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On a partial differential equation related to the diamond Bessel Klein-Gordon operator

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Abstract

In this paper, we consider the equation

$$\Diamond_{B,m}^{k} u(x) = \sum_{r=0}^{t} c_r \Diamond_{B,m}^{r} \delta$$

where $\Diamond_{B,m}^{k}$ is the operator iterated k-times and is defined by

$$\Diamond_{B,m}^{k} = \left(\left(\left(\sum_{i=1}^{p} B_{x_i} \right)^2 + \frac{m^2}{2} \right)^2 - \left(\left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 - \frac{m^2}{2} \right)^2 \right)^k,$$

where $p+q=n, x=(x_1,\ldots,x_n) \in \mathbb{R}_n^+, B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}, v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}$ [6], $x_i > 0, i = 1,2,\ldots,n, c_r$ is a constant, k is a nonnegative integer, δ is the Dirac-delta distribution, $\Diamond_{B,m}^{0}\delta = \delta$ and n is the dimension of $\mathbb{R}_n^+$. It is shown that, depending on the relationship between k and t, the solution to this equation can be ordinary functions, tempered distributions, or singular distributions.

Keywords: Bessel diamond operator, Diamond Bessel Klein Gordon, Dirac-delta distribution.

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1 Introduction

Kananthai [1] has studied the partial differential operator $\diamond^k$ and is named as the diamond operator iterated k-times, which is defined by

$$\diamond^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \quad p+q=n \tag{1.1}$$

where n is the dimension of the space $\mathbb{R}^n$, for $x=(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ and k is a non-negative integer. The operator $\diamond^k$ can be expressed in the form $\diamond^k = \square^k \triangle^k = \triangle^k \square^k$, where the operator $\triangle^k$ is the Laplace operator and which is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k \tag{1.2}$$

and the operator $\Box^k$ is the ultra-hyperbolic operator and which is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k \tag{1.3}$$

Satsanit [12] has showed that

$$\left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k = \left(\left(\frac{\triangle + \Box}{2}\right)^2 + \left(\frac{\triangle - \Box}{2}\right)^2\right)^k = \left(\frac{\triangle^2 + \Box^2}{2}\right)^k \tag{1.4}$$

Satsanit [13] has studied the diamond Bessel Klein - Gordon operator related to linear differential equation of the form $(\diamondsuit_B + m^2)^k u(x) = \delta$, we obtain $u(x) = W_{2k}(x, m)$ is the elementary solution of the diamond Bessel Klein - Gordon operator, δ is the Dirac - delta distribution. Lunnaree and Nonlaopon [4] have introduced the operator $(\diamondsuit + m^2)^k$, that is named as the diamond Klein-Gordon operator, which is defined by

$$(\diamondsuit + m^2)^k = \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right)^k,$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, m$ is a nonnegative real number and k is a nonnegative integer, see [2, 3, 8, 9] for more details. Later, Yildirim, Sarikaya and Ozturk [7] have studied the Bessel diamond operator, which is defined by

$$\diamond_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right)^k \tag{1.5}$$

$$= \left(\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right)^k \left(\sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j}\right)^k.$$

Yildirim, Sarikaya and Ozturk have showed that the function $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution for the operator $\diamond_B^k$, where $*$ indicates convolution, $R_{2k}(x)$ and $S_{2k}(x)$ are defined by (2.2) and (2.3) respectively, that is,

$$\diamond_B^k \left((-1)^k S_{2k}(x) * R_{2k}(x)\right) = \delta,$$

$x \in \mathbb{R}_n^+ = \{x : x = (x_1, \ldots, x_n), x_1 > 0, \ldots, x_n > 0\}$. For $k = 1$ the operator $\diamond_B$ can be expressed in the form $\diamond_B = \triangle_B \Box_B = \Box_B \triangle_B$ where $\Box_B$ is the Bessel ultra-hyperbolic operator,

$$\Box_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}}$$

where $p + q = n$ and $\triangle_B$ is the Laplace Bessel operator,

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} + B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}}$$

Bupasiri [11] has introduced the elementary solution of the n-dimensional $\oplus_B^k$ operator and showed that the solution of the convolution form $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ is a unique elementary solution of the equation $\oplus_B^k u(x) = \delta$, where

$$\oplus_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i}\right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^4\right)^k.$$

