Graduate J. Math. 9, Issue 1 (2024), 40 – 50

# Stochastic comparisons of fail-safe systems comprising dependent components with Archimedean copula as joint distribution

Ali Akbar Hosseinzadeh, Ghobad Saadat Kia (Barmalzan) and Narayanaswamy Balakrishnan

**Abstract**

In this paper, we compare the lifetimes of two fail-safe systems consisting of dependent components with Archimedean copula as joint distribution by means of usual stochastic, hazard rate and likelihood ratio orderings. We also present some examples to illustrate the established results.

*Keywords:* Stochastic Orders; Fail-Safe Systems; Weak Supermajorization Order; Archimedean Copula.

MSC 2020: 90B25, 60E15

## 1 Introduction

Fail-safe systems have been commonly adopted in many day-to-day applications of reliability structures. A fail-safe is specifically a design feature that, when a failure occurs, will respond in a way that no harm happens to the system itself. The brake system on a railway train is a good example of a fail-safe system in which the brakes are held in off-position by air pressure and if a brake line splits or a carriage becomes de-coupled, the air pressure will be lost and the brakes get applied by a local air reservoir. Another example of a fail-safe system is an elevator in which brakes are held off brake pads by tension and if the tension gets lost, the brakes latch on the rails in the shaft thus preventing the elevator from falling. There are many other fail-safe systems in common use, of course.

A $k$-out-of-$n$ system, with $n$ components, would work iff at least $k$ components work; it includes parallel, fail-safe and series systems all as special cases with $k = 1$, $k = n-1$ and $k = n$, respectively. If $X_1, \cdots, X_n$ denote the lifetimes of components of a system and $X_{1:n} \le \cdots \le X_{n:n}$ the corresponding order statistics, then $X_{n-k+1:n}$ is evidently the lifetime of the $k$-out-of-$n$ system. Hence, the theory of order statistics becomes essential for studying $(n-k+1)$-out-of-$n$ systems. For detailed discussions on order statistics and their applications, interested readers may refer to the handbooks on order statistics by Balakrishnan and Rao (1998a,b).

Balakrishnan et al. (2015) established necessary and sufficient conditions for comparing two fail-safe systems with independent homogeneous exponential components, in the sense of mean residual life, dispersive, hazard rate and likelihood ratio orders. Their results specifically show how one can compare an $(n-1)$-out-of-$n$ system consisting of heterogeneous components with exponential lifetimes with any $(m-1)$-out-of-$m$ system consisting of homogeneous components with exponential lifetimes. In a similar vein, Zhang et al. (2018) presented sufficient (and necessary) conditions on lifetimes of components and their survival probabilities from random shocks for comparing the lifetimes of two fail-safe systems by means of usual stochastic, hazard rate and likelihood ratio orders.

In this work, we consider fail-safe systems in which the components are dependent with their joint distribution being an Archimedean copula. We then compare the lifetimes of two such fail-safe systems in this general setting in terms of usual, hazard rate and likelihood ratio orderings.

The rest of this paper proceeds as follows. In Section 2, we first briefly review some basic concepts and notions that are used in the subsequent sections. In Section 3, we discuss the usual stochastic order of fail-safe systems with dependent components. The hazard rate order of these systems is then discussed in Section 4. In Section 5, the lifetimes of two fail-safe systems are compared by means of the likelihood ratio order. Finally, some concluding remarks are made in Section 6. Some examples are presented through out to illustrate all the results established in this work.

## 2 Preliminaries

We briefly introduce in this section some known concepts about stochastic orders, majorization and copulas. Throughout the discussion here, we shall use 'increasing' to mean 'non-decreasing', and similarly 'decreasing' to mean 'non-increasing'.

### 2.1 Stochastic Orders

Suppose $X$ and $Y$ are two non-negative random variables with density functions $f_X$ and $f_Y$, distribution functions $F_X$ and $F_Y$, survival functions $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$, and hazard rates $r_X = f_X/\bar{F}_X$ and $r_Y = f_Y/\bar{F}_Y$, respectively.

**Definition 2.1.** Let $X$ and $Y$ be two non-negative continuous random variables. Then, $X$ is said to be smaller than $Y$ in the

(i) usual stochastic order (denoted by $X \le_{st} Y$) if $\bar{F}_X(x) \le \bar{F}_Y(x)$ for all $x \in \mathbb{R}^+$, which is equivalent to saying that $\mathbb{E}(\phi(X)) \le \mathbb{E}(\phi(Y))$ for all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$, when the involved expectations exist;

(ii) hazard rate order (denoted by $X \le_{hr} Y$) if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}^+$. In fact, $X \le_{hr} Y$ if and only if $r_Y(x) \le r_X(x)$ for all $x \in \mathbb{R}^+$;

(iii) likelihood ratio order (denoted by $X \le_{lr} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}^+$.

The implications

$$X \le_{lr} Y \Longrightarrow X \le_{hr} Y \Longrightarrow X \le_{st} Y$$

are well-known in the literature. One may refer to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) for extensive discussions on various stochastic orderings, their properties and applications.

