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Fractional stochastic heat equation with Hermite noise

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Abstract

We analyze the existence and several trajectorial and distributional properties of the solution to the fractional heat equation driven by a multiparameter Hermite process. We give a necessary and sufficient condition for the existence of the mild solution and we study the scaling property and the regularity of the trajectories of this solution. We also obtain a decomposition of the solution as the sum of a self-similar process with stationary increments and of another process with nice sample paths. This decomposition is used to get the limit behavior of the temporal p-variation of the solution. Via the p-variation method, we define a consistent estimator for the drift parameter of this equation.

Keywords: fractional Laplacian, Wiener chaos, Hermite process, stochastic heat equation, multiple stochastic integrals, multiparameter stochastic processes.

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1 Introduction

The Hermite processes are self-similar stochastic processes with stationary increments and with long memory. The class of Hermite processes includes the fractional Brownian motion (which is the only Gaussian Hermite process) and the Rosenblatt process. In the last two decades, this class of stochastic processes has been extensively studied, due to their potential to model various phenomena and also due to their interesting theoretical properties. The reader may consult the recent monographs [16] or [22] for a complete presentation of the Hermite processes.

The development of stochastic analysis of Hermite processes naturally led to the study of stochastic (partial) differential equations driven by these stochastic processes. Some examples of stochastic (partial) differential equations with Hermite noise can be found in [1, 2, 3, 4, 5, 7, 14, 19, 21].

The purpose of this work is to study the fractional stochastic heat equation driven by the multiparameter Hermite process. This pursues the very dynamic research direction on stochastic partial differential equations (SPDEs in the sequel) with more and more general random noises, initiated by the seminal paper [24]. The term "fractional" is related to the presence of the fractional Laplacian operator in the heat equation, which replaces the usual Laplacian. The case of the stochastic heat equation with standard Laplacian has been considered in e.g. [19, 20, 5]. We give a necessary and sufficient condition for the existence of the solution, and we study several other properties of the solution, such as the self-similarity, the regularity of the sample paths. We also state and prove a decomposition of the solution as the sum of a self-similar (non-Gaussian) process with stationary increments and of another process with very regular trajectories. This decomposition will be used to obtain the behaviour of the temporal p-variations of the solution to the fractional stochastic heat equation with Hermite noise and then to estimate the drift parameter of this equation.

Although several properties of the solution can be obtained by following the lines of the proofs in the case of the standard Laplacian (as in e.g. [19]), the statistical inference for the stochastic heat equation (or even in general for stochastic partial equations) with Hermite noise has never been considered. We point out that several results are available for stochastic differential equations driven by Hermite processes, especially for the Hermite Ornstein-Uhlenbeck process, see e.g. [1] or [14]. The method to estimate the drift parameter, based on some transform of the solution that allows to move this parameter in front of the random noise and then to employ the p-variation techniques, constitutes a first step for the statistical inference in Hermite-driven models and it also works for the case of the standard Laplacian operator. Our techniques are based on the properties of the random processes in a Wiener chaos of fixed order and on some estimates involving the Green kernel of the fractional Laplacian operator.

This work is organized as follows: in Section 2, we present the basics on multiparameter Hermite process and we recall the construction of the Wiener integral with respect to them. In Section 3, we consider the solution to the fractional stochastic heat equation with Hermite noise and we analyze several properties: its existence, its scaling property, the regularity of its trajectories and the p-variation. We apply these results to estimate the drift parameter in Section 4. Section 5 is an Appendix which contains the basic properties of multiple stochastic integrals.

2 Preliminaries

Let us start by introducing the definition and the basic properties of the multiparameter Hermite processes and by recalling the construction of the Wiener integral with respect to the Hermite sheet. Since we are dealing with multi-indices stochastic process, we will introduce some notation, needed in order to facilitate the reading. For $d \in \mathbb{N}\setminus\{0\}$, if $\mathbf{a} = (a_1, a_2, \dots, a_d)$, $\mathbf{b} = (b_1, b_2, .., b_d)$ and $\alpha = (\alpha_1, .., \alpha_d)$ are vectors in $\mathbb{R}^d$, we set

$$\begin{aligned}
&\mathbf{ab} = \prod_{i=1}^{d} a_i b_i, \quad |\mathbf{a}-\mathbf{b}|^{\alpha} = \prod_{i=1}^{d} |a_i - b_i|^{\alpha_i},\\
&\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, \dots, a_d/b_d), \quad [\mathbf{a},\mathbf{b}] = \prod_{i=1}^{d}[a_i, b_i], \quad (\mathbf{a},\mathbf{b}) = \prod_{i=1}^{d}(a_i, b_i),\\
&\sum_{\mathbf{i}=0}^{\mathbf{N}} a_{\mathbf{i}} = \sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\dots\sum_{i_d=0}^{N_d} a_{i_1,i_2,\dots,i_d}, \quad \mathbf{a}^{\mathbf{b}} = \prod_{i=1}^{d} a_i^{b_i} \quad, \text{ if } \ \mathbf{N} = (N_1, .., N_d)
\end{aligned} \tag{2.1}$$

We also set $\mathbf{a} < \mathbf{b}$ if $a_1 < b_1, a_2 < b_2, \dots, a_d < b_d$ (analogously for the other inequalities). We write $\mathbf{a} - \mathbf{n}$ to indicate the product $\prod_{i=1}^{d}(a_i - n)$. By β we denote the Beta function $\beta(p, q) = \int_0^1 z^{p-1}(1-z)^{q-1}dz, p, q > 0$ and we use the notation

$$\beta(\mathbf{a},\mathbf{b}) = \prod_{i=1}^{d} \beta(a_i, b_i)$$

if $\mathbf{a} = (a_1, .., a_d)$ and $\mathbf{b} = (b_1, .., b_d)$ belong to $(0, \infty)^d$.

2.1 The Multiparameter Hermite Process

Let $\mathbf{H} = (H_1, \dots, H_d) \in \left(\frac{1}{2}, 1\right)^d$ and let $d, q \geq 1$ be integer numbers. The d-parameter Hermite process (or the Hermite sheet) $\left(Z^{(q)}_{\mathbf{H},d}(t), t \in \mathbb{R}^d_+\right)$ of order $q \geq 1$ and with Hurst parameter $\mathbf{H}$ is defined, for every $t \in \mathbb{R}_+$, as

$$Z^{(q)}_{\mathbf{H},d}(t) = c(\mathbf{H}, q)\int_{\mathbb{R}^{dq}}\left[\int_{[0,t]}(s-y_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}\dots(s-y_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}ds\right]dW(y_1)\dots dW(y_q) \tag{2.2}$$

where $x_+ = \max(x,0)$ and $(W(y), y \in \mathbb{R}^d)$ is a d-parameter Brownian sheet, i.e. it is a centered Gaussian process with covariance

$$\mathbf{E}W(y)W(z) = \prod_{j=1}^{d} \left(\frac{1}{2} \left(|y_j| + |z_j| - |y_j - z_j| \right) \right)$$

if $y = (y_1, ..., y_d), z = (z_1, .., z_d) \in \mathbb{R}^d$. The stochastic integral $dW(y_1) \dots dW(y_q)$ should be understood as a multiple stochastic integral of order q with respect to the Gaussian process $(W(y), y \in \mathbb{R}^d)$, as described in the Appendix (Section 5). The notation $(s - y_1)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{q}\right)}$ is understood as $\prod_{j=1}^{d} (s_j - y_{1,j})^{-\left(\frac{1}{2} - \frac{1-H_j}{q}\right)}$ if $s = (s_1, ..., s_d)$ and $y_1 = (y_{1,1}, y_{1,2}, ..., y_{1,d})$. The constant $c(\mathbf{H}, q)$ is a strictly positive constant chosen such that $\mathbf{E}(Z^{(q)}_{\mathbf{H},d}(t))^2 = t^{2\mathbf{H}}$ for every $t \in \mathbb{R}^d_+$. Using the notation in the Appendix, we can write, for every $t \in \mathbb{R}^d_+$,

$$Z^{(q)}_{\mathbf{H},d}(t) = I_q(L_t)$$

where, for $y_1, \dots, y_q \in \mathbb{R}^d$, the kernel L_t is given by

$$L_t(y_1, \dots, y_q) = c(\mathbf{H}, q) \left[\int_{[0,t]} (s - y_1)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{q}\right)} \dots (s - y_q)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{q}\right)} ds \right]. \tag{2.3}$$