We consider the equation

$$\diamond_{B,m}^k u(x) = \sum_{r=0}^{t} c_r \diamond_{B,m}^r \delta$$

where $\diamond_{B,m}^k$ is the operator and which is defined by

$$\diamond_{B,m}^k = \left(\left(\left(\sum_{i=1}^{p} B_{x_i} \right)^2 + \frac{m^2}{2} \right)^2 - \left(\left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 - \frac{m^2}{2} \right)^2 \right)^k$$
$$= \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 + m^2 \right)^k \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right)^k$$
$$= (\diamond_B + m^2)^k \circledcirc_B^k \tag{1.6}$$

where

$$(\diamond_B + m^2)^k = \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 + m^2 \right)^k \tag{1.7}$$

$$\circledcirc_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right)^k = \left(\left(\frac{\Delta_B + \Box_B}{2} \right)^2 + \left(\frac{\Delta_B - \Box_B}{2} \right)^2 \right)^k$$
$$= \left(\frac{\Delta_B^2 + \Box_B^2}{2} \right)^k \tag{1.8}$$

From (1.6) with $q = m = 0$ and $k = 1$, we obtain $\oplus_B = \Delta_{B,p}^4$, where

$$\Delta_{B,p} = B_{x_1} + B_{x_2} + \cdots + B_{x_p}. \tag{1.9}$$

The purpose of this article is finding the solution to the equation

$$\diamond_{B,m}^k u(x) = \sum_{r=0}^{t} c_r \diamond_{B,m}^r \delta$$

by using B-convolutions of the generalized function, where $p + q = n$, $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \ldots, x_n), x_1 > 0, \ldots, x_n > 0\}$, c_r is a constant, δ is the Dirac-delta distribution, and $\diamond_{B,m}^0 \delta = \delta$.

The following is the main result of this paper, a proof of which is given in §3.1.

Theorem 1.1. *Consider the linear differential equation*

$$\diamond_{B,m}^k u(x) = \sum_{r=0}^{t} c_r \diamond_{B,m}^r \delta, \tag{1.10}$$

where $p + q = n$, $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \ldots, x_n), x_1 > 0, \ldots, x_n > 0\}$, c_r *is a constant,* δ *is the Dirac-delta distribution, and* $\diamond_{B,m}^0 \delta = \delta$. *Then the type of solution to* (1.10) *depends on the relationship between* k *and* t*, according to the following cases:*

(1) *If* $t < k$ *and* $t = 0$*, then* (1.10) *has the solution*

$$u(x) = W_{2k}(x, m) * c_0 \left((-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1} \right)$$

which is the elementary solution of the operator $\diamond_{B,m}^k$ *in Proposition* 3.1*, is an ordinary function when* $2k \geq n + |v|$ *and* $4k \geq n + |v|$ *and is a temper distribution when* $2k < n + |v|$ *and* $4k < n + |v|$.

(2) *If* $t < k$ *and* $t = m = 0$*, then* (1.10) *has the solution*

$$u(x) = c_0 \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right)$$

which is the elementary solution of the operator $\oplus_B^k$*, is an ordinary function when* $6k \geq n + |v|$ *and is a temper distribution when* $6k < n + |v|$.

(3) *If* $0 < t < k$ *then the solution of* (1.10) *is*

$$u(x) = \sum_{r=1}^{t} W_{2(k-r)}(x, m) * c_r \left((-1)^{2(k-r)} S_{4(k-r)}(x) * R_{4(k-r)}(x) * (C^{*(k-r)}(x))^{*-1} \right)$$

which is an ordinary function when $2k - 2r \geq n + |v|$ *and* $4k - 4r \geq n + |v|$ *and is a tempered distribution when* $2k - 2r < n + |v|$ *and* $4k - 4r < n + |v|$.

(4) *If* $t \geq k$ *and* $k \leq t \leq M$*, then* (1.10) *has the solution*

$$u(x) = \sum_{r=k}^{M} c_r \diamond_{B,m}^{r-k} \delta$$

which is only a singular distribution.