### 2.2 Majorization Order

**Definition 2.2.** Let $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ be two vectors with increasing arrangements $a_{(1)} \le \cdots \le a_{(n)}$ and $b_{(1)} \le \cdots \le b_{(n)}$, respectively. Then:

(i) Vector $\mathbf{a}$ is said to be majorized by vector $\mathbf{b}$ (denoted by $\mathbf{a} \overset{m}{\preceq} \mathbf{b}$) if $\sum_{j=1}^{i} a_{(j)} \ge \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \cdots, n-1$, and $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$;

(ii) Vector $\mathbf{a}$ is said to be weakly supermajorized by vector $\mathbf{b}$ (denoted by $\mathbf{a} \overset{w}{\preceq} \mathbf{b}$) if $\sum_{j=1}^{i} a_{(j)} \ge \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \cdots, n$.

**Definition 2.3.** A real-valued function $\phi$, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$, is said to be Schur-convex (Schur-concave) on $\mathbb{A}$ if $\mathbf{a} \overset{m}{\preceq} \mathbf{b}$ implies $\phi(\mathbf{a}) \leq (\geq)\phi(\mathbf{b})$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{A}$.

One may refer to Marshall et al. (2011) for an elaborate discussion on majorization and Schur functions. The necessary and sufficient conditions for the characterization of Schur-convex and Schur-concave functions are presented in the following lemma, as presented in Marshall et al. (2011, p. 84).

**Lemma 2.4.** *Suppose $J \subset \mathbb{R}$ is an open interval and $\phi : J^n \to \mathbb{R}$ is continuously differentiable. Then, necessary and sufficient conditions for $\phi$ to be Schur-convex (Schur-concave) on $J^n$ are*

*(i) $\phi$ is symmetric on $J^n$;*

*(ii) for all $i \neq j$ and all $\mathbf{z} \in J^n$,*

$$(z_i - z_j)\left(\frac{\partial\phi(\mathbf{z})}{\partial z_i} - \frac{\partial\phi(\mathbf{z})}{\partial z_j}\right) \geq 0\,(\leq 0),$$

*where $\partial\phi(\mathbf{z})/\partial z_i$ denotes the partial derivative of $\phi$ with respect to its $i$-th argument.*

The following lemma, taken from Marshall et al. (2011, p. 87), presents some conditions for the characterization of vector functions preserving weak supermajorization order.

**Lemma 2.5.** *Consider the real-valued function $\varphi$, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$. Then, $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$ implies $\varphi(\boldsymbol{a}) \geq \varphi(\boldsymbol{b})$ if and only if $\varphi$ is decreasing and Schur-convex on $\mathbb{A}$.*

## 2.3 Archimedean Copula

Numerous stochastic comparisons between univariate random variables have been defined and discussed in many different contexts, as can be seen in the books of Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). Most of them are based on independence of underlying random variables. Recently, some authors have established stochastic ordering results by considering the random variables to be dependent with an Archimedean copula as joint distribution.

Archimedean copulas possess mathematical tractability and also have the ability to capture a wide range of dependence. For a decreasing and continuous function $\psi : [0, \infty) \longrightarrow [0, 1]$ such that $\psi(0) = 1$ and $\psi(+\infty) = 0$ and $\phi = \psi^{-1}$ being the pseudo-inverse,

$$C_\psi(u_1, \cdots, u_n) = \psi(\phi(u_1) + \cdots + \phi(u_n)) \qquad for\ all\ u_i \in [0, 1],\ \ i = 1, \cdots, n,$$

is said to be an Archimedean copula with generator $\psi$ if $(-1)^k\psi^{[k]}(x) \geq 0$ for $k = 0, \cdots, n-2$, and $(-1)^{n-2}\psi^{[n-2]}(x)$ is decreasing and convex. Here, $\psi^{[k]}(x)$ denotes the $k$-th derivative of the function $\psi(x)$. This copula family includes many well-known copulas such as independence (product) copula, Clayton copula, Gumbel-Hougaard copula and Ali-Mikhail-Haq (AMH) copula.

A function $f$ is said to be superadditive if $f(x + y) \geq f(x) + f(y)$ for all $x$ and $y$ in the domain of $f$. Then, based on Lemma A.1 of Li and Fang (2015), it is known that for two $n$-dimensional Archimedean copulas $C_{\psi_1}(u)$ and $C_{\psi_2}(u)$ with respective generators $\psi_1$ and $\psi_2$ and pseudo-inverses $\phi_1$ and $\phi_2$, if $\phi_2 \circ \psi_1$ is supperadditive, then $C_{\psi_1}(u) \leq C_{\psi_2}(u)$ for all $u \in [0, 1]^n$. Interested readers may refer to Nelsen (2006) for elaborate discussion on copulas, their properties and applications.

# 3 Usual Stochastic Order

In this section, we establish the usual stochastic order of fail-safe systems with dependent components having Archimedean copula as joint distribution.

For the results in this section, we consider the following general set-up. We have a fail-safe system with $n$ dependent components whose joint distribution is an Archimedean copula described earlier in

Section 2.3. Moreover, the marginal survival functions of the lifetimes of the components are given by $\bar{F}^{\alpha_i}(x)$, for $\alpha_i > 0,\ i = 1, \cdots, n$, where $\bar{F}(x)$ is some baseline survival function. Note that this specification of marginal distributions corresponds to proportional hazard model, denoted by PH here, as the corresponding hazard functions are given by $\alpha_i r_F(x)$, where $r_F(x) = \frac{f(x)}{\bar{F}(x)}$ is the hazard function of the baseline distribution $F(x)$; see, for example, Marshall and Olkin (2007).