It can be checked that L_t belongs to $L^2(\mathbb{R}^{dq})$ for every $t \geq 0$ and this justifies the fact that the multiple stochastic integral in (2.2) is well-defined. We can also write, taking into account the notation (2.1),

$$\begin{aligned} Z^{(q)}_{\mathbf{H},d}(t) &= C(\mathbf{H}, q) \int_{\mathbb{R}^{dq}} dW(y_{1,1}, ..., y_{1,d}) \dots dW(y_{q,1}, ..., y_{q,d}) \left[\int_0^{t_1} \dots \int_0^{t_d} ds_d \dots ds_1 \right. \\ & \left. \prod_{j=1}^{d} (s_1 - y_{1,j})_+^{-\left(\frac{1}{2} + \frac{1-H_j}{2}\right)} \dots \prod_{j=1}^{d} (s_d - y_{d,j})_+^{-\left(\frac{1}{2} + \frac{1-H_j}{2}\right)} \right]. \end{aligned}$$

if $t = (t_1, ..., t_d), s = (s_1, ..., s_d), y_i = (y_{i,1}, ..., y_{i,d})$ for $i = 1, ..., q$.

The Hermite sheet is an $\mathbf{H}$-self-similar stochastic process and it has stationary increments and its paths are Hölder continuous of order $\boldsymbol{\delta} < \mathbf{H}$ (see [16] or [22] for the description of these concepts in the multidimensional context). Its covariance is the same for every $q \geq 1$ and it coincides with the covariance of the d-parameter fractional Brownian motion, i.e.

$$\mathbf{E}Z^{(q)}_{\mathbf{H},d}(t) Z^{(q)}_{\mathbf{H},d}(s) = \prod_{i=1}^{d} \left(\frac{1}{2} \left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right) \right) =: R_{\mathbf{H}}(t, s), \quad t_i, s_i \geq 0. \tag{2.4}$$

if $t = (t_1, ..., t_d)$ and $s = (s_1, ..., s_d)$.

2.2 The Wiener-Hermite Integral

The definition of the solution to the fractional stochastic heat equation with Hermite noise involves a Wiener integral with respect to the Hermite sheet. We recall below the definition of the Wiener integral with respect to the multiparameter Hermite process. This integral will be called in the sequel *the Wiener-Hermite integral*. The first step is to introduce the space of suitable integrands. Let $\left|\mathcal{H}(\mathbb{R}^d)\right|$ be the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dudv |f(u)||f(v)||u - v|^{2\mathbf{H}-2} < \infty.$$

If $f \in \left|\mathcal{H}(\mathbb{R}^d)\right|$, the Wiener integral of f with respect to $Z^{(q)}_{\mathbf{H},d}$ is defined as

$$\int_{\mathbb{R}^d} f(s) dZ^{(q)}_{\mathbf{H},d}(s) = I_q(Jf) \tag{2.5}$$

where I_q denotes the multiple integral of order q with respect to the d-parameter standard Brownian field $\Big(B(\mathbf{y}), \mathbf{y} \in \mathbb{R}^d\Big)$ and the transfer operator $Jf \in L^2(\mathbb{R}^{d.q})$ is given by, for every $y_1, \dots, y_q \in \mathbb{R}^d$,

$$Jf(y_1, ..., y_q) = c(\mathbf{H}, q) \int_{\mathbb{R}^d} du f(u)(u - y_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \tag{2.6}$$

The Wiener-Hermite integral satisfies the following isometry

$$\mathbf{E}\left(\int_{\mathbb{R}^d} f(s) dZ^{(q)}_{\mathbf{H},d}(s) \int_{\mathbb{R}^d} g(s) dZ^{(q)}_{\mathbf{H},d}(s)\right) = \langle f, g\rangle_{\mathcal{H}(\mathbb{R}^d)} \tag{2.7}$$

for every $f, g \in \left|\mathcal{H}(\mathbb{R}^d)\right|$, where we used the notation

$$\langle f, g\rangle_{\mathcal{H}(\mathbb{R}^d)} = \alpha_{\mathbf{H}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u) g(v) |u - v|^{\mathbf{2H-2}} du dv$$

with $\alpha_{\mathbf{H}} = \mathbf{H}(\mathbf{2H-1}) = \prod_{j=1}^d \left(H_j(2H_j - 1)\right)$.

3 Fractional Heat Equation with Hermite Noise

Consider the stochastic partial differential equation

$$\frac{\partial}{\partial t} u(t, x) = -(-\Delta)^{\frac{\alpha}{2}} u(t, x) + \dot{Z}^{(q)}_{(H_0,\mathbf{H}),d+1}(t, x), \qquad t \geq 0, x \in \mathbb{R}^d. \tag{3.1}$$

with vanishing initial condition $u(0, x) = 0$ for every $x \in \mathbb{R}^d$. In (3.1), $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with exponent $\frac{\alpha}{2}$, $\alpha \in (0, 2]$ and $\dot{Z}^{(q)}_{(H_0,\mathbf{H}),d+1}$ denotes the formal derivative of the $d+1$-parameter Hermite sheet of order $q \geq 1$ with Hurst parameter $(H_0, \mathbf{H}) \in \left(\frac{1}{2}, 1\right)^{d+1}$ where $\mathbf{H} = (H_1, \dots, H_d) \in \left(\frac{1}{2}, 1\right)^d$.

The *mild* solution can be expressed as

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} G_\alpha(t - s, x - y) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds, dy) \tag{3.2}$$

where G_α is the Green kernel associated to the fractional Laplacian, i.e. it satisfies

$$\frac{\partial u}{\partial t}(t, x) = -(-\Delta)^{\frac{\alpha}{2}} u(t, x)$$

and $Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds, dy)$ means the Wiener integral with respect to the Hermite sheet, see Section 2.2. We refer to [8], [9] for the precise definition and other properties of the fractional Laplacian operator. The simplest way to define the Green kernel G_α is via its Fourier transform

$$\mathcal{F}G_\alpha(t, \cdot)(\xi) = e^{-t|\xi|^\alpha} \tag{3.3}$$

i.e. for $t \geq 0, x \in \mathbb{R}^d$,

$$G_\alpha(t, x) = \int_{\mathbb{R}^d} e^{-\langle x, \xi\rangle} e^{-t|\xi|^\alpha} d\xi. \tag{3.4}$$

Due to the formula (2.5), we can write the random variable $u(t, x)$ given by (3.2) as a multiple stochastic integral in the qth Wiener chaos, i.e. for every $t \geq 0, x \in \mathbb{R}^d$,

$$u(t, x) = I_q(H_{t,x})$$

with

$$H_{t,x}\left((s_1,z_1),\ldots,(s_q,z_q)\right) \tag{3.5}$$

$$= c(\mathbf{H},q)\int_0^t du\int_{\mathbb{R}^d} dyG_\alpha(t-u,x-y)(u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)}\ldots(u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)}$$
$$(y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}\ldots(y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \tag{3.6}$$

for every $((s_1,z_1),\ldots,(s_q,z_q))\in\left(\mathbb{R}_+\times\mathbb{R}^d\right)^q$.