2 Preliminaries

Denoted by T_x^y the generalized shift operator acting according to the law [6]

$$T_x^y \varphi(x) = C_v^* \int_0^\pi \dots \int_0^\pi \varphi \left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, \dots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n} \right)$$
$$\times \left(\Pi_{i=1}^n \sin^{2v_i - 1} \right) d\theta_1 \dots d\theta_n,$$

where $x, y \in \mathbb{R}_n^+, C_v^* = \Pi_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [6].

$$\frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}$$
$$U(x, 0) = f(x),$$
$$U_y(x, 0) = 0.$$

The convolution operator determined by T_x^y is as follow:

$$(f * \varphi) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\Pi_{i=1}^n y_i^{2v_i} \right) dy. \tag{2.1}$$

Convolution (2.1) is known as a B-convolution. We note the following properties for the B-convolution and the generalized shift operator:

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+), g(x)$ is a bounded function, $x > 0$ and

$$\int_0^\infty |f(x)| \left(\Pi_{i=1}^n x_i^{2v_i} \right) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\Pi_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\Pi_{i=1}^n y_i^{2v_i} \right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$,

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\Pi_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\Pi_{i=1}^n y_i^{2v_i} \right) dy$$

(e) $(f * g)(x) = (g * f)(x)$.

Lemma 2.1. *If $\Box_B^k u(x) = \delta$ for $x \in \Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0 \ldots, x_n > 0$ and $V > 0\}$, where $\Box_B^k$ is the Bessel-ultra hyperbolic operator iterated k-times . Then $u(x) = R_{2k}(x)$ is the elementary solution of the operator $\Box_B^k$, where*

$$\Box_B^k = \left(\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right)^k, p + q = n$$

$$R_{2k}(x) = \frac{V^{\frac{2k-n-|v|}{2}}}{K_n(2k)} = \frac{\left(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \right)^{\left(\frac{2k-n-|v|}{2}\right)}}{K_n(2k)} \tag{2.2}$$

for

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+2k-n-2|v|}{2}\right) \Gamma\left(\frac{1-2k}{2}\right) \Gamma(2k)}{\Gamma\left(\frac{2+2k-p-2|v|}{2}\right) \Gamma\left(\frac{p-2k}{2}\right)}.$$

Lemma 2.2. *Given the equation $\triangle_B^k u(x) = \delta$ for $x \in \mathbb{R}_n^+$, where $\triangle_B^k$ is the Laplace Bessel operator iterated k-times. Then $u(x) = (-1)^k S_{2k}(x)$ is the elementary solution of the operator $\triangle_B^k$, where*

$$\triangle_B^k = \left(\sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j} \right)^k,$$

$$S_{2k}(x) = \frac{|x|^{2k-n-2|v|}}{w_n(2k)}, \qquad p + q = n, \tag{2.3}$$

and

$$w_n(2k) = \frac{\Pi_{i=1}^n 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \Gamma(k)}{2^{n+2|v|-4k} \Gamma\left(\frac{n+2|v|-2k}{2}\right)}. \tag{2.4}$$

Lemma 2.3. *The convolution $R_{2k}(x) * (-1)^k S_{2k}(x)$ is the elementary solution for the Bessel diamond operator $\diamond_B^k$ iterated k-times and is defined by* (1.5).

Lemma 2.4. *$R_{2k}(x)$ and $S_{2k}(x)$ are homogeneous distributions of order $(2k - n - 2|v|)$.*

We need to show that $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$ satisfy the Euler equation; that is,

$$(2k - n - 2|v|) R_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{2k}(x)$$

and

$$(2k - n - 2|v|) S_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_{2k}(x).$$

Lemma 2.5. *(The B-convolution of tempered distribution). $R_{2k}(x) * S_{2k}(x)$ exists and is a tempered distribution.*

Proof. For the proofs of Lemma 2.1- Lemma 2.5, see ([7], p.378-383). □

Lemma 2.6. *(The B-convolution of $R_{2k}(x)$ and $S_{2k}(x)$). Let $R_{2k}(x)$ and $S_{2k}(x)$ defined by (2.2) and (2.3) respectively, then we obtain:*

(1) $S_{2k}(x) * S_{2m}(x) = S_{2k+2m}(x)$, where k and m are nonnegative integers.

(2) $R_{2k}(x) * R_{2m}(x) = R_{2k+2m}(x)$, where k and m are nonnegative integers.

