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Some rings of invariants that are Gorenstein

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Abstract

Let G be a finite subgroup of $SL(V)$ and let V be a 3-dimensional vector space over a finite field $\mathbb{F}$ of positive characteristic p, which divides $|G|$. We denote by $S(V)$ the symmetric algebra and by $S(V)^G$ the subring of G-invariants. Let $T(G)$ be the transvections group. In this paper, we classify the Gorenstein rings of the form $S(V)^G$, where V is a decomposable G-module of the form $V = \mathbb{F}v \oplus W$ with $\mathbb{F}$v and W being G-submodules with $\dim_{\mathbb{F}} W = 2$. There are several cases for $T(G)$ and W, so for each of them we provide a sufficient and necessary condition (G as above) to ensure the Gorenstein property of $S(V)^G$.

Keywords: Cohen-Macaulay rings, Gorenstein rings, Invariant Theory, Transvection groups.

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1 Introduction

The Gorenstein rings play an important part, as Noetherian rings do, in commutative algebra and algebraic geometry. The study of the relationship between injective dimension, global dimension of rings and modules, and homological algebra has occupied most of the studies on Gorenstein rings [4].

Let $G \subset SL(V)$ be a finite subgroup of $SL(V)$. Let $\mathbb{F}$ be a field with characteristic char $\mathbb{F} = p > 0$ dividing $|G|$ (the order of G), V a 3-dimensional $\mathbb{F}$-vector space and $S(V)$ the symmetric algebra of V. We denote by $S(V)^G$ the subring of G-invariants. In this paper, we consider V, a decomposable G-module of the form

$$V = \mathbb{F}v \oplus W$$

where $\mathbb{F}v$, W are G-submodules with $\dim_{\mathbb{F}} W = 2$. Our aim is to give conditions for when $S(V)^G$ is Gorenstein (Definitions in §2).

Eagon and Hochster proved in [6] that if $G \subset GL(V)$ is a finite group, then $S(V)^G$ is Cohen Macaulay ring for all nonmodular groups, i.e., when $|G|$ is prime to p, but $S(V)^G$ often fails to be Cohen-Macaulay in the modular case, i.e., when p divides $|G|$. A special example is given in [8, Theorem 1.2]: if $\mathbb{F}$ is a field of positive characteristic p, V is a faithful representation of a non-trivial p-group P, and mV denotes the faithful representation of G formed by taking the direct sum of m copies of V, then $S(mV)^P$ is not Cohen-Macaulay when $m \geq 3$. It is known however that if $G \subset GL(V)$, and $\dim_{\mathbb{F}} V = 3$, then $S(V)^G$ is always a Cohen-Macaulay ring, even in the modular case [11, Proposition 5.6.10]. This fact makes the 3-dimensional case special since the Cohen-Macaulay property is a necessary condition to be Gorenstein.

Recall that $g \in GL(V)$ is a pseudo-reflection if g has a finite order and $\operatorname{rank}(g - I) = 1$, where I denotes the unit matrix. A pseudo-reflection is called a transvection if it is not diagonalizable. Hence, if $G \subset SL(V)$, then all the pseudo-reflections are transvections. We denote by $T(G)$ the G-subgroup generated by all transvections in G and by $W(G)$ the G-subgroup generated by all pseudo-reflections in G. Given a basis $\{v_1, ..., v_n\}$ of V, and $g \in G$, then $g(v_i) = \sum_{j=1}^n a_{ij} v_j, a_{ij} \in \mathbb{F}$ where the matrix $A := (a_{ij})$, representing g, acts on coordinates (=row vectors) from the right. Consequently, if we fix a basis $\{v, w_1, w_2\}$ for V such that $W = \mathbb{F}w_1 + \mathbb{F}w_2$, then for every $g \in G$, since $V = \mathbb{F}v \oplus W$, the matrix representation of g with respect to this basis has the following form:

$$g = \begin{pmatrix} \lambda_1(g) & 0 & 0 \\ 0 & \lambda_2(g) & \lambda_3(g) \\ 0 & \lambda_4(g) & \lambda_5(g) \end{pmatrix},$$

where $\lambda_i(g) \in F$ for $1 \leq i \leq 5$. This matrix acts on the basis $\{v, w_1, w_2\}$ as follows:

$$\begin{aligned} g(v) &= \lambda_1(g)v, \\ g(w_1) &= \lambda_2(g)w_1 + \lambda_3(g)w_2, \text{ and} \\ g(w_2) &= \lambda_4(g)w_1 + \lambda_5(g)w_2 \end{aligned}$$

