.1, $u\in L^\infty(\mathbb{R}^+,V)$ and applying Lemma 2.5, we infer that $u(t)\to u_\infty$ strongly in H, which in view of [5, Corollary 3.6] implies that $u(t)$ converges strongly to u_∞ in V. Finally, Theorem 1.1 ensures that $u_\infty\in\arg\min\Phi$. The proof is complete. □

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