Lemma 2.7. *The function $R_{-2k}(x)$ and $(-1)^k S_{-2k}(x)$ are the inverses in the B-convolution algebra of $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$, respectively. That is,*

$$R_{-2k}(x) * R_{2k}(x) = R_{-2k+2k}(x) = R_0(x) = \delta,$$
$$(-1)^k S_{-2k}(x) * (-1)^k S_{2k}(x) = S_{-2k+2k}(x) = S_0(x) = \delta$$

Proof. For the proofs of Lemma 2.6 and Lemma 2.7, see [10]. □

Definition 2.8. Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbb{R}_n^+$, the function $W_\alpha(x, m)$ is defined by

$$W_\alpha(x,m) = \sum_{r=0}^{\infty} \binom{-\alpha/2}{r} (m^2)^r (-1)^{\alpha/2+r} S_{\alpha+2r}(x) * R_{\alpha+2r}(x), \tag{2.5}$$

where α is a complex parameter, m is a nonnegative real number, $R_{\alpha+2r}(x)$ and $S_{\alpha+2r}(x)$ are defined by (2.2) and (2.3) respectively.

From the definition of $W_\alpha(x, m)$ and by putting $\alpha = -2k$, we have

$$W_{-2k}(x,m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r (-1)^{-k+r} S_{2(-k+r)}(x) * R_{2(-k+r)}(x).$$

Since the operator $(\diamond_B + m^2)^k$ defined in equation (1.7) is a linearly continuous and has $1-1$ mapping, then it has inverse. From Lemma 2.3 we obtain

$$\begin{aligned} W_{-2k}(x,m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \diamond_B^{-k-r} \delta \\ &= (\diamond_B + m^2)^k \delta. \end{aligned} \tag{2.6}$$

By putting $k = 0$ in (2.6) , we have $W_0(x, m) = \delta$. By putting $\alpha = 2k$ into (2.5), we have

$$\begin{aligned} W_{2k}(x,m) = &\binom{-k}{0} (m^2)^0 (-1)^{k+0} S_{2k+0}(x) * R_{2k+0}(x) \\ &+ \sum_{r=1}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} S_{2k+2r}(x) * R_{2k+2r}(x). \end{aligned} \tag{2.7}$$

The second summand of the right-hand member of (2.7) vanishes for $m = 0$ and then, we have

$$W_{2k}(x, m=0) = (-1)^k S_{2k}(x) * R_{2k}(x) \tag{2.8}$$

which is the elementary solution of the Bessel diamond operator.

Lemma 2.9. *[13] Given the equation*

$$(\diamond_B + m^2)^k u(x) = \delta$$

for $x \in \mathbb{R}_n^+$ and $(\diamond_B + m^2)^k$ is the diamond Bessel Klein Gordon operator iterated k-times defined by (1.7), we obtain

$$u(x) = W_{2k}(x,m)$$

is the elementary solution or Green function of the operator $(\diamond_B + m^2)^k$ and $W_{2k}(x, m)$ is defined by (2.5) with $\alpha = 2k$. The function $W_{2k}(x, m)$ has the following properties $W_0(x, m) = \delta$ and

$$(\diamond_B + m^2)^k W_\alpha(x,m) = W_{\alpha-2k}(x,m).$$

Lemma 2.10. *[11] Given the equation*

$$\circledcirc_B^k G(x) = \left(\frac{\triangle_B^2 + \Box_B^2}{2}\right)^k G(x) = \delta \tag{2.9}$$

for $x \in \mathbb{R}_n^+$, where $\circledcirc_B^k$ is the operator iterated k-times is defined by (1.8) . Then we obtain $G(x)$ is the elementary solution of (2.9), where

$$G(x) = (R_{4k}(x) * (-1)^{2k} S_{4k}(x)) * (C^{*k}(x))^{*-1}$$

where

$$C(x) = \frac{1}{2} R_4(x) + \frac{1}{2}(-1)^2 S_4(x).$$

Here $C^{*k}(x)$ denotes the convolution of $C(x)$ itself k-times, $(C^{*k}(x))^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $G(x)$ is a tempered distribution.

3 Main Results

Proposition 3.1. *Given the equation*

$$\diamond_{B,m}^k u(x) = \delta, \tag{3.1}$$

where $\diamond_{B,m}^k$ is the operator iterated k-times defined by (1.6) , δ is the Dirac-delta distribution, $x \in \mathbb{R}_n^+$ and k is a nonnegative integer. Then we obtain

$$u(x) = W_{2k}(x,m) * \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{3.2}$$

is the elementary solution for the operator $\diamond_{B,m}^k$. In particular, for $m = 0$ then (3.1) becomes

$$\oplus_B^k u(x) = \delta, \tag{3.3}$$

we obtain

$$u(x) = R_{6k}(x) * (-1)^{3k} S_{6k}(x) * (C^{*k}(x))^{*-1}$$

is the elementary solution of the equation (3.3), for $q = m = 0$ then (3.1) becomes

$$\Delta_{B,p}^{4k} u(x) = \delta, \tag{3.4}$$

we obtain

$$u(x) = S_{8k}(x)$$

is the elementary solution of (3.4), where $\Delta_{B,p}^{4k}$ is the Laplace Bessel operator of p-dimension, iterated 4k-times which is defined by (1.9).