We say that the mild solution to the fractional heat equation (3.1) exists if the Wiener-Hermite integral in (3.2) is well-defined. In the next paragraph, we give a necessary and sufficient condition for the existence of the mild solution (3.2).

A basic tool for our calculation is the Parseval identity

$$\alpha_{\mathbf{H}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\psi(y)|x-y|^{2\mathbf{H}-2}dydx = a_{\mathbf{H}}\int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi}(\xi)|\xi|^{\mathbf{1}-2\mathbf{H}}d\xi \tag{3.7}$$

with some positive constant $a_{\mathbf{H}}$, for every φ,ψ such that

$$\int_{\mathbb{R}^d}|\varphi(x)\psi(y)||x-y|^{2\mathbf{H}-2}dydx<\infty.$$

3.1 Existence of the Solution

Let us give the condition for the existence of the mild solution (3.2). We have the following result.

Proposition 3.1. *Let $\left(u(t,x),t\geq 0,x\in\mathbb{R}^d\right)$ be given by (3.2). For every $t\geq 0,x\in\mathbb{R}^d$, the random variable $u(t,x)$ belongs to $L^2(\Omega)$ if and only if*

$$d<\alpha H_0+H_1+\ldots+H_d. \tag{3.8}$$

Moreover, under (3.8), for every $T>0,p\geq 2$,

$$\sup_{t\in[0,T],x\in\mathbb{R}^d}\mathbf{E}|u(t,x)|^p\leq C_{p,T} \tag{3.9}$$

where $C_{p,T}>0$ is a constant that depends only on T,p.

Proof. For $t\geq 0,x\in\mathbb{R}^d$, we have by the Wiener-Hermite isometry (2.7)

$$\mathbf{E}u(t,x)^2 = \alpha_{H_0}\alpha_{\mathbf{H}}\int_0^t\int_0^t dudv|u-v|^{2H_0-2}$$
$$\times\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}dydzG_\alpha(t-u,x-y)G_\alpha(t-v,x-z)|y-z|^{2\mathbf{H}-2}$$

where we recall that (see convention (2.1))

$$|y-z|^{2\mathbf{H}-2}=\prod_{j=1}^d|y_j-z_j|v^{2H_j-2}\text{ if }y=(y_1,..,y_d),z=(z_1,...,z_d)\in\mathbb{R}^d.$$

By using Parseval's identity (3.7) and the expression of the Fourier transform of the Green kernel (3.3)

$$\begin{aligned}\mathbf{E}u(t,x)^2 &= C\int_0^t\int_0^t dudv|u-v|^{2H_0-2}\\ &\int_{\mathbb{R}^d} d\xi \mathcal{F}G_\alpha(t-u,x-\cdot)(\xi)\overline{\mathcal{F}G_\alpha(t-v,x-\cdot)}(\xi)|\xi|^{\mathbf{1}-2\mathbf{H}}\\ &= C\int_0^t\int_0^t dudv|u-v|^{2H_0-2}\int_{\mathbb{R}^d} d\xi e^{-u|\xi|^\alpha}e^{-v|\xi|^\alpha}|\xi|^{\mathbf{1}-2\mathbf{H}}\\ &= C\int_0^t\int_0^t dudv|u-v|^{2H_0-2}\int_{\mathbb{R}^d} d\xi e^{-(u+v)|\xi|^\alpha}\prod_{j=1}^d|\xi_i|^{1-2H_j}\end{aligned}$$

where $C>0$ may change from line to line. Now we use the change of variables $\tilde{\xi_i}=(u+v)^{\frac{1}{\alpha}}\xi_i$ for $i=1,...,d$ and we obtain

$$\begin{aligned}&\mathbf{E}u(t,x)^2\\ &= C\int_0^t\int_0^t dudv|u-v|^{2H_0-2}(u+v)^{-\frac{d}{\alpha}}(u+v)^{\frac{1}{\alpha}(2(H_1+\ldots H_d)-d)}\int_{\mathbb{R}^d} d\xi e^{-|\xi|^\alpha}|\xi|^{\mathbf{1}-2\mathbf{H}}\\ &= C\int_0^t\int_0^t dudv|u-v|^{2H_0-2}(u+v)^{-\frac{d}{\alpha}}(u+v)^{-\frac{1}{\alpha}(\sum_{j=1}^d(1-2H_j))}\\ &= C\int_0^t du\int_0^u dv|u-v|^{2H_0-2}(u+v)^{-\frac{d}{\alpha}}(u+v)^{\frac{1}{\alpha}(2(H_1+\ldots H_d)-d)}.\end{aligned}$$

where we used the fact that $\int_{\mathbb{R}^d} d\xi e^{-|\xi|^\alpha}|\xi|^{\mathbf{1}-2\mathbf{H}}<\infty$ for $\mathbf{H}\in\left(\frac{1}{2},1\right)^d$. By the change of variables $\frac{v}{u}=z$, we can write

$$\begin{aligned}\mathbf{E}u(t,x)^2 &= C\int_0^1 duu^{2H_0-1}u^{-2\frac{d}{\alpha}+\frac{2}{\alpha}(H_1+\ldots H_d)}\int_0^1 dz(1-z)^{2H_0-2}(1+z)^{-2\frac{d}{\alpha}+\frac{2}{\alpha}(H_1+\ldots H_d)}\\ &= C\int_0^t duu^{2H_0-1}u^{-2\frac{d}{\alpha}+\frac{2}{\alpha}(H_1+\ldots H_d)} \qquad (3.10)\end{aligned}$$

because the integral $\int_0^1 dz(1-z)^{2H_0-2}(1+z)^{-2\frac{d}{\alpha}+\frac{2}{\alpha}(H_1+\ldots H_d)}$ is finite for $H_0>\frac{1}{2}$. The integral du in (3.10) is finite if and only if

$$2H_0-2\frac{d}{\alpha}+\frac{2}{\alpha}(H_1+\ldots H_d)>0$$

or

$$d<\alpha H_0+(H_1+\ldots+H_d).$$

If (3.8) holds, then by (3.10) we find

$$\mathbf{E}|u(t,x)|^2=Ct^{2H_0-2\frac{d}{\alpha}+\frac{2}{\alpha}(H_1+\ldots H_d)}\le CT^{2H_0-2\frac{d}{\alpha}+\frac{2}{\alpha}(H_1+\ldots H_d)}:=C_T<\infty$$

and then (3.9) follows due to the hypercontractivity property (5.3), because the random variable $u(t,x)$ belongs to the qth Wiener chaos. □

Remark 3.2.
1. For $\alpha=2$, we retrieve the condition from [19], namely

$$d<2H_0+H_1+\ldots+H_d.$$

2. When $\alpha>1$ and $H_0,H_1,...,H_d$ are all very close to $\frac{1}{2}$, then (3.8) becomes $d<\frac{\alpha}{2}+\frac{d}{2}$ which means $d<\alpha$ so we can consider only the situation when the spatial dimension is $d=1$. If all the Hurst indices are very close to 1, then (3.8) always holds true and then we can consider any dimension in space.

3. If $\alpha\in(0,1)$ is small enough then the solution may not exist even in dimension $d=1$.

3.2 Self-Similarity

In the sequel, we assume (3.8). The next step is to analyze the scaling property of the process $(u(t,x), t \geq 0)$ when the spatial variable $x \in \mathbb{R}^d$ is fixed. We will prove the following self-similarity property.