Our characterization of the Gorenstein property of $S(V)^G$ will proceed in steps subdivided into 3 cases:

1. W is an irreducible and primitive $T(G)$-module.

2. W is a reducible $T(G)$-module.

3. W is an irreducible and imprimitive $T(G)$-module.

In our main Proposition 3.1, we provide a sufficient and necessary condition for G as above to ensure the Gorenstein property of $S(V)^G$. By means of this condition, we handle each of the three cases above, and provide new conditions on $S(V)^G$ to be Gorenstein. In other words, we translate separately the meaning of the condition stated in Proposition 3.1 when we focus on each case. This is presented in Propositions 3.2, 3.3, 3.4 and 3.5 which are the new results of this paper.

This work is part of my M.Sc. thesis under the supervision of Professor Amiram Braun. The proofs are based on some computations and specific techniques by A. Braun in [3].

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2 Preliminaries

If R is a commutative Noetherian ring and M is a finitely generated R-module, an element $a \in R$ is *regular* for M provided that $0 \neq M \neq aM$ and if $am = 0$ for $m \in M$ then $m = 0$ (i.e., a is not a divisor on M). A sequence $x_1, ..., x_r \in R$ is a *regular sequence* for M if each x_i is regular for $M/(x_1M + ... + x_{i-1}M)$. The *depth* of M is the length of the longest regular sequence for M. One defines the ring R or the module M to be *Cohen-Macaulay* if its depth is equal to its *Krull* dimension.

Recall that an ideal I in a commutative ring is called irreducible if whenever $I = I' \cap I''$ for ideals I', I'', then either $I = I'$ or $I = I''$. If R is Noetherian, a *parameter ideal* for R is an ideal generated by a system of parameters for R. A commutative Noetherian ring R is called *Gorenstein* if it is *Cohen-Macaulay* and every parameter ideal is irreducible. As for regular rings, the Gorenstein rings can be characterized in terms of homological algebra [4].

We start with the following useful result of Fleischmann - Woodcock and A. Braun.

Theorem 2.1. *(A.Braun [2], Fleischmann-Woodcock [7]) Suppose that $S(V)^G$ is Cohen-Macaulay and $S(V)^{W(G)}$ is a polynomial ring. Then $S(V)^G$ is Gorenstein if and only if $G/W(G) \subseteq SL(m/m^2)$, where m is the unique homogenous maximal ideal of $S(V)^{W(G)}$ and $W(G)$ is the $G-$subgroup generated by all pseudo-reflections (of all types).*

The next two results are useful.

Lemma 2.2. *Suppose that $G \subset SL(V)$. Then W is a faithful G-module.*

Proof. Let $1_G \neq g \in G$, with $g|W = Id$. Then

$$g = \begin{pmatrix} \lambda(g) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $g(v) = \lambda(g)v$, for all $g \in G$. Since $G \subset SL(V)$, $\lambda(g) = 1$ and therefore $g = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, a contradiction. □

The next result does not require the assumption $V = \mathbb{F}v \oplus W$.

Lemma 2.3. *Suppose that $G \subset SL(V)$ and U a 1-dimensional $G-$submodule. Then $U \subseteq ker(\sigma - I)$ for all transvections σ in G. Equivalently, $T(G)$ acts trivially on U.*

Proof. Let $\sigma \in T(G)$ be a transvection, $\sigma \neq I$, and $M = ker(\sigma - I)$. Suppose that $Fu = U \not\subset M$. Then with respect to a basis $\{u, m_1, m_2\}, m_1, m_2 \in M$, we have:

$$\sigma = \begin{pmatrix} \lambda(\sigma) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\sigma(u) = \lambda(\sigma)u$ and $\sigma|M = Id$. But $G \subset SL(V)$, therefore $\lambda(\sigma) = 1$, implying that $\sigma = I$, a contradiction. □

3 Main Results

Our first Proposition provides a sufficient and necessary condition for G, with the assumptions in §2, to ensure the Gorenstein property of $S(V)^G$.