Proof. From (1.6) and (3.1), we have

$$(\diamond_B + m^2)^k \left(\frac{\triangle_B^2 + \Box_B^2}{2}\right)^k u(x) = \delta. \tag{3.5}$$

B-convolving both sides of (3.5) by $W_{2k}(x,m) * \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1}$, we obtain

$$\left(W_{2k}(x,m) * \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x) * (C^{*k}(x))^{*-1}\right)\right) * (\diamond_B + m^2)^k \left(\frac{\triangle_B^2 + \Box_B^2}{2}\right)^k u(x)$$
$$= W_{2k}(x,m) * \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \delta.$$

By properties of B-convolutions

$$(\diamond_B + m^2)^k W_{2k}(x,m) * \left(\frac{\triangle_B^2 + \Box_B^2}{2}\right)^k \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x) * (C^{*k}(x))^{*-1}\right) * u(x)$$
$$= W_{2k}(x,m) * \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1}.$$

By Lemma 2.9 and Lemma 2.10, we obtain,

$$\delta * \delta * u(x) = u(x) = W_{2k}(x,m) * \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x) * (C^{*k}(x))^{*-1}\right) \tag{3.6}$$

is the elementary solution of operator $\diamond_{B,m}^k$. In particular, for $m = 0$ then (3.1) becomes

$$\diamond_{B,0}^k u(x) = \oplus_B^k u(x) = \delta,$$

by Lemma 2.6, equations (2.8) and (3.6) we obtain

$$\begin{aligned} u(x) &= \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x) * (C^{*k}(x))^{*-1}\right) * W_{2k}(x,0) \\ &= \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x) * (C^{*k}(x))^{*-1}\right) * ((-1)^k S_{2k}(x) * R_{2k}(x)) \\ &= \left(R_{6k}(x) * (-1)^{3k} S_{6k}(x)\right) * (C^{*k}(x))^{*-1} \end{aligned}$$

is the elementary solution of the operator $\oplus_B^k$, for $q = m = 0$ then (3.1) becomes

$$\Delta_{B,p}^{4k} u(x) = \delta, \tag{3.7}$$

where $\Delta_{B,p}^{4k}$ is the Laplace Bessel operator of p-dimension iterated $4k$-times. By Lemma 2.2, we have

$$u(x) = (-1)^{4k} S_{8k}(x) = S_{8k}(x)$$

is the elementary solution of (3.7). □

Proposition 3.2. *For $0 < r < k$,*

$$\diamond_{B,m}^k \left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)$$
$$= W_{2(k-r)}(x,m) * \left((-1)^{2(k-r)} S_{4(k-r)}(x) * R_{4(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right)$$

and for $k \leq t$,

$$\diamond_{B,m}^t \left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right) = \diamond_{B,m}^{t-k} \delta.$$

Proof. For $0 < r < k$, by Proposition 3.1,

$$\diamond_{B,m}^k \left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right) = \delta.$$

Thus,

$$\diamond_{B,m}^{k-r} \diamond_{B,m}^r \left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right) = \delta$$

or

$$\diamond_{B,m}^{k-r} \delta * \diamond_{B,m}^r \left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right) = \delta.$$

B- convolving both sides by $W_{2(k-r)}(x,m) * (-1)^{2(k-r)} S_{4(k-r)}(x) * R_{4(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}$, we obtain

$$\diamond_{B,m}^{k-r} \left(W_{2(k-r)}(x,m) * (-1)^{2(k-r)} S_{4(k-r)}(x) * R_{4(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right)$$
$$* \diamond_{B,m}^r \left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)$$
$$= W_{2(k-r)}(x,m) * (-1)^{2(k-r)} S_{4(k-r)}(x) * R_{4(k-r)}(x) * (C^{*(k-r)}(x))^{*-1} * \delta.$$

By Proposition 3.1,

$$\begin{aligned}&\delta * \diamond^r_{B,m}\left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&= W_{2(k-r)}(x,m) * (-1)^{2(k-r)} S_{4(k-r)}(x) * R_{4(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}.\end{aligned}$$