Proposition 3.3. *The process $(u(t,x), t \geq 0)$ given by (3.2) is self-similar of order*

$$\gamma = H_0 - \frac{d}{\alpha} + \frac{\sum_{i=1}^d H_i}{\alpha} \tag{3.11}$$

Proof. Using the definition of the Wiener-Hermite integral (see (2.5) and (2.6)),

$$\begin{aligned} u(t,x) &= c(\mathbf{H},q) \int_{\mathbb{R}} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}} \int_{\mathbb{R}^d} W(ds_1, z_1) \dots W(ds_q, z_q) \\ & \int_0^t du \int_{\mathbb{R}^d} dy G_\alpha(t-u, x-y)(u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \dots (u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \end{aligned}$$

Let $a > 0$. Then

$$\begin{aligned} u(at,x) &= c(\mathbf{H},q) \int_{\mathbb{R}} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}} \int_{\mathbb{R}^d} W(ds_1, z_1) \dots W(ds_q, z_q) \\ & \int_0^{at} du \int_{\mathbb{R}^d} dy G_\alpha(at-u, x-y)(u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \dots (u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \\ &= c(\mathbf{H},q) a \int_{\mathbb{R}} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}} \int_{\mathbb{R}^d} W(ds_1, z_1) \dots W(ds_q, z_q) \\ & \int_0^t du \int_{\mathbb{R}^d} dy G_\alpha(a(t-u), x-y)(au-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \dots (au-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (y-z_q)^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \\ &= c(\mathbf{H},q) a \int_{\mathbb{R}} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}} \int_{\mathbb{R}^d} W(d(as_1), z_1) \dots W(d(as_q), z_q) \\ & \int_0^t du \int_{\mathbb{R}^d} dy G_\alpha(a(t-u), x-y)(au-as_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \dots (au-as_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \end{aligned}$$

Using the fact that the Brownian sheet W is $\frac{1}{2}$-self-similar with respect to its time variable and its increments are stationary in space, we get (we denote everywhere by $\equiv^{(d)}$ the equivalence of the finite dimensional distributions),

$$\begin{aligned} u(at,x) &\equiv^{(d)} c(\mathbf{H},q) a a^{-q\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} a^{\frac{q}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}} \int_{\mathbb{R}^d} W(ds_1, z_1) \dots W(ds_q, z_q) \\ & \int_0^t du \int_{\mathbb{R}^d} dy G_\alpha(a(t-u), -y)(u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \dots (u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \end{aligned}$$

We use the following property of the Green kernel (see e.g. [6])

$$G_\alpha(at, x) = a^{-\frac{d}{\alpha}} G_\alpha(t, a^{-\frac{1}{\alpha}} x).$$

Then

$$\begin{aligned} u(at,x) \quad \equiv^{(d)} \quad & c(\mathbf{H},q)a^{H_0}a^{-\frac{d}{\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}}\int_{\mathbb{R}^d}W(ds_1,z_1)\dots W(ds_q,z_q) \\ & \int_0^t du\int_{\mathbb{R}^d}dyG_\alpha(t-u,-a^{-\frac{1}{\alpha}}y)(u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)}\dots(u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}\dots(y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \end{aligned}$$

Next, we make the change of variables $a^{-\frac{1}{\alpha}}y_i=\tilde{y}_i$ for $i=1,...,d$ and then $\tilde{z}_i=a^{-\frac{1}{\alpha}}z_i$ for $i=1,...,d$. We will obtain

$$\begin{aligned} u(at,x) \quad \equiv^{(d)} \quad & c(\mathbf{H},q)a^{H_0}a^{-\frac{d}{\alpha}}a^{\frac{d}{\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}}\int_{\mathbb{R}^d}W(ds_1,d(a^{\frac{1}{\alpha}}z_1))\dots W(ds_q,d(a^{\frac{1}{\alpha}}z_q)) \\ & \int_0^t du\int_{\mathbb{R}^d}dyG_\alpha(t-u,-y)(u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)}\dots(u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (a^{\frac{1}{\alpha}}(y-z_1))_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}\dots(a^{\frac{1}{\alpha}}(y-z_q))_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \\ = \quad & c(\mathbf{H},q)a^{H_0}a^{-\frac{q}{\alpha}\sum_{i=1}^d\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} \\ & \int_{\mathbb{R}}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}}\int_{\mathbb{R}^d}W(ds_1,d(a^{\frac{1}{\alpha}}z_1))\dots W(ds_q,d(a^{\frac{1}{\alpha}}z_q)) \\ & \int_0^t du\int_{\mathbb{R}^d}dyG_\alpha(t-u,-y)(u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)}\dots(u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}\dots(y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \end{aligned}$$

Finally, we use the spatial scaling property of the Brownian sheet W to get

$$\begin{aligned} u(at,x) \quad \equiv^{(d)} \quad & c(\mathbf{H},q)a^{H_0}a^{-\frac{q}{\alpha}\sum_{i=1}^d\left(\frac{1}{2}+\frac{1-H_i}{q}\right)}a^{\frac{dq}{2\alpha}} \\ & \int_{\mathbb{R}}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}}\int_{\mathbb{R}^d}W(ds_1,z_1)\dots W(ds_q,z_q) \\ & \int_0^t du\int_{\mathbb{R}^d}dyG_\alpha(t-u,-y)(u-s_1)^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)}\dots(u-s_q)^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}\dots(y-z_q)^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \end{aligned}$$

so for $x\in\mathbb{R}$ fixed,

$$(u(at,x),t\geq 0)\equiv^{(d)}\left(a^{H_0-\frac{d}{\alpha}+\frac{\sum_{i=1}^d H_i}{\alpha}}u(t,x),t\geq 0\right)=(a^\gamma u(t,x),t\geq 0)$$

with γ in (3.11). □

Remark 3.4. Notice that the self-similarity index γ given by (3.11) is strictly positive, due to condition (3.8). Also, we have $\gamma<H_0<1$ since

$$\gamma=H_0-\frac{d-\sum_{i=1}^d H_i}{\alpha}\leq H_0<1.$$

For $\alpha=2$, we found the scaling index from [19].

3.3 Regularity of the Sample Paths

Here, we analyze the regularity of the sample paths $t\to u(t,x)$. Let us start by estimating the $L^2(\Omega)$-norm of the increment $u(t,x)-u(s,x)$ with $0\leq s<t$ and $x\in\mathbb{R}^d$.

Proposition 3.5. *Let u be defined by (3.2) and assume (3.8). Then for every $0 \le s < t$ and $x \in \mathbb{R}^d$, we have*

$$\mathbf{E}\,|u(t,x) - u(s,x)|^2 \le C|t-s|^{2\gamma} \tag{3.12}$$

with γ given by (3.11) and $C > 0$ independent of s, t, x.

Proof. For $0 \le s < t$, we can write

$$\begin{aligned} u(t,x) - u(s,x) &= \int_s^t \int_{\mathbb{R}^d} G_\alpha(t-v, x-y) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(dv,dy) \\ &\quad + \int_0^s \int_{\mathbb{R}^d} \left(G_\alpha(t-v,x-y) - G_\alpha(s-v,x-y)\right) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(dv,dy) \\ &= T_1(s,t) + T_2(s,t) \end{aligned}$$

so

$$\mathbf{E}\,|u(t,x) - u(s,x)|^2 \le 2\left(\mathbf{E}|T_1(s,t)|^2 + \mathbf{E}|T_2(s,t)|^2\right).$$

Now, for $0 \le s < t$,

$$\begin{aligned} & E|T_1(s,t)^2| \\ =\ & C \int_s^t \int_s^t dudv |u-v|^{2H_0-2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dydx G_\alpha(t-u,x-y) G_\alpha(t-v,x-z) |y-z|^{\mathbf{2H-2}} \\ =\ & C \int_s^t \int_s^t dudv |u-v|^{2H_0-2} \int_{\mathbb{R}^d} d\xi e^{-(t-u)|\xi|^\alpha} e^{-(t-v)|\xi|^\alpha} |\xi|^{\mathbf{1-2H}} \\ =\ & C \int_0^{t-s} \int_0^{t-s} dudv |u-v|^{2H_0-2} \int_{\mathbb{R}^d} d\xi e^{-(u+v)|\xi|^\alpha} |\xi|^{\mathbf{1-2H}}. \end{aligned}$$