Proposition 3.1. *Let $G \subset SL(V)$ be a finite group and $V = \mathbb{F}v \oplus W$ a decomposition of V into $G-$submodules with $dim_F W = 2$. Then the following are equivalent:*

1. *$S(V)^G$ is Gorenstein.*

2. *$det(g^W) = det(g^{m/m^2})$, for each $g \in G$, where m is the unique homogenous maximal ideal of $S(W)^{T(G)}$, and g^W, g^{m/m^2} are the restrictions of g on W and m/m^2 respectively.*

Proof. By Lemma 2.3 $S(V)^{T(G)} = S(W)^{T(G)}[v]$. Now by [10], $S(W)^{T(G)} = F[a_1, a_2]$ is a polynomial ring, where $m = (a_1, a_2)$. Let P be the unique homogenous maximal ideal of $S(V)^{T(G)}$, then $P = (a_1, a_2, v)$. By Theorem 2.1 $S(V)^G$ is Gorenstein if and only if

$$1 = det(g^{P/P^2}) = \lambda(g)det(g^{m/m^2}),$$

where $g(v) = \lambda(g)v$, for all $g \in G$. Since $G \subset SL(V)$, $det(g^W) = \lambda(g)^{-1}$, hence $S(V)^G$ is Gorenstein if and only if $det(g^W) = det(g^{m/m^2})$, for each $g \in G$. □

Now we handle each of the three cases that we mentioned in the introduction, and translate the previous condition of Proposition 3.1 to other conditions separately.

Case 1: W is an irreducible and primitive T(G)-module

Denote by $T_W(G) = T(G)|W$ the restriction of $T(G)$ on W. The transvection groups $T_W(G)$ have been classified by Kantor whenever $T_W(G)$ is an irreducible primitive linear group, see [9, Theorem 1.5] and [12]. So with the above notation one of the following holds:

(i) $T_W(G) = SL(2, \mathbb{F}_q)$, where $p|q$ (p divides q).

(ii) $T_W(G) \cong SL(2, \mathbb{F}_5)$, $T_W(G) \subset SL(2, \mathbb{F}_9)$ and $\mathbb{F}_9 \subseteq \mathbb{F}$.

We now consider the above item (i).

Proposition 3.2. *Suppose $G \subset SL(V)$ is a finite group with $T_W(G) = SL(2, \mathbb{F}_q)$, where $q = p^s$ and $\mathbb{F}_q \subseteq F$. Then $S(V)^G$ is Gorenstein if and only if $G_W \subset GL(2, \mathbb{F}_{q^2})$, where G_W is the restriction of G on W.*

Proof. Recall from [1, Theorem 8.2.1] that

$$S(W)^{T(G)} = S(W)^{SL(2,\mathbb{F}_q)} = F[u, c_{21}],$$

where $u = xy^q - yx^q$, $deg(u) = q+1$ and $c_{21} = \dfrac{xy^{q^2} - yx^{q^2}}{u}$, $deg(c_{21}) = q^2 - q$. Let $g \in G$ and set $g^W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the restriction of g on W with respect to the basis $\{x, y\}$ of W. So $(g^{-1})^W = \frac{1}{det(g|W)}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Since $T(G)$ is a normal subgroup of G, we get after restriction on W that

$$\begin{aligned} (g^W)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(g^{-1})^W &= \frac{1}{det(g^W)}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \\ &= \frac{1}{det(g^W)}\left(\begin{smallmatrix} ad-bc-ac & a^2 \\ -c^2 & ad-bc+ac \end{smallmatrix}\right) \end{aligned}$$

is in $SL(2, \mathbb{F}_q)$. Similarly,

$$\begin{aligned} &(g^W)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)(g^{-1})^W \\ =\ & \frac{1}{det(g^W)}\left(\begin{smallmatrix} ad-bc+bd & -b^2 \\ d^2 & ad-bc-bd \end{smallmatrix}\right) \in SL(2, \mathbb{F}_q). \end{aligned}$$

Set $e = det(g^W) = ad - bc$. Then

$$\frac{ac}{e}, \frac{a^2}{e}, \frac{c^2}{e}, \frac{b^2}{e}, \frac{d^2}{e}, \frac{bd}{e} \in \mathbb{F}_q.$$

Assume firstly that $a \neq 0$. Then $\gamma := \dfrac{c}{a} = \dfrac{(\frac{c^2}{e})}{(\frac{ac}{e})} \in \mathbb{F}_q$.