It follows that

$$\begin{aligned}&\diamond^r_{B,m}\left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&= W_{2(k-r)}(x,m) * (-1)^{2(k-r)} S_{4(k-r)}(x) * R_{4(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\end{aligned}$$

as required. For $k \leq t$,

$$\begin{aligned}&\diamond^t_{B,m}\left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&= \diamond^{t-k}_{B,m}\diamond^k_{B,m}\left(W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&= \diamond^{t-k}_{B,m}\delta\end{aligned}$$

by Proposition 3.1. That completes the proofs. □

We now come to the proof of our main result.

3.1 Proof of Theorem 1.1

(1) For $t = 0$, we have $\diamond^k_{B,m} u(x) = c_0\delta$, and by Proposition 3.1 we obtain

$$u(x) = W_{2k}(x,m) * c_0\left((-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right).$$

Now, $W_{2k}(x,m)$,$(-1)^{2k}S_{4k}(x)$ and $R_{4k}(x)$ are the analytic functions for $2k \geq n + |v|$ and $4k \geq n + |v|$ and also $W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}$ exits and is an analytic function by (3.2). It follows that $W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}$ is an ordinary function for $2k \geq n + |v|$ and $4k \geq n + |v|$. By Lemma 2.5, $W_{2k}(x,m)$, $(-1)^{2k}S_{4k}(x)$ and $R_{4k}(x)$ are tempered distributions with $2k < n + |v|$ and $4k < n + |v|$, we obtain $W_{2k}(x,m) * (-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}$ exits and is a tempered distribution.

(2) For $t = m = 0$, we have $\diamond^k_{B,0} u(x) = \oplus^k_B u(x) = c_0\delta$, and by Proposition 3.1, Lemma 2.6 and equation (2.8) we obtain

$$\begin{aligned}u(x) &= W_{2k}(x,0) * c_0\left((-1)^{2k} S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&= c_0\left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}\right).\end{aligned}$$

Now, $(-1)^{3k}S_{6k}(x)$ and $R_{6k}(x)$ are the analytic functions for $6k \geq n+|v|$ and also $(-1)^{3k}S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ exits and is an analytic function by (3.2). It follows that $(-1)^{3k}S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ is an ordinary function for $6k \geq n + |v|$. By Lemma 2.5, $(-1)^{3k}S_{6k}(x)$ and $R_{6k}(x)$ are tempered distributions with $6k < n + |v|$, we obtain $(-1)^{3k}S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ exits and is a tempered distribution.

(3) For the case $0 < t < k$, we have

$$\diamond^k_{B,m} u(x) = c_1 \diamond_{B,m}\delta + c_2 \diamond^2_{B,m}\delta + \cdots + c_t \diamond^t_{B,m}\delta.$$

We convolved both sides of the above equation by $W_{2k}(x,m)*(-1)^{2k}S_{4k}(x)*R_{4k}(x)*(C^{*k}(x))^{*-1}$ to obtain

$$\begin{aligned}&\diamond^k_{B,m} W_{2k}(x,m) * \left((-1)^{2k}S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right) * u(x)\\&= c_1 \diamond_{B,m} \left(W_{2k}(x,m) * (-1)^{2k}S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&+ c_2 \diamond^2_{B,m} \left(W_{2k}(x,m) * (-1)^{2k}S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&+ \cdots + c_t \diamond^t_{B,m} \left(W_{2k}(x,m) * (-1)^{2k}S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right).\end{aligned}$$

By Proposition 3.2, we obtain

$$\begin{aligned}u(x) &= c_1 \left(W_{2(k-1)}(x,m) * (-1)^{2(k-1)}S_{4(k-1)}(x) * R_{4(k-1)}(x) * (C^{*(k-1)}(x))^{*-1}\right)\\&+ c_2 \left(W_{2(k-2)}(x,m) * (-1)^{2(k-2)}S_{4(k-2)}(x) * R_{4(k-2)}(x) * (C^{*(k-2)}(x))^{*-1}\right)\\&+ \cdots + c_t \left(W_{2(k-t)}(x,m) * (-1)^{2(k-t)}S_{4(k-t)}(x) * R_{4(k-t)}(x) * (C^{*(k-t)}(x))^{*-1}\right).\end{aligned}$$

or

$$u(x) = \sum_{r=1}^{t} c_r \left(W_{2(k-r)}(x,m) * (-1)^{2(k-r)}S_{4(k-r)}(x) * R_{4(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}\right).$$

Similarly, as in the case (1), $u(x)$ is an ordinary function for $2k-2r \geq n+|v|$ and $4k-4r \geq n+|v|$ and is a tempered distribution for $2k-2r < n+|v|$ and $4k-4r < n+|v|$.