By the change of variables $(u+v)^{\frac{1}{\alpha}} \xi_i = \tilde{\xi}_i$ for $i = 1, ..., d$,

$$\begin{aligned} & \mathbf{E}|T_1(s,t)^2| \\ =\ & C \int_0^{t-s} \int_0^{t-s} dudv |u-v|^{2H_0-2} (u+v)^{-\frac{d}{\alpha}} (u+v)^{-\frac{1}{\alpha}\sum_{i=1}^d (1-2H_i)} \int_{\mathbb{R}^d} d\xi e^{-|\xi|^\alpha} |\xi|^{\mathbf{1-2H}} \\ =\ & C \int_0^{t-s} du \times u^{2H_0-1} u^{-\frac{d}{\alpha}} u^{-\frac{1}{\alpha}\sum_{i=1}^d (1-2H_i)} = C \int_0^{t-s} du u^{2\gamma-1} \\ =\ & C|t-s|^{2\gamma}. \end{aligned}$$

Concerning $T_2(s,t)$, we have

$$\begin{aligned} \mathbf{E}|T_2(s,t)|^2 &= C \int_0^s \int_0^s dudv |u-v|^{2H_0-2} \int_{\mathbb{R}^d} d\xi |\xi|^{\mathbf{1-2H}} \\ &\quad \left(e^{-(t-u)|\xi|^\alpha} - e^{-(s-u)|\xi|^\alpha}\right)\left(e^{-(t-v)|\xi|^\alpha} - e^{-(s-v)|\xi|^\alpha}\right) \\ &= C|t-s|^{2H_0} \int_0^{\frac{s}{t-s}} \int_0^{\frac{s}{t-s}} dudv |u-v|^{2H_0-2} \int_{\mathbb{R}^d} d\xi |\xi|^{\mathbf{1-2H}} \\ &\quad \left(e^{-(t-s)(1+u)|\xi|^\alpha} - e^{-(t-s)u|\xi|^\alpha}\right)\left(e^{-(t-s)(1+v)|\xi|^\alpha} - e^{-(t-s)v|\xi|^\alpha}\right) \end{aligned}$$

where we performed the chage of variables $\tilde{u} = \frac{s-u}{t-s}, \tilde{v} = \frac{s-v}{t-s}$. in the next step, we set $\tilde{\xi}_i = (t-s)^{\frac{1}{\alpha}} \xi_i$ for every $i = 1, ..., d$. In this way,

$$
\begin{aligned}
& \mathbf{E}|T_2(s,t)|^2 \\
= \; & C|t-s|^{2H_0}(t-s)^{-\frac{d}{\alpha}+\sum_{i=1}^{d}(1-2H_i)} \int_0^{\frac{s}{t-s}} \int_0^{\frac{s}{t-s}} dudv|u-v|^{2H_0-2} \int_{\mathbb{R}^d} d\xi |\xi|^{\mathbf{1-2H}} \\
& \left(e^{-(2+u+v)|\xi|^\alpha} - 2e^{-(1+u+v)|\xi|^\alpha} + e^{-(u+v)|\xi|^\alpha} - 2\right) \\
\leq \; & C(t-s)^{2\gamma} \int_0^\infty \int_0^\infty dudv|u-v|^{2H_0-2} \int_{\mathbb{R}^d} d\xi |\xi|^{\mathbf{1-2H}} \\
& \left(e^{-(2+u+v)|\xi|^\alpha} - 2e^{-(1+u+v)|\xi|^\alpha} + e^{-(u+v)|\xi|^\alpha}\right) \\
= \; & C(t-s)^{2\gamma} \int_{\mathbb{R}^d} d\xi |\xi|^{\mathbf{1-2H}} e^{-|\xi|^\alpha} \int_0^\infty \int_0^\infty dudv|u-v|^{2H_0-2} \\
& \left[(2+u+v)^{-\frac{d}{\alpha}-\frac{1}{\alpha}\sum_{i=1}^{d}(1-2H_i)} - 2(1+u+v)^{-\frac{d}{\alpha}-\frac{1}{\alpha}\sum_{i=1}^{d}(1-2H_i)} + (u+v)^{-\frac{d}{\alpha}-\frac{1}{\alpha}\sum_{i=1}^{d}(1-2H_i)}\right] \\
= \; & C(t-s)^{2\gamma} \int_0^\infty \int_0^\infty dudv|u-v|^{2H_0-2} \\
& \left[(2+u+v)^{2\gamma-2H_0} - 2(1+u+v)^{2\gamma-2H_0} + (u+v)^{2\gamma-2H_0}\right].
\end{aligned}
$$

For u, v close to infinity, we have

$$
(2+u+v)^{2\gamma-2H_0} - 2(1+u+v)^{2\gamma-2H_0} + (u+v)^{2\gamma-2H_0} \leq C|u+v|^{2\gamma-2H_0-2}
$$

and the above integral $dudv$ is convergent at infinity because $2\gamma < 2$ or equivalently

$$
H_0 - \frac{1}{2\alpha}\left(d - \sum_{i=1}^{d} H_i\right) \leq 1.
$$

□

As an immediate consequence of Proposition 3.5, we obtain the Hölder regularity in time of the solution.

Corollary 3.6. *The mapping $t \to u(t,x)$ is Hölder continuous of order δ for every $\delta \in (0,\gamma)$, with γ given by (3.11).*

Proof. From (3.12) and by using the hypercontractivity (5.3), we have for $p \geq 2$,

$$
\mathbf{E}\,|u(t,x) - u(s,x)|^p \leq C_p|t-s|^{p\gamma}
$$

for every $0 \leq s < t$ and $x \in \mathbb{R}^d$. The conclusion follows by Kolmogorov's continuity criterion. □

3.4 A Decomposition of the Solution

Consider the random field $(u(t,x), t \geq 0, x \in \mathbb{R}^d)$ which is mild solution to (3.1) and assume (3.8) is satisfied. We have seen in Section 3.2 that the the random field u is self-similar in time and it is pretty obvious that it has no stationary increments with respect to the time variable. The purpose is to give a decomposition of the solution to the fractional stochastic heat equation (3.1) as a sum of a self-similar process in time with temporal stationary increments and of another process with very nice sample paths in time. This decomposition is useful, among others, to get the behavior of the $p-$ variation of the solution.

We will use the so-called *pinned string method* introduced in [12] and then used by several authors (in e.g. [23], [10], [13], [19]). Let us set, for $t \geq 0, x \in \mathbb{R}^d$,

$$
U(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(G_\alpha((t-s)_+, x-y) - G_\alpha((-s)_+, x-y)\right) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy). \tag{3.13}
$$

This is called the *pinned string process.* We can also write

$$\begin{aligned} U(t,x) &= \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy) \\ &\quad + \int_{-\infty}^0 \int_{\mathbb{R}} \left(G_\alpha(t-s,x-y) - G_\alpha(-s,x-y)\right) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy). \end{aligned}$$

We will first show that U has the scaling property in time and it also has stationary temporal increments. Then we will show that the difference $u(t,x) - U(t,x)$ has C^∞ -sample paths with respect to the time variable.