We than have the following result for the usual stochastic ordering between the lifetimes of the two fail-safe systems in the general set-up outlined above.

**Theorem 3.1.** *Let $X_i \sim PH(\alpha_i)$ $(i = 1, \cdots, n)$ have their joint distribution as Archimedean copula with generator $\psi_1$ and $Y_i \sim PH(\beta_i)$ have their joint distribution as Archimedean copula with generator $\psi_2$. Further, suppose $\phi_2 \circ \psi_1$ is superadditive and $t\phi_1'(t)$ is an increasing function. Then,*

$$(\beta_1, \cdots, \beta_n) \overset{w}{\succeq} (\alpha_1, \cdots, \alpha_n) \Longrightarrow X_{2:n} \geq_{st} Y_{2:n}.$$

**Proof.** The survival functions of $X_{2:n}$ and $Y_{2:n}$ are given by

$$\bar{F}_{X_{2:n}}(x) = \sum_{l=1}^{n} \psi_1 \left( \sum_{k=1, k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x)) \right) - (n-1)\ \psi_1 \left( \sum_{k=1}^{n} \phi_1(\bar{F}^{\alpha_k}(x)) \right), \qquad x > 0,$$

$$\bar{F}_{Y_{2:n}}(x) = \sum_{l=1}^{n} \psi_2 \left( \sum_{k=1, k\neq l}^{n} \phi_2(\bar{F}^{\beta_k}(t)) \right) - (n-1)\ \psi_2 \left( \sum_{k=1}^{n} \phi_2(\bar{F}^{\beta_k}(x)) \right), \qquad x > 0,$$

respectively. The superadditivity of $\phi_2 \circ \psi_1$ implies that

$$\begin{aligned} I(\phi_1, \boldsymbol{\beta}) &= \sum_{l=1}^{n} \psi_1 \left( \sum_{k=1, k\neq l}^{n} \phi_1(\bar{F}^{\beta_k}(x)) \right) - (n-1)\ \psi_1 \left( \sum_{k=1}^{n} \phi_1(\bar{F}^{\beta_k}(x)) \right) \\ &\leq \sum_{l=1}^{n} \psi_2 \left( \sum_{k=1, k\neq l}^{n} \phi_2(\bar{F}^{\beta_k}(x)) \right) - (n-1)\ \psi_2 \left( \sum_{k=1}^{n} \phi_2(\bar{F}^{\beta_k}(x)) \right) \\ &= I(\phi_2, \boldsymbol{\beta}). \end{aligned}$$

So, to prove the desired result, it is sufficient to show that $I(\phi_1, \boldsymbol{\alpha}) \leq I(\phi_1, \boldsymbol{\beta})$. According to Lemma 2.5, we only need to show that $I(\phi_1, \boldsymbol{\alpha}) = \sum_{l=1}^{n} \psi_1 \left( \sum_{k=1, k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x)) \right) - (n-1)\ \psi_1 \left( \sum_{k=1}^{n} \phi_1 \left( \bar{F}^{\alpha_k}(x) \right) \right)$ is decreasing and Schur-convex in $(\alpha_1, \cdots, \alpha_n)$, for any fixed $x > 0$. Taking the derivative of $I(\phi_1, \boldsymbol{\alpha})$ with respect to $\alpha_j$, we have

$$\begin{aligned} \frac{\partial I(\phi_1, \boldsymbol{\alpha})}{\partial \alpha_j} &= \bar{F}^{\alpha_j}(x) \ln \left( \bar{F}(x) \right) \phi_1'(\bar{F}^{\alpha_j}(x)) \\ &\times \left[ \sum_{l=1, l\neq j}^{n} \psi_1' \left( \sum_{k=1, k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x)) \right) - (n-1)\ \psi_1' \left( \sum_{k=1}^{n} \phi_1 \left( \bar{F}^{\alpha_k}(x) \right) \right) \right]. \end{aligned} \tag{3.1}$$

Because $\sum_{k=1, k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x)) \leq \sum_{k=1}^{n} \phi_1 \left( \bar{F}^{\alpha_k}(x) \right)$ and $\psi'(x)$ is an increasing function, we have

$$\psi_1' \left( \sum_{k=1, k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x)) \right) = \psi_1' \left( \sum_{k=1}^{n} \phi_1 \left( \bar{F}^{\alpha_k}(x) \right) \right), \ \forall l \in \{1, 2, \ldots, n\}/\{j\}, \tag{3.2}$$

which implies that

$$\sum_{l=1, l\neq j}^{n} \psi_1' \left( \sum_{k=1, k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x)) \right) \leq (n-1)\ \psi_1' \left( \sum_{k=1}^{n} \phi_1 \left( \bar{F}^{\alpha_k}(x) \right) \right). \tag{3.3}$$