(4) For the case $t \geq k$ and $k \leq t \leq M$, we have

$$\diamond^k_{B,m} u(x) = c_k \diamond^k_{B,m} \delta + c_{k+1} \diamond^{k+1}_{B,m} \delta + \cdots + c_M \diamond^M_{B,m} \delta.$$

B-convolved both sides of the above equation by $W_{2k}(x,m)*(-1)^{2k}S_{4k}(x)*R_{4k}(x)*(C^{*k}(x))^{*-1}$ to obtain

$$\begin{aligned}&\diamond^k_{B,m} \left(W_{2k}(x,m) * (-1)^{2k}S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right) * u(x)\\&= c_k \diamond^k_{B,m} \left(W_{2k}(x,m) * (-1)^{2k}S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&+ c_{k+1} \diamond^{k+1}_{B,m} \left(W_{2k}(x,m) * (-1)^{2k}S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right)\\&+ \cdots + c_M \diamond^M_{B,m} \left(W_{2k}(x,m) * (-1)^{2k}S_{4k}(x) * R_{4k}(x) * (C^{*k}(x))^{*-1}\right).\end{aligned}$$

By Proposition 3.2 again, we obtain

$$u(x) = c_k\delta + c_{k+1} \diamond_{B,m} \delta + c_{k+2} \diamond^2_{B,m} \delta + \cdots + c_M \diamond^{M-k}_{B,m} \delta = \sum_{r=k}^{M} c_r \diamond^{r-k}_{B,m} \delta.$$

Since $\diamond^{r-k}_{B,m}\delta$ is a singular distribution, hence $u(x)$ is only the singular distribution. That completes the proof.

References

[1] A. Kananthai, *On the solutions of the n-dimensional diamond operator*, Appl. Math. Comput. **88** (1997), 27–37.

[2] A. Liangprom, K. Nonlaopon, *On the convolution equation related to the Klein-Gordon operator*, International Journal of Pure and Applied Mathematics **71** (2011), 67–82.

[3] A. Liangprom, K. Nonlaopon, *On the convolution equation related to the diamond Klein-Gordon operator*, Abstract and Applied Analysis **2011** (2011), 1–14.

[4] A. Lunnaree, K. Nonlaopon, *On the Fourier transform of the diamond Klein - Gordon kernel*, International Journal of Pure and Applied Mathematics **68** (2011), 85–97.

[5] A.H. Zemanian, *Distribution and Transform Analysis,* McGraw-Hill, New York, 1965.

[6] B.M. Levitan, *Expansion in Fourier series and integrals with Bessel functions*, Uspeki Mat. Nauka (N.S.) **6**(42)(1951), 102–143.

[7] H. Yildirim, M.Z. Sarikaya, S. Ozturk, *The solutions of the n-dimensional Bessel diamond operator and the Fourier-Bessel transform of their convolution*, Proc. Indian Acad. Sci. (Math.Sci.), **114**(4), (2004), 375–387.

[8] K. Nonlaopon, *On the inverse ultra-hyperbolic Klein-Gordon kernel*, Mathematics **7** (2019), 534.

[9] K. Nonlaopon, *On the solution of the n-dimensional diamond Klein-Gordon operator and its convolution*, Far East Journal of Mathematical Sciences, **63**(2012), 203–220.

[10] M.Z. Sarikaya, H. Yildirim, *On the B-convolutions of the Bessel diamond kernel of Riesz*, Appl. Math. Comput. **208**(2009), 18–22.

[11] S. Bupasiri, *On the solution of the n-dimensional $\oplus_B^k$ operator*, Applied Mathematical Sciences **9**(10)(2015), 469–479.

[12] W. Satsanit, *Green function and Fourier transform for o-plus operator*, Electronic J. of Diff. Eq. **2010**(2010), 1–14.

[13] W. Satsanit, *On the diamond Bessel Klein Gordon operator related to linear differential equation*, Journal of non linear science and applications **12** (2019), 552–561.

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