Proposition 3.7. *Let $x \in \mathbb{R}^d$ be fixed. Then the process $(U(t,x), t \geq 0)$ defined by (3.13) is γ-self-similar and it has stationary increments.*

Proof. The self-similarity follows as above in the proof of Proposition 3.3. Let us show that for every $h > 0$, the stochastic processes

$$(U(t+h,x) - U(h,x), t \geq 0) \text{ and } (U(t,x)), t \geq 0)$$

have the same finite dimensional distributions. We can write, for $h > 0$,

$$\begin{aligned} & U(t+h,x) - U(h,x) \\ = & \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(G_\alpha((t+h-s)_+, x-y) - G_\alpha((h-s)_+, x-y)\right) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy) \\ \equiv^{(d)} & \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(G_\alpha((t-s)_+, x-y) - G_\alpha((-s)_+, x-y)\right) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy) \\ = & U(t,x) \end{aligned}$$

where we used the fact that the process $(Z^{(q)}_{(H_0,\mathbf{H}),d+1}(t,x), t \geq 0)$ has stationary increments in time. □

From the above result, it follows (see e.g. [22]) that the covariance of the process $(U(t,x), t \geq 0)$ is given by

$$\mathbf{E}U(t,x)U(s,x) = \frac{\mathbf{E}U(1,x)^2}{2} \left(t^{2\gamma} + s^{2\gamma} - |t-s|^{2\gamma}\right), \quad t,s \geq 0.$$

On the other hand, since $(U(t,x), t \geq 0)$ is not a Gaussian process, the covariance did not determine the probability distribution of this stochastic process.

Set, for $t > 0$ and $x \in \mathbb{R}^d$,

$$Y(t,x) = u(t,x) - U(t,x)$$

so

$$Y(t,x) = -\int_{-\infty}^0 \int_{\mathbb{R}^d} \left(G_\alpha(t-s,x-y) - G_\alpha(-s,x-y)\right) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy). \tag{3.14}$$

We will show that the random field Y has smooth sample paths with respect to the time variable.

Proposition 3.8. *Let $(Y(t,x), t > 0, x \in \mathbb{R}^d)$ be given by (3.14)and assume (3.8). Then for every $x \in \mathbb{R}^d$, the sample paths $t \to Y(t,x)$ are absolutely continuous and of class C^∞ on $(0,\infty)$.*

Proof. Set

$$Y'(t,x) = -\int_{-\infty}^0 \int_{\mathbb{R}^d} \frac{\partial}{\partial t} G_\alpha(t-s,x-y) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy), \quad t > 0, x \in \mathbb{R}^d, \tag{3.15}$$

the formal derivative of Y with respect to the time variable t. We will have (again $C>0$ denotes a generic constant) via (3.7)

$$\begin{aligned}\mathbf{E}|Y'(t,x)|^2 &= C\int_{-\infty}^0\int_{-\infty}^0 dudv|u-v|^{2H_0-2}\int_{\mathbb{R}^d} d\xi|\xi|^{\mathbf{1-2H}}\\ &\left(\frac{\partial}{\partial t}e^{-(t-u)|\xi|^\alpha}\right)\left(\frac{\partial}{\partial t}e^{-(t-v)|\xi|^\alpha}\right)\\ &= C\int_{-\infty}^0\int_{-\infty}^0 dudv|u-v|^{2H_0-2}\int_{\mathbb{R}^d} d\xi|\xi|^{1-2H+2\alpha}\\ &\times e^{-(t-u)|\xi|^\alpha}e^{-(t-v)|\xi|^\alpha}\\ &= C\int_t^\infty\int_t^\infty dudv|u-v|^{2H_0-2}\int_{\mathbb{R}^d} d\xi|\xi|^{1-2H+2\alpha}e^{-(u+v)|\xi|^\alpha}\end{aligned}$$

and by setting $(u+v)^{\frac{1}{\alpha}}\xi_i=\tilde{\xi}_i$ for $i=1,...,d$, we get

$$\begin{aligned}\mathbf{E}|Y'(t,x)|^2 &= C\int_t^\infty\int_t^\infty dudv|u-v|^{2H_0-2}(u+v)^{-\frac{d}{\alpha}}(u+v)^{-\frac{1}{\alpha}\sum_{i=1}^d(1+2\alpha-2H_i)}\\ &\int_{\mathbb{R}^d} d\xi|\xi|^{1-2H+2\alpha}e^{-|\xi|^\alpha}\\ &= C\int_t^\infty\int_t^\infty dudv|u-v|^{2H_0-2}(u+v)^{-\frac{d}{\alpha}}(u+v)^{-\frac{1}{\alpha}\sum_{i=1}^d(1+2\alpha-2H_i)}.\end{aligned}$$

Thus, with $z=\frac{u}{v}$

$$\begin{aligned}&\mathbf{E}|Y'(t,x)|^2\\ =\ & C\int_t^\infty dv\int_v^\infty du(u-v)^{2H_0-2}(u+v)^{-\frac{2d}{\alpha}-2d+\frac{2}{\alpha}\sum_{i=1}^d H_i}\\ =\ & C\int_t^\infty dvv^{2H_0-1-\frac{2d}{\alpha}-2d+\frac{2}{\alpha}\sum_{i=1}^d H_i}\int_1^\infty dz(z-1)^{2H_0-1}(1+z)^{-\frac{2d}{\alpha}-2d+\frac{2}{\alpha}\sum_{i=1}^d H_i}\\ =\ & C\int_t^\infty dvv^{2\gamma-1-2d}\int_1^\infty(1-z)^{2H_0-2}(1+z)^{2\gamma-2H_0-2d}.\end{aligned}$$

The integral dv is finite because $2\gamma-2d<0$ (see Remark 3.4) and the integral dx is finite at 1 because $2H_0>1$ and at infinity because $2\gamma-2d-1<0$ (again by Remark 3.4). Therefore $(Y'(t,x),t>0)$ is a well defined random field and consequently $t\to Y(t,x)$ is absolute continuous and of class C^1 on $(0,\infty)$. Similarly (see [13], [19] or [23] for details), we can deal with the nth derivative and we can show that $t\to Y(t,x)$ is of class C^∞ on $(0,\infty)$. □

3.5 p-Variation

We will use the above decomposition result in order to obtain the p-variation in time of the solution u. Let us first define the concept of p-variation. Consider $0\le A_1<A_2$ two real numbers and let

$$t_i=A_1+\frac{i}{N}(A_2-A_1),\quad i=0,...,N \tag{3.16}$$

be a partition of the interval $[A_1,A_2]$. Let $(v(t,x),t\ge 0,x\in\mathbb{R}^d)$ a general random field and define, for $x\in\mathbb{R}^d$, $p>0$ and $N\ge 1$

$$S^{N,p}_{[A_1,A_2]}(v(\cdot,x))=\sum_{i=0}^{N-1}|v(t_{i+1},x)-v(t_i,x)|^p. \tag{3.17}$$

We will say that v admits a temporal p-variation over the interval $[A_1,A_2]$ if the sequence $\left(S^{N,p}_{[A_1,A_2]}(v(\cdot,x)),N\ge 1\right)$ converges in probability as $N\to\infty$.

For the solution to the fractional stochastic heat equation, we have the following result. Recall that γ is given by (3.11).

Theorem 3.9. *Let $(u(t,x), t \geq 0, x \in \mathbb{R}^d)$ be defined by (3.2) and assume (3.8). Then*

$$S^{N,\frac{1}{\gamma}}_{[A_1,A_2]}(u(\cdot,x)) \to_{N\to\infty} \mathbf{E}\left|U(1,0)\right|^{\frac{1}{\gamma}}(A_2 - A_1) \text{ in probability}$$

where U is given by (3.13).