In view of (3.3), from (3.1), we have $\frac{\partial I(\phi_1,\boldsymbol{\alpha})}{\partial \alpha_j} \leq 0$, which implies that $I(\phi_1,\boldsymbol{\alpha})$ is a decreasing function. Also,

$$\begin{aligned} A(x) &= \left(\frac{\partial I(\phi_1,\boldsymbol{\alpha})}{\partial \alpha_j} - \frac{\partial I(\phi_1,\boldsymbol{\alpha})}{\partial \alpha_i}\right) \\ &= \bar{F}^{\alpha_j}(x)\ \ln\left(\bar{F}(x)\right) \phi_1'(\bar{F}^{\alpha_j}(x)) \left[\sum_{l=1,l\neq j}^{n} \psi_1'\left(\sum_{k=1,k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x))\right) - (n-1)\ \psi_1'\left(\sum_{k=1}^{n} \phi_1\left(\bar{F}^{\alpha_k}(x)\right)\right)\right] \\ &\quad -\bar{F}^{\alpha_i}(x)\ \ln\left(\bar{F}(x)\right) \phi_1'(\bar{F}^{\alpha_i}(x)) \left[\sum_{l=1,l\neq i}^{n} \psi_1'\left(\sum_{k=1,k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x))\right) - (n-1)\ \psi_1'\left(\sum_{k=1}^{n} \phi_1\left(\bar{F}^{\alpha_k}(x)\right)\right)\right]. \end{aligned}$$

If $\alpha_i > (<)\alpha_j$, since $\phi'(x) \leq 0$ and $\psi''(x) \geq 0$, we get

$$\sum_{l=1,l\neq j}^{n} \psi_1'\left(\sum_{k=1,k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x))\right) \ \leq (\geq) \ \sum_{l=1,l\neq i}^{n} \psi_1'\left(\sum_{k=1,k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x))\right).$$

Therefore, we have

$$\begin{aligned} A(x) \ &\leq (\geq) \ \ln\left(\bar{F}(x)\right) \left(\bar{F}^{\alpha_j}(x)\phi_1'(\bar{F}^{\alpha_j}(x)) - \bar{F}^{\alpha_i}(x)\phi_1'(\bar{F}^{\alpha_i}(x))\right) \\ &\quad \times \left[\sum_{l=1,l\neq i}^{n} \psi_1'\left(\sum_{k=1,k\neq l}^{n} \phi_1(\bar{F}^{\alpha_k}(x))\right) - (n-1)\ \psi_1'\left(\sum_{k=1}^{n} \phi_1\left(\bar{F}^{\alpha_k}(x)\right)\right)\right]. \end{aligned}$$

Hence, if $t\phi_1'(t)$, for $t \in (0,1]$, is an increasing function, then $A(x) \leq (\geq)\, 0$, and consequently

$$(\alpha_j - \alpha_i)\left(\frac{\partial I(\phi_1,\boldsymbol{\alpha})}{\partial \alpha_j} - \frac{\partial I(\phi_1,\boldsymbol{\alpha})}{\partial \alpha_i}\right) \geq 0, \qquad \forall i \neq j,$$

as required. $\square$

**Remark 3.2.** It should be mentioned that the condition "$\phi_2 \circ \psi_1$ is superadditive " in Theorem 3.1 is quite general and hold for many Archimedean copulas. For example, consider the Gumbel-Hougaard copula with generator $\psi(t) = e^{1-(1+t)^{\theta}}$, for $\theta \in (0,\infty)$, and set $\psi_1(t) = e^{1-(1+t)^{\alpha}}$ and $\psi_2(t) = e^{1-(1+t)^{\beta}}$. It can be observed that $\phi_2 \circ \psi_1(t) = (1+t)^{\alpha/\beta-1}$, and differentiating it twice with respect to $t$, we obtain $[\phi_2 \circ \psi_1(t)]'' = (\frac{\alpha}{\beta})(\frac{\alpha}{\beta}-1)(1+t)^{\alpha/\beta-1} \geq 0$ for $\alpha > \beta$, implying the superadditivity of $\phi_2 \circ \psi_1(t)$. $\square$

**Remark 3.3.** It may also be noted that the condition " $t\phi_1'(t)$ is increasing" in Theorem 3.1 is quite general and holds for many Archimedean copulas. For example, we observe the following:

(i) If $\phi(t) = (1-lnt)^{\frac{1}{\beta}} - 1$, we have $t\phi'(t) = -\frac{1}{\beta}(1-lnt)^{\frac{1}{\beta}-1}$ is increasing in $t \in (0,1]$, for $\beta \in (0,1]$

(ii) If $\phi_1(t) = (-lnt)^{\theta}$, $\theta \geq 1$, we have $t\phi_1'(t) = -\theta(-lnt)^{\theta-1}$ to be increasing in $t \in (0,1]$;

(iii) If $\phi_1(t) = \frac{t^{-\theta}-1}{\theta}$, $\theta > 0$, we have $t\phi_1'(t) = -t^{-\theta}$ to be increasing in $t \in (0,1]$;

(iv) If $\phi_1(t) = e^{-t^{-\theta}} - e$, $\theta > 0$, we have $t\phi_1'(t) = -\theta t^{-\theta} e^{-t^{-\theta}}$ to be increasing in $t \in (0,1]$.