If $b \neq 0$ then $\mu := \dfrac{d}{b} = \dfrac{(\frac{bd}{e})}{(\frac{b^2}{e})} \in \mathbb{F}_q$. Hence

$$e = ad - bc = ab\mu - ba\gamma = ab(\mu - \gamma)$$

and therefore

$$\beta := \frac{b}{a} = \frac{ab(\mu-\gamma)}{a^2(\mu-\gamma)} = \frac{e}{a^2(\mu-\gamma)} \in \mathbb{F}_q,$$

as well as $\delta := \dfrac{d}{a} = \dfrac{\mu b}{a} = \mu\beta \in \mathbb{F}_q$. If $b = 0$ then take $\beta = 0$ and $\delta := \dfrac{d}{a} = \dfrac{ad}{a^2} = \dfrac{e}{a^2} \in \mathbb{F}_q$. So in both cases $b = a\beta, c = a\gamma, d = a\delta$, where $\beta, \gamma, \delta \in \mathbb{F}_q$. Therefore $g^W = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ \gamma & \delta \end{smallmatrix}\right) = aI_W\left(\begin{smallmatrix} 1 & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, with $h := \left(\begin{smallmatrix} 1 & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in GL(2, \mathbb{F}_q)$.

If $a = 0$ then $e = det(g^W) = -bc$ implies $b \neq 0 \neq c$. So we get equality $g^W = \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \beta \\ 1 & \delta \end{smallmatrix}\right)$ with c replacing a and $\beta, \delta \in \mathbb{F}_q$. Recall from [11, Ex.1, P. 104] that c_{21} is a Dickson invariant of $SL(V)^{GL(2,\mathbb{F}_q)}$. Consequently if $a \neq 0$ we get

$$g^W(c_{21}) = (aI_W)(h(c_{21})) = a^{q^2-q}c_{21}.$$

If $a^q = a, b^q = b, c^q = c, d^q = d$, we get

$$\begin{aligned} g^W(u) &= g^W(x)g^W(y)^q - g^W(y)g^W(x)^q \\ &= (ax+by)(cx+dy)^q - (cx+dy)(ax+by)^q \\ &= (ad-bc)(xy^q - yx^q) = det(g^W)u. \end{aligned}$$

Let $M = \mathbb{F}u + \mathbb{F}c_{21}$ be the 2−dimensional subspace of the polynomial ring $S(W)^{T(G)} = \mathbb{F}[u, c_{21}]$, $m = (u, c_{21})$, so $m/m^2 \cong M$. Then the matrix representing g^M, the restriction of g on M, with respect to the basis $\{u, c_{21}\}$ is $\left(\begin{smallmatrix} det(g^W) & 0 \\ 0 & a^{q^2-q} \end{smallmatrix}\right)$. Therefore since $a^q = a$, the condition of Proposition 3.1 : $det(g^W) = det(g^{m/m^2}) = det(g^M)$ is equivalent to:

$$det(g^W) = det(g^W)a^{q^2-q} = a^{q^2-1}det(g^W).$$

Hence it is equivalent to $a^{q^2-1} = 1$, namely $a \in \mathbb{F}_{q^2}$. Using $h \in GL(2, \mathbb{F}_q)$, this is also equivalent to $g^W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{F}_{q^2})$. A similar conclusion is obtained if $a = 0$. □

We next consider the above item (ii).

Proposition 3.3. *Suppose $G \subset SL(3, \mathbb{F})$ is a finite group with $T_W(G) \cong SL(2, \mathbb{F}_5), T_W(G) \subseteq SL(2, \mathbb{F}_9)$ and $\mathbb{F}_9 \subseteq F$. Then $S(V)^G$ is Gorenstein if and only if $G_W \subseteq < T_W(G), \eta I_W >$, where η is a 20^{th}-primitive root of unity.*

Proof. Recall from [9] that

$$S(W)^{SL(2,\mathbb{F}_5)} = \mathbb{F}[f_{10}, f_{12}],$$

where

$$\begin{aligned} f_{10} &= x_1^9x_2 - x_1x_2^9, \\ f_{12} &= x_1^{12} + x_1^{10}x_2^2 - x_1^6x_2^6 + x_1^2x_2^{10} - x_2^{12}. \end{aligned}$$