Proof. In a first step, we will show that the p-variation of the random field Y given by (3.14) vanishes, for every $p \geq \frac{1}{\gamma}$. Indeed, for $p \geq \frac{1}{\gamma} > 1$

$$\sum_{i=0}^{N-1} |Y(t_{i+1},x) - Y(t_i,x)|^p \leq \sup_{|a-b|\leq \frac{A_2-A_1}{N}} |Y(a,x) - Y(b,x)|^{p-1} \sum_{i=0}^{N-1} |Y(t_{i+1},x) - Y(t_i,x)|.$$

Recall from Proposition 3.8 that Y has absolute continuous temporal sample paths. The continuity of Y with respect to the time variable and $p > 1$ implies that

$$\sup_{|a-b|\leq \frac{A_2-A_1}{N}} |Y(a,x) - Y(b,x)|^{p-1} \to_{N\to\infty} 0$$

pointwise while the quantity $\sum_{i=0}^{N-1} |Y(t_{i+1},x) - Y(t_i,x)|$ is bounded by the total variation of $t \to Y(t,x)$ over the interval $[A_1, A_2]$. Consequently,

$$\sum_{i=0}^{N-1} |Y(t_{i+1},x) - Y(t_i,x)|^p \to_{N\to\infty} 0 \text{ pointwise.} \tag{3.18}$$

Let us now analyze the $\frac{1}{\gamma}$-variation of the random field U. We have, by the scaling propery in time of U obtained in Proposition 3.7, if " $=^{(d)}$ " stands for the equality in distribution,

$$\begin{aligned}\sum_{i=0}^{N-1} |U(t_{i+1},x) - U(t_i,x)|^p &=^{(d)} (A_2 - A_1)^{\gamma p} \frac{1}{n^{\gamma p}} \sum_{i=0}^{N-1} |U(i+1,x) - U(i,x)|^p \\ &= (A_2 - A_1)^{\gamma p} \frac{1}{n^{\gamma p - 1}} V_N \end{aligned} \tag{3.19}$$

with

$$V_N = \frac{1}{N} \sum_{i=0}^{N-1} |U(i+1,x) - U(i,x)|^p.$$

The sequence $(U(i+1,x) - U(i,x), i \geq 0)$ is stationary due to the fact that $(U(t,x), t \geq 0)$ has stationary increments, see Proposition 3.7. On the other hand, $U(t,x)$ is an element of the qth Wiener chaos, for every $t \geq 0, x \in \mathbb{R}^d$. Actually, we have

$$U(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} G_{t,x}\left((s_1,z_1), \ldots, (s_q,z_q)\right) W(ds_1, dz_1) \ldots W(ds_q, dz_q)$$

where

$$\begin{aligned} & G_{t,x}\left((s_1,z_1), \ldots (s_q,z_q)\right) \\ = \; & c(\mathbf{H},q) \int_{\mathbb{R}} du \int_{\mathbb{R}^d} dy \left(G_\alpha((t-u)_+, x-y) - G_\alpha((-u)_+, x-y)\right) \\ & (u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \ldots (u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \ldots (y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}. \end{aligned}$$

Moreover,

$$\begin{aligned} & G_{t,x}\left((s_1,z_1),\dots(s_q,z_q)\right) - G_{s,x}\left((s_1,z_1),\dots(s_q,z_q)\right) \\ = \; & c(\mathbf{H},q)\int_{\mathbb{R}} du \int_{\mathbb{R}^d} dy\, (G_\alpha((t-u)_+,x-y) - G_\alpha((s-u)_+,x-y)) \\ & (u-s_1)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \dots (u-s_q)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \\ = \; & c(\mathbf{H},q)\int_{\mathbb{R}} du \int_{\mathbb{R}^d} dy\, (G_\alpha((t-s-u)_+,x-y) - G_\alpha((-u)_+,x-y)) \\ & \times (u-(s_1-s))_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \dots (u-(s_q-s)_+^{-\left(\frac{1}{2}+\frac{1-H_0}{q}\right)} \\ & (y-z_1)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \dots (y-z_q)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \end{aligned}$$

so

$$\begin{aligned} & G_{t,x}\left((s_1,z_1),\dots(s_q,z_q)\right) - G_{s,x}\left((s_1,z_1),\dots(s_q,z_q)\right) \\ = \; & G_{t-s,x}\left((s_1-s,z_1),\dots(s_q-s,z_q)\right). \end{aligned} \tag{3.20}$$

Since the kernel $G_{t,x}$ satisfies the shifting property (3.20), it follows from Theorem 8.3.1 in [18] that the sequence $(U(i+1,x)-U(i,x), i \geq 0)$ is also mixing. Theorefore (see e.g. Chapter 2 in [18])

$$V_N \to_{N\to\infty} \mathbf{E}\, |U(1,0)|^p \text{ almost surely and in } L^1(\Omega). \tag{3.21}$$

By (3.19) and (3.21) we obtain (since the convergence in law to a constant implies the convergence in probability)

$$\sum_{i=0}^{N-1} |U(t_{i+1},x)-U(t_i,x)|^p \to_{N\to\infty} \begin{cases} 0, \text{ if } p > \frac{1}{\gamma} \\ \mathbf{E}|U(1,0)|^{\frac{1}{\gamma}}(A_2-A_1) \text{ if } p = \frac{1}{\gamma} \\ +\infty \text{ if } p < \frac{1}{\gamma} \end{cases} \tag{3.22}$$

in probability. Now, by using Minkovski's inequality

$$\begin{aligned} & \left(\sum_{i=0}^{N-1} |U(t_{i+1},x)-U(t_i,x)|^p\right)^{\frac{1}{p}} - \left(\sum_{i=0}^{N-1} |Y(t_{i+1},x)-Y(t_i,x)|^p\right)^{\frac{1}{p}} \leq S^{N,p}_{[A_1,A_2]}(u(\cdot,x))) \\ \leq \; & \left(\sum_{i=0}^{N-1} |U(t_{i+1},x)-U(t_i,x)|^p\right)^{\frac{1}{p}} + \left(\sum_{i=0}^{N-1} |Y(t_{i+1},x)-Y(t_i,x)|^p\right)^{\frac{1}{p}}. \end{aligned}$$

To get the conclusion, it suffices to use (3.18) and (3.22). □

Remark 3.10. From the proof of Theorem 3.9, we notice that the solution (3.2) has zero p variation in time on any interval $[A_1, A_2]$ if $p > \frac{1}{\gamma}$.

4 Drift Parameter Estimation

Now, we consider the parametrized fractional stochastic heat equation with Hermite noise

$$\frac{\partial u_\theta}{\partial t}(t,x) = -\theta(-\Delta)^{\frac{\alpha}{2}} u_\theta(t,x) + \dot{Z}^{(q)}_{(H_0,\mathbf{H}),d+1}(t,x), \quad t \geq 0, x \in \mathbb{R}^d \tag{4.1}$$

with vanishing initial condition $u_\theta(0,x) = 0$ for every $x \in \mathbb{R}^d$. Our purpose is to estimate the drift parameter $\theta > 0$ in (4.1) based on the observation of the solution u_θ at discrete times and at a fixed point in space. That is, we assume that we have at our disposal the observations $(u_\theta(t_i,x), i -$

$0,1,...,N)$ with $t_i = \frac{i}{N}$ for $i = 0,1,...,N$ and $x \in \mathbb{R}^d$. Again, the noise $Z^{(q)}_{(H_0,\mathbf{H}),d+1}$ is a $d+1$-parameter Hermite process with Hurst index $(H_0, \mathbf{H}) \in \left(\frac{1}{2},1\right)^{d+1}$ where $\mathbf{H} = (H_1,\ldots,H_d)$.

Since the Green kernel that solves $\frac{\partial u_\theta}{\partial t}(t,x) = -\theta\Delta u_\theta(t,x)$ is $G_\alpha(\theta t, x)$ with G_α defined by (3.4), the mild solution to (4.1) can be written as

$$u_\theta(t,x) = \int_0^t \int_{\mathbb{R}^d} G_\alpha(\theta(t-s), x-y) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy) \tag{4.2}$$

where the integral $Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy)$ is a Wiener-Hermite integral described in Section 2.2. It follows from Proposition 3.1 that (4.2) is well-defined if and only if condition (3.8) holds.