Suppose $X_i \sim Exp(\alpha_i)$ $(i = 1,2,3)$ and $Y_i \sim Exp(\beta_i)$ $(i = 1,2,3)$. Set $(\alpha_1,\alpha_2,\alpha_3) = (6,7,9)$ and $(\beta_1,\beta_2,\beta_3) = (3,5,6)$. It is then easy to observe that $(\alpha_1,\alpha_2,\alpha_3) \overset{w}{\preceq} (\beta_1,\beta_2,\beta_3)$. Now, let us consider the Gumbel-Hougaard copula with parameters $\theta_1 = 0.5$ and $\theta_2 = 0.2$. Then, the survival functions of $X_{2:3}$ and $Y_{2:3}$ are given by, respectively,

$$\begin{aligned}
\bar{F}_{X_{2:3}}(x) &= \exp\left\{1 - \left[-1 + (1+\alpha_2 x)^{1/\theta_1} + (1+\alpha_3 x)^{1/\theta_1}\right]^{\theta_1}\right\} \\
&+ \exp\left\{1 - \left[-1 + (1+\alpha_1 x)^{1/\theta_1} + (-1+\alpha_3 x)^{1/\theta_1}\right]^{\theta_1}\right\} \\
&+ \exp\left\{1 - \left[-1 + (1+\alpha_1 x)^{1/\theta_1} + (1+\alpha_2 x)^{1/\theta_1}\right]^{\theta_1}\right\} \\
&- 2\exp\left\{1 - \left[-2 + (1+\alpha_1 x)^{1/\theta_1} + (1+\alpha_2 x)^{1/\theta_1} + (1+\alpha_3 x)^{1/\theta_1}\right]^{\theta_1}\right\},
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{Y_{2:3}}(x) &= \exp\left\{1 - \left[-1 + (1+\beta_2 x)^{1/\theta_2} + (1+\beta_3 x)^{1/\theta_2}\right]^{\theta_2}\right\} \\
&+ \exp\left\{1 - \left[-1 + (1+\beta_1 x)^{1/\theta_2} + (-1+\beta_3 x)^{1/\theta_2}\right]^{\theta_2}\right\} \\
&+ \exp\left\{1 - \left[-1 + (1+\beta_1 x)^{1/\theta_2} + (1+\beta_2 x)^{1/\theta_2}\right]^{\theta_2}\right\} \\
&- 2\exp\left\{1 - \left[-2 + (1+\beta_1 x)^{1/\theta_2} + (1+\beta_2 x)^{1/\theta_2} + (1+\beta_3 x)^{1/\theta_1}\right]^{\theta_2}\right\}.
\end{aligned}$$

Figure 1 plots these survival functions of $X_{2:3}$ and $Y_{2:3}$, from which it can be observed that $\bar{F}_{Y_{2:3}}(x)$

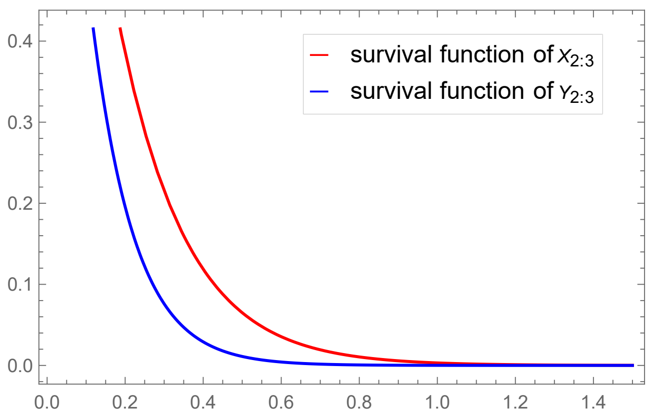


Fig. 1: Plots of survival functions of $X_{2:3}$ and $Y_{2:3}$.

is always below $\bar{F}_{Y_{2:3}}(x)$, for $x > 0$, thus validating the result of Theorem 3.1. □

**Remark 3.4.** It should be mentioned that the superadditivity of $\phi_2 \circ \psi_1$ has a nice interpretation as follows: For two $n$-dimensional Archimedean copulas $C_{\psi_1}(u)$ and $C_{\psi_2}(u)$ with respective generators $\psi_1$ and $\psi_2$ and pseudo-inverses $\phi_1$ and $\phi_2$, if $\phi_2 \circ \psi_1$ is supperadditive, then $C_{\psi_1}(u) \le C_{\psi_2}(u)$ for all $u \in [0,1]^n$. In this case, for many sub-families of Archimedean copulas, the superadditivity of $\phi_2 \circ \psi_1$ can be roughly interpreted as follows: Kendall's $\tau$ of the copula with generator $\psi_2$ is larger than that with generator $\psi_1$, and is therefore more positive dependent.

## 4 Hazard Rate Order

In this section, we establish the hazard rate order of fail-safe systems with dependent components having Archimedean copula as joint distribution. In addition, for convenience, we use $a \overset{sgn}{=} b$ to denote that both sides of an equality have the same sign.