Let $g \in G$, choose $\xi \in \bar{\mathbb{F}}$ such that $\xi^2 det(g^W) = 1$. Hence $\hat{g}^W := \xi I_W g^W \in SL(2, \bar{\mathbb{F}})$ where $\bar{\mathbb{F}}$ is the algebraic closure of F. It is proved in [3, Proposition 3.7] that the normalizer of $T_W(G)$ in $SL(2,\mathbb{F})$ is $T_W(G)$. This is true for any field $\mathbb{F}_q \subset F$ and in particular for $\bar{\mathbb{F}}$. Now $\hat{g}^W$ normalizes $T_W(G)$ since g^W and ξI_W are such. Also $\hat{g}^W \in SL(2,\bar{\mathbb{F}})$, so $\hat{g}^W \in T_W(G)$. Hence $\left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi \end{smallmatrix}\right) = \hat{g}^W (g^{-1})^W \in (T_W(G))(G_W) = G_W$.

Consequently $G_W \subseteq \langle T_W(G), \left(\begin{smallmatrix} \eta & 0 \\ 0 & \eta \end{smallmatrix}\right) | \eta \in \bar{\mathbb{F}}\rangle$. Since $\left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi \end{smallmatrix}\right) g^W = \hat{g}^W \in T_W(G) = SL(2,\mathbb{F}_5)$, we get that $\hat{g}^W(f_{10}) = f_{10}$ and $\hat{g}^W(f_{12}) = f_{12}$, which implies that $g^W(f_{10}) = \xi^{-10} f_{10}$ and $g^W(f_{12}) = \xi^{-12} f_{12}$. Let $U = \mathbb{F}f_{10} + \mathbb{F}f_{12}$ be the 2−dimensional subspace of the polynomial ring $S(W)^{SL(2,\mathbb{F}_5)} = \mathbb{F}[f_{10}, f_{12}]$, $m = (f_{10}, f_{12})$, so $m/m^2 \cong U$. Then g^U, the matrix representing the restriction of g on U with respect to the basis $\{f_{10}, f_{12}\}$ is $\left(\begin{smallmatrix} \xi^{-10} & 0 \\ 0 & \xi^{-12} \end{smallmatrix}\right)$. Therefore, by Proposition 3.1 , $det(g^W) = det(g^{m/m^2}) = det(g^U)$ is translated into $\xi^{-2} = \xi^{-22}$ or $\xi^{20} = 1$. Therefore $S(V)^G$ is Gorenstein if and only if $G_W \subseteq \langle T_W(G), \eta I_W \rangle$, where $\eta \in \bar{\mathbb{F}}$ is a primitive 20^{th} root of unity. □

Case 2: W is a reducible T(G)-module

We now consider case 2, namely the possibility of W being a reducible $T(G)$-module.

Proposition 3.4. *Let $G \subset SL(3,\mathbb{F})$ be a finite group. Assume that W is a reducible $T(G)$-submodule. Then $S(V)^G$ is Gorenstein if and only if*

$$G_W \subseteq \{\left(\begin{smallmatrix} a & e \\ 0 & d \end{smallmatrix}\right) | a^{p^n-1} = 1, a, e, d \in \mathbb{F}\},$$

where p is the characteristic of $\mathbb{F}$ and

$$T_W(G) = \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_n \rangle.$$

Proof. Let $g^W = \left(\begin{smallmatrix} a & e \\ c & d \end{smallmatrix}\right) \in G_W \subset GL(2,W)$, and set

$$\hat{g}^W := \xi I_W g^W,$$

with $det(\hat{g}^W) = \xi^2 det(g^W) = 1$. Since W is a reducible $T_W(G)$-module, there exists a basis $\{x_1, x_2\}$ in which $T_W(G) \subseteq \left(\begin{smallmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{smallmatrix}\right)$. The computations of A.Braun in [3, Theorem 3.1, p. 244-245] with $\hat{g}^W = \left(\begin{smallmatrix} \xi a & \xi e \\ \xi c & \xi d \end{smallmatrix}\right)$ show that $\xi c = 0$, hence $c = 0$ and $\hat{g}^W = \left(\begin{smallmatrix} \xi a & \pi \\ 0 & \xi d \end{smallmatrix}\right)$, where $\pi \in \mathbb{F}$, and $\hat{g}^{m/m^2} = \left(\begin{smallmatrix} (\xi a)^{p^n} & 0 \\ 0 & \xi d \end{smallmatrix}\right)$. We have by [3, Theorem 3.1, p. 244] that $T_W(G) = \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_n \rangle$ is elementary abelian group, where σ_i is a transvection of the form $\left(\begin{smallmatrix} 1 & \alpha_i \\ 0 & 1 \end{smallmatrix}\right)$, for $i = 1, ..., n$, and $\{\alpha_1, ..., \alpha_n\}$ are linearly independent over $\mathbb{F}_p$. Here $m = (b, x_2)$ is the maximal homogenous ideal of $S(W)^{T(G)}$, where b is defined in [3, Theorem 3.1, p. 245] as an invariant of degree p^n. Recall now that $g^W = \left(\begin{smallmatrix} a & e \\ 0 & d \end{smallmatrix}\right)$, and $det(g^W) = ad$. Hence we have the following:

$$\begin{cases} \hat{g}^W(b) = (\xi a)^{p^n} b = \xi^{p^n} a^{p^n} b \\ g^W(b) = (\xi^{-1} I_W) \hat{g}^W(b) = \xi^{-p^n} \xi^{p^n} a^{p^n} b = a^{p^n} b \\ \hat{g}^W(x_2) = (\xi d)(x_2) \\ g^W(x_2) = (\xi^{-1} I_W \hat{g}^W)(x_2) = (\xi^{-1} \xi d) x_2 = d x_2 \end{cases}$$

Therefore $g^{m/m^2} = \left(\begin{smallmatrix} a^{p^n} & 0 \\ 0 & d \end{smallmatrix}\right)$. Consequently

$$det(g^{m/m^2}) = a^{p^n} d = a^{p^n-1}(ad) = a^{p^n-1} det(g^W).$$

Hence, by Proposition 3.1 $S(V)^G$ is Gorenstein if and only if $a^{p^n-1} = 1$. In other words $S(V)^G$ is Gorenstein if and only if $G_W \subseteq \{\left(\begin{smallmatrix} a & e \\ 0 & d \end{smallmatrix}\right) | a^{p^n-1} = 1, b, e \in \mathbb{F}\}$. □

Case 3: W is an irreducible and imprimitive T(G)-module

It is known that in this case, $T_W(G)$ is a monomial subgroup (see [13]). Recall that $H \subset GL(W)$ is called monomial if W has a basis with respect to which the matrix of each element of H has exactly one non-zero entry in each row and column. If $char F \neq 2$, then by [3, Lemma 3.9.] $T_W(G) = 1$. This contradicts the assumption that W is an irreducible $T_W(G)$−module. Hence we only deal with $p = 2$.

Proposition 3.5. *Assume that $p = 2$. Let $G \subset SL(3,\mathbb{F})$ be a finite group with $T_W(G)$ acting imprimitively and irreducibly on the submodule W. Then $S(V)^G$ is Gorenstein if and only if $G_W \subseteq < T_W(G), \delta I_W >$, where δ is a primitive $d-th$ root of unity and d is defined in the proof.*

Proof. Let $\{x_1, x_2\}$ be a basis of W. Let $\{g_1, ..., g_n\}$ be the set of transvections generating $T_W(G)$. Since W is an irreducible $T(G)$-module, $n \geq 2$. Since g_i is a monomial, then either $g_i = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$, for $i = 1, ..., n$. The first possibility leads, as in [3, Lemma 3.9.] to $g_i = I_W$. So $g_i = \left(\begin{smallmatrix} 0 & \alpha_i \\ \alpha_i^{-1} & 0 \end{smallmatrix}\right)$, for $i = 1, ..., n$, with respect to the basis $\{x_1, x_2\}$ of W. Let n_{ij} be the minimal number such that $(g_i g_j)^{n_{ij}} = 1$, where $i \neq j$, clearly $g_i^{-1} = g_i$ for $i = 1, ..., n$. Also $n_{ij} = n_{ji} > 1$, and since char $\mathbb{F} = 2$, n_{ij} is odd. Recall that $S(W)^{T(G)}$ is a polynomial ring [10, Theorem 2.4]. We next compute the actual generators of $S(W)^{T(G)}$.