The key observation is that, via a trivial transform of the solution, we can move the drift θ in front of the random noise. This remark will allow to apply the p-variation method in order to estimate the drift parameter. This idea has been used in [11] or [17].

Let us define, for $t \geq 0$ and $x \in \mathbb{R}^d$,

$$v_\theta(t,x) = u_\theta\left(\frac{t}{\theta}, x\right). \tag{4.3}$$

We have the following result for v_θ.

Proposition 4.1. *The random field $(v_\theta(t,x), t \geq 0, x \in \mathbb{R}^d)$ satisfies, in the mild sense, the following SPDE*

$$\frac{\partial v_\theta}{\partial t} = -(-\Delta)^{\frac{\alpha}{2}} v_\theta(t,x) + \theta^{-H_0} \dot{Z}^{(q)}_{(H_0,\mathbf{H}),d+1}(t,x), \quad t \geq 0, x \in \mathbb{R}^d \tag{4.4}$$

with vanishing initial condition $v_\theta(0,x) = 0$ for every $x \in \mathbb{R}^d$

Proof. We have, by the definition of the mild solution (4.2), for $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} v_\theta(t,x) &= \int_0^{\frac{t}{\theta}} \int_{\mathbb{R}^d} G_\alpha(t-\theta s, x-y) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy) \\ &= \int_0^t \int_{\mathbb{R}^d} G_\alpha(t-s, x-y) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(d(\frac{s}{\theta}),dy) \end{aligned}$$

and by using the H_0-self-similarity in time of $Z^{(q)}_{(H_0,\mathbf{H}),d+1}$, we obtain

$$\begin{aligned} v_\theta(t,x) &\equiv^{(d)} \theta^{-H_0} \int_0^t \int_{\mathbb{R}^d} G_\alpha(t-s, x-y) Z^{(q)}_{(H_0,\mathbf{H}),d+1}(ds,dy) \\ &= \theta^{-H_0} u_1(t,x). \end{aligned} \tag{4.5}$$

which means that v_θ verifies (4.4) in the mild sense. □

From Theorem 3.9, we deduce immediately the behavior of the p-variation in time for the random field u_θ. Recall the notation (3.17), for $p > 0, x \in \mathbb{R}^d$,

$$S^{N,p}_{[A_1,A_2]}(u_\theta(\cdot,x)) = \sum_{i=0}^{N-1} |u_\theta(t_{i+1},x) - u_\theta(t_i,x)|^p$$

with $t_i, i = 0,1,...,N$ given by (3.16). If $[A_1,A_2] = [0,t]$, then we use the notation $S^{N,p}_{[A_1,A_2]} = S^{N,p}_t$.

Proposition 4.2. *Let u_θ be the solution to the parametrized heat equation defined by (4.2) and assume (3.8). Then*

$$S^{N,\frac{1}{\gamma}}_{[A_1,A_2]}(u_\theta(\cdot,x)) \to_{N\to\infty} \theta^{1-\frac{H_0}{\gamma}} (A_2 - A_1) \mathbf{E}\,|U(1,0)|^{\frac{1}{\gamma}} \text{ in probability.}$$

Proof. We have, for every $x \in \mathbb{R}^d$ and $N \geq 1$, via (4.3),

$$
\begin{aligned}
S^{N,\frac{1}{\gamma}}_{[A_1,A_2]}(u_\theta(\cdot,x)) &= \sum_{i=0}^{N-1} |u_\theta(t_{i+1},x) - u_\theta(t_i,x)|^{\frac{1}{\gamma}} \\
&= \sum_{i=0}^{N-1} |v_\theta(\theta t_{i+1},x) - v_\theta(\theta t_i,x)|^{\frac{1}{\gamma}} \\
&=^{(d)} \theta^{-\frac{H_0}{\gamma}} \sum_{i=0}^{N-1} |u_1(\theta t_{i+1},x) - u_1(\theta t_i,x)|^{\frac{1}{\gamma}}
\end{aligned}
$$

where for the last equality we used (4.5). Notice that $\theta t_i, i = 0, \ldots, N$ constitutes a partition of the interval $[\theta A_1, \theta A_2]$. Then, by Theorem 3.9,

$$
\sum_{i=0}^{N-1} |u_1(\theta t_{i+1},x) - u_1(\theta t_i,x)|^{\frac{1}{\gamma}} \to_{N\to\infty} (\theta A_2 - \theta A_1)\mathbf{E}\,|U(1,0)|^{\frac{1}{\gamma}}.
$$

Consequently,

$$
S^{N,\frac{1}{\gamma}}_{[A_1,A_2]}(u_\theta(\cdot,x)) \to_{N\to\infty} \theta^{1-\frac{H_0}{\gamma}}(A_2 - A_1)\mathbf{E}\,|U(1,0)|^{\frac{1}{\gamma}} \text{ in probability}
$$

with U given by (3.13). □

Remark 4.3. If $H_0 = \frac{1}{2}, d = 1$ and $H_1 = \frac{1}{2}$ (which corresponds to the situation of the space-time white noise), we have $\gamma = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)$ and $1 - \frac{H_0}{\gamma} = -\frac{1}{\alpha-1}$. We retrieve a result in [11].

Assume that we have the observations $(u_\theta(t_i,x), i = 0.1,..,N)$ with t_i given by (3.16), i.e. the solution is discretely observed during the time interval $[A_1.A_2]$ at some fixed point in space. From Proposition 4.2, we can define the following natural estimator for the drift parameter θ

$$
\widehat{\theta}_N = \left(\frac{1}{(A_2 - A_1)\mathbf{E}\,|U(1,0)|^{\frac{1}{\gamma}}} S^{N,\frac{1}{\gamma}}_{[A_1,A_2]}(u_\theta(\cdot,x)) \right)^{\frac{\gamma}{\gamma - H_0}}. \tag{4.6}
$$

We deduce the following asymptotic behavior for (4.6).

Proposition 4.4. *The estimator $\widehat{\theta}_N$ given by (4.6) is consistent, i.e. $\widehat{\theta}_N$ converges in probability to θ as $N \to \infty$.*

Proof. It is an immediate consequence of Proposition 4.2. □

Remark 4.5. The asymptotic distribution of the estimator $\widehat{\theta}_N$ is rather complex, since it involves the power increments of the Hermite sheet. The intuition is that the limit behavior of (4.6) is not Gaussian, since, for instance, the limit behavior of the quadratic variations of the Hermite process is not Gaussian (see e.g. [22]).

5 Appendix: Multiple Stochastic Integrals

Here, we shall only recall some elementary facts concerning the multiple stochastic integrals; our main reference is [15]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle .,. \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by:

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{5.1}$$

The isometry of multiple integrals can be written as: for $p,\ q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbf{E}\Big(I_p(f) I_q(g)\Big) = \begin{cases} q! \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

It also holds that:

$$I_q(f) = I_q\Big(\tilde{f}\Big),$$

where $\tilde{f}$ denotes the canonical symmetrization of f and it is defined by:

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

in which the sum runs over all permutations σ of $\{1, \ldots, q\}$.

An important property of finite sums of multiple integrals is the hypercontractivity. Namely, if $F = \sum_{k=0}^{n} I_k(f_k)$ with $f_k \in \mathcal{H}^{\otimes k}$ then

$$\mathbf{E}|F|^p \leq C_p \left(\mathbf{E} F^2\right)^{\frac{p}{2}}. \tag{5.3}$$

for every $p \geq 2$. In this work, the role of the Hilbert space $\mathcal{H}$ will be played by $L^2\left(\left(\mathbb{R}_+ \times \mathbb{R}^d\right)^q\right)$.

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