**Theorem 4.1.** *Let $X_i$ be non-negative random variables having common distribution $F_1$ $(i = 1, \cdots, n)$, and $Y_i$ be non-negative random variables having common distribution $F_2$ $(i = 1, \cdots, n)$, with their joint distributions as a common Archimedean copula with generator $\psi$. If*

$$t \ \ln' \left[n\psi\left[(n-1)\phi(t)\right] - \ (n-1)\psi\left[n\phi(t)\right]\right]$$

*is decreasing in $t$, then*

$$X_1 \geq_{hr} X_2 \Longrightarrow X_{2:n} \geq_{hr} Y_{2:n}.$$

**Proof.** The survival functions of $X_{2:n}$ and $Y_{2:n}$ are given by

$$\bar{F}_{X_{2:n}}(x) = n\psi\left[(n-1)\phi(\bar{F}_1(x))\right] - (n-1)\ \psi\left[n\phi(\bar{F}_1(x))\right], \qquad x > 0,$$

$$\bar{F}_{Y_{2:n}}(x) = n\psi\left[(n-1)\phi(\bar{F}_2(x))\right] - (n-1)\ \psi\left[n\phi(\bar{F}_2(x))\right], \qquad x > 0,$$

respectively. For obtaining the required result, it is sufficient to show that the ratio $\bar{F}_{X_{2:n}}(x)/\bar{F}_{Y_{2:n}}(x)$ is increasing in $x \in \mathbb{R}_+$. For this purpose, let us consider

$$\Delta(x) \quad = \quad \frac{\bar{F}_{X_{2:n}}(x)}{\bar{F}_{Y_{2:n}}(x)} = \frac{n\psi\left[(n-1)\phi(\bar{F}_1(x))\right] - (n-1)\ \psi\left[n\phi(\bar{F}_1(x))\right]}{n\psi\left[(n-1)\phi(\bar{F}_2(x))\right] - (n-1)\ \psi\left[n\phi(\bar{F}_2(x))\right]}.$$

The condition $X_1 \geq_{hr} X_2$ implies $r_{F_1}(x) \leq r_{F_2}(x)$ and $\bar{F}_1(x) \geq \bar{F}_2(x)$, for $x \in \mathbb{R}_+$. Therefore,

$$\begin{aligned}
\Delta'(x) \quad &\overset{sgn}{=} \quad r_{F_2}(x)\bar{F}_2(x)\left[\frac{n(n-1)\phi'(\bar{F}_2(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_2(x))\right] - \ \psi'\left[n\phi(\bar{F}_2(x))\right]\right\}}{n\psi\left[(n-1)\phi(\bar{F}_2(x))\right] - \ (n-1)\psi\left[n\phi(\bar{F}_2(x))\right]}\right] \\
&\quad -r_{F_1}(x)\bar{F}_1(x)\left[\frac{n(n-1)\phi'(\bar{F}_1(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_1(x))\right] - \ \psi'\left[n\phi(\bar{F}_1(x))\right]\right\}}{n\psi\left[(n-1)\phi(\bar{F}_1(x))\right] - \ (n-1)\psi\left[n\phi(\bar{F}_1(x))\right]}\right] \\
&\geq \quad r_{F_1}(x)\bar{F}_2(x)\left[\frac{n(n-1)\phi'(\bar{F}_2(x))\left(\left\{\psi'\left[(n-1)\phi(\bar{F}_2(x))\right] - \ \psi'\left[n\phi(\bar{F}_2(x))\right]\right\}\right.}{n\psi\left[(n-1)\phi(\bar{F}_2(x))\right] - \ (n-1)\psi\left[n\phi(\bar{F}_2(x))\right]}\right] \\
&\quad -r_{F_1}(x)\bar{F}_1(x)\left[\frac{n(n-1)\phi'(\bar{F}_1(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_1(x))\right] - \ \psi'\left[n\phi(\bar{F}_1(x))\right]\right\}}{n\psi\left[(n-1)\phi(\bar{F}_1(x))\right] - \ (n-1)\psi\left[n\phi(\bar{F}_1(x))\right]}\right] \\
&\overset{sgn}{=} \quad t\ \ln'\left[n\psi\left[(n-1)\phi(t)\right] - \ (n-1)\psi\left[n\phi(t)\right]\right]\Big|_{t=\bar{F}_2(x)} \\
&\qquad -t\ \ln'\left[n\psi\left[(n-1)\phi(t)\right] - \ (n-1)\psi\left[n\phi(t)\right]\right]\Big|_{t=\bar{F}_1(x)} \\
&\geq \quad 0,
\end{aligned}$$

where the last inequality follows from the decreasing property of $t\ \ln'\left[n\psi\left[(n-1)\phi(t)\right] - \ (n-1)\psi\left[n\phi(t)\right]\right]$ with respect to $t \in (0, 1)$. $\square$

**Remark 4.2.** *It needs to be mentioned that the condition " $t\ \ln'\left[n\psi\left[(n-1)\phi(t)\right] - \ (n-1)\psi\left[n\phi(t)\right]\right]$ is decreasing" in Theorem 4.1 is general and holds for many Archimedean copulas. For example, we observe the following:*