Let $[(T_W(G)), (T_W(G))] = \langle g_i g_j g_i g_j \rangle$ be the commutator subgroup of $T_W(G)$. We have $(g_1 g_2)^{n_{12}} = 1$. Therefore $g_2 = (g_1 g_2)^{n_{12}} g_2 = (g_1 g_2)^{n_{12}-1}(g_1 g_2) g_2 = (g_1 g_2)^{n_{12}-1} g_1$. But since n_{12} is odd, $n_{12} - 1$ is even. Hence

$$g_2 = (g_1 g_2 g_1 g_2)^{\frac{n_{12}-1}{2}} g_1 \in [T_W(G), T_W(G)] g_1.$$

This similarly holds for $g_i, i \geq 3$, hence

$$\frac{|(T_W(G))|}{|([T_W(G), T_W(G)])|} = 2.$$

Let $d = lcm\{n_{ij}|i < j\}$. Since

$$g_i g_j g_i g_j = \begin{pmatrix} \alpha_i\alpha_j^{-1}\alpha_i\alpha_j^{-1} & 0 \\ 0 & \alpha_i^{-1}\alpha_j\alpha_i^{-1}\alpha_j \end{pmatrix},$$

it follows that $[(T_W(G)),(T_W(G))] = \{\left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1}\end{smallmatrix}\right)|\zeta^d = 1\}$. Since $(\alpha_i\alpha_j^{-1})^{n_{ij}} = 1$ it follows that $\alpha_i^{n_{ij}} = \alpha_j^{n_{ij}}$. Consequently $\alpha_i^d := \beta$ for $i = \{1,2,3,..,n\}$.

Now we have:

1. $g_i(x_1^d + \alpha_i^d x_2^d) = g_i(x_1)^d + \alpha_i^d g_i(x_2)^d$
   $= \alpha_i^d x_2^d + \alpha_i^d(\alpha_i^{-1}x_1)^d = x_1^d + \alpha_i^d x_2^d$.
2. $g_i(x_1x_2) = g_i(x_1)g_i(x_2) = \alpha_i x_2 \cdot \alpha_i^{-1}x_1 = x_1x_2$.

Hence $\{x_1x_2, x_1^d + \beta x_2^d\} \in S(W)^{T(G)}$ and

$$\deg(x_1x_2)\deg(x_1^d + \beta x_2^d) = 2d = |T_W(G)|.$$

So by [5, Theorem 3.7.5] we get that

$$S(W)^{T(G)} = \mathbb{F}[x_1x_2, x_1^d + \beta x_2^d]$$

is a polynomial ring. Let $g^W = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and set $\hat{g}^W = \xi I_W g^W$, where $\xi^2 det(g^W) = 1$. Now the computations of A. Braun in the proof of [3, Proposition 3.10], with $\hat{g}^W \in SL(2,\mathbb{F})$ show that $\hat{g}^W \in T_W(G)$. Therefore we have:

(i) $g^W(x_1x_2) = \left(\begin{smallmatrix} \xi^{-1} & 0 \\ 0 & \xi^{-1}\end{smallmatrix}\right)\hat{g}^W(x_1x_2) = \xi^{-2}x_1x_2$.

(ii) $g^W(x_1^d + \beta x_2^d) = \left(\begin{smallmatrix} \xi^{-1} & 0 \\ 0 & \xi^{-1}\end{smallmatrix}\right)\hat{g}^W(x_1^d + \beta x_2^d)$
$= \left(\begin{smallmatrix} \xi^{-1} & 0 \\ 0 & \xi^{-1}\end{smallmatrix}\right)(x_1^d + \beta x_2^d) = \xi^{-d}(x_1^d + \beta x_2^d)$.

Consequently, $g^{m/m^2} = \left(\begin{smallmatrix} \xi^{-2} & 0 \\ 0 & \xi^{-d}\end{smallmatrix}\right)$, where

$$m/m^2 = U = \mathbb{F}(x_1x_2) + \mathbb{F}(x_1^d + \beta x_2^d).$$

So

$$det(g^{m/m^2}) = \xi^{-2}\xi^{-d} = \det(g^W)\xi^{-d},$$

and $\det(g^W) = det(g^{m/m^2})$ if and only if $\xi^{-d} = 1$. Consequently, by Proposition 3.1 $S(V)^G$ is Gorenstein if and only if $G_W \subseteq \langle T_W(G), \delta I_W\rangle$ where δ is a primitive $d-th$ root of unity.

□

Corollary 3.6. *Suppose* $G \subset SL(3,\mathbb{F}_p)$, *and* $V = \mathbb{F}_p v \oplus W$ *a decomposition of* $V = \mathbb{F}_p^3$ *into G-submodules. Then* $S(V)^G$ *is Gorenstein.*

Proof. This is a consequence of Proposition 3.2, Proposition 3.3, Proposition 3.4 and Proposition 3.5. □

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