*(i) If $\psi(t) = \frac{1}{2e^t - 1}$, for $n = 2$, we have*

$$t\ \ln'\left[2\psi\left[\phi(t)\right] - \ \psi\left[2\phi(t)\right]\right] = \frac{(t^4 - 4t^3 + 2t + 1)}{(t^2 - 2t - 1)\,(t^2 - t - 1)}$$

*to be increasing in $t \in (0, 1)$;*

*(ii)* *If* $\psi(t) = (\theta t + 1)^{-\frac{1}{\theta}}$, $\theta \in \mathbb{R}^+$, *for* $n = 2$, *we have*

$$t\ \ln'\left[2\psi\left[\phi(t)\right] - \ \psi\left[2\phi(t)\right]\right] = \frac{2t - 2t^{-\theta}\left(2t^{-\theta} - 1\right)^{-\frac{1}{\theta}-1}}{2t - \left(2t^{-\theta} - 1\right)^{-\frac{1}{\theta}}}$$

*to be decreasing in* $t \in (0, 1)$;

*(iii)* *If* $\psi(t) = \frac{1}{\sqrt{t}+1}$, *for* $n = 2$, *we have*

$$t\ \ln'\left[2\psi\left[\phi(t)\right] - \ \psi\left[2\phi(t)\right]\right] = -\sqrt{2}\left(\frac{-\left(\sqrt{2}\left(\sqrt{2}\left(1-t\right)+t\right)^2\right)+1}{\left(\sqrt{2}\left(1-t\right)+x\right)\left(2\left(\sqrt{2}\left(1-t\right)+t\right)-1\right)}\right)$$

*to be decreasing in* $t \in (0, 1)$;

*(iv)* *If* $\psi(t) = e^{-x^{\theta}}$, $\theta \in (0, 1]$, *for* $n = 4$, *we have*

$$t\ \ln'\left[4\psi\left[3\phi(t)\right] - \ 3\psi\left[4\phi(t)\right]\right] = \frac{4\left(3^{\theta}t^{3^{\theta}}\right) - 3\left(4^{\theta}t^{4^{\theta}}\right)}{4t^{3^{\theta}} - 3t^{4^{\theta}}}$$

*to be decreasing in* $t \in (0, 1)$;

*(v)* *If* $\psi(t) = e^{1-(1+t)^{\theta}}$, $\beta \in [1, \infty)$, *for* $n = 3$, *we have*

$$t\ \ln'\left[3\psi\left[2\phi(t)\right] - \ 2\psi\left[3\phi(t)\right]\right] =$$
$$\frac{6\left(1-\ln t\right)^{\frac{1}{\beta}-1}\left(\left(-1+2\left(1-\ln t\right)^{\frac{1}{\beta}}\right)^{\beta-1}e^{\left(1-\left(-1+2(1-\ln t)^{\frac{1}{\beta}}\right)^{\beta}\right)} - \left(-2+3\left(1-\ln t\right)^{\frac{1}{\beta}}\right)^{\beta-1}e^{\left(1-\left(-2+3(1-\ln t)^{\frac{1}{\beta}}\right)^{\beta}\right)}\right)}{3e^{\left(1-\left(-1+2(1-\ln t)^{\frac{1}{\beta}}\right)^{\beta}\right)} - 2e^{\left(1-\left(-2+3(1-\ln t)^{\frac{1}{\beta}}\right)^{\beta}\right)}}$$

*to be decreasing in* $t \in (0, 1)$

Suppose $\bar{F}_1(x) = e^{-4x}$ and $\bar{F}_2(x) = e^{-7x}$, for $x > 0$. Next, consider $\psi(x) = e^{-x^{\theta}}$, $n = 4$ and $\theta = 0.8$. It is then easy to observe that $F_1 \geq_{hr} F_2$. The ratio of the survival functions is

$$\frac{\bar{F}_{X_{2:4}}(x)}{\bar{F}_{Y_{2:4}}(x)} = \frac{4[\exp(-4x)]^{3^{\theta}} - 3[\exp(-4x)]^{4^{\theta}}}{4[\exp(-7x)]^{3^{\theta}} - 3[\exp(-7x)]^{4^{\theta}}}, \qquad x > 0.$$

As seen in Figure 2, the ratio function is monotone for $x > 0$, thus validating the result of Theorem 4.1. $\square$

## 5 Likelihood Ratio Order

In this section, we establish the likelihood ratio order of fail-safe systems with dependent components having Archimedean copula as joint distribution.

**Theorem 5.1.** *Let* $X_i$ *be non-negative random variables having common distribution* $F_1$ $(i = 1, \cdots, n)$, *and* $Y_i$ *be non-negative random variables having common distribution* $F_2$ $(i = 1, \cdots, n)$, *with their joint distributions as a common Archimedean copula with generator* $\psi$. *If* $t\ \ln'\left[\frac{\psi'[(n-1)\phi(t)] - \ \psi'[n\phi(t)]}{\psi'(\phi(t))}\right]$ *is decreasing in* $t$, *then*

$$X_1 \geq_{hr} X_2 \Longrightarrow X_{2:n} \geq_{lr} Y_{2:n}.$$

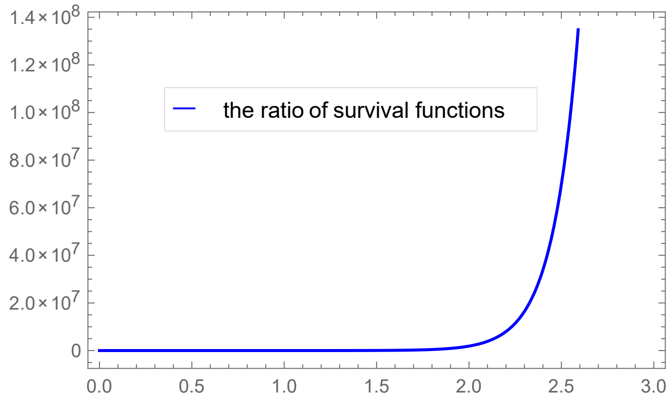


Fig. 2: The ratio of survival functions of $X_{2:4}$ and $Y_{2:4}$.

**Proof.** The density functions of $X_{2:n}$ and $Y_{2:n}$ are given by

$$f_{X_{2:n}}(x) = n(n-1)f_1(x)\phi'(\bar{F}_1(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_1(x))\right] - \psi'\left[n\phi(\bar{F}_1(x))\right]\right\}, \qquad x>0,$$

$$f_{Y_{2:n}}(x) = n(n-1)f_2(x)\phi'(\bar{F}_2(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_2(x))\right] - \psi'\left[n\phi(\bar{F}_2(x))\right]\right\}, \qquad x>0,$$

respectively. For obtaining the desired result, it is sufficient to show that the ratio $f_{X_{2:n}}(x)/f_{Y_{2:n}}(x)$ is increasing in $x \in \mathbb{R}_+$. For this purpose, let us consider

$$\frac{f_{X_{2:n}}(x)}{f_{Y_{2:n}}(x)} = \frac{n(n-1)f_1(x)\phi'(\bar{F}_1(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_1(x))\right] - \psi'\left[n\phi(\bar{F}_1(x))\right]\right\}}{n(n-1)f_2(x)\phi'(\bar{F}_2(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_2(x))\right] - \psi'\left[n\phi(\bar{F}_2(x))\right]\right\}}.$$

It is enough to show that

$$\Phi(x) = \frac{\phi'(\bar{F}_1(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_1(x))\right] - \psi'\left[n\phi(\bar{F}_1(x))\right]\right\}}{\phi'(\bar{F}_2(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_2(x))\right] - \psi'\left[n\phi(\bar{F}_2(x))\right]\right\}}$$

is increasing in $x \in \mathbb{R}_+$. The condition $X_1 \geq_{hr} X_2$ implies that $r_{F_1}(x) \leq r_{F_2}(x)$ and $\bar{F}_1(x) \geq \bar{F}_2(x)$, for $x \in \mathbb{R}_+$. So, let us now set

$$\begin{aligned} A &= \bar{F}_2(x)\Big[\phi''(\bar{F}_2(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_2(x))\right] - \psi'\left[n\phi(\bar{F}_2(x))\right]\right\} \\ &+ \phi'^2(\bar{F}_2(x))\left\{(n-1)\psi''\left[(n-1)\phi(\bar{F}_2(x))\right] - n\psi''\left[n\phi(\bar{F}_2(x))\right]\right\}\Big] \end{aligned}$$

and

$$\begin{aligned} B &= \bar{F}_1(x)\Big[\phi''(\bar{F}_1(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_1(x))\right] - \psi'\left[n\phi(\bar{F}_1(x))\right]\right\} \\ &+ \phi'^2(\bar{F}_1(x))\left\{(n-1)\psi''\left[(n-1)\phi(\bar{F}_1(x))\right] - n\psi''\left[n\phi(\bar{F}_1(x))\right]\right\}\Big]. \end{aligned}$$

Then, we have

$$\begin{aligned}
\Phi'(x) &= r_{F_2}(x)\frac{A}{\phi'(\bar{F}_2(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_2(x))\right]-\ \psi'\left[n\phi(\bar{F}_2(x))\right]\right\}}\\
&\quad -r_{F_1}(x)\frac{B}{\phi'(\bar{F}_1(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_1(x))\right]-\ \psi'\left[n\phi(\bar{F}_1(x))\right]\right\}}\\
&\geq r_{F_1}(x)\frac{A}{\phi'(\bar{F}_2(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_2(x))\right]-\ \psi'\left[n\phi(\bar{F}_2(x))\right]\right\}}\\
&\quad -r_{F_1}(x)\frac{B}{\phi'(\bar{F}_1(x))\left\{\psi'\left[(n-1)\phi(\bar{F}_1(x))\right]-\ \psi'\left[n\phi(\bar{F}_1(x))\right]\right\}}\\
&\overset{sgn}{=} t\ \ln'\left[\frac{\psi'\left[(n-1)\phi(t)\right]-\ \psi'\left[n\phi(t)\right]}{\psi'\left(\phi(t)\right)}\right]\Big|_{t=\bar{F}_2(x)} - t\ \ln'\left[\frac{\psi'\left[(n-1)\phi(t)\right]-\ \psi'\left[n\phi(t)\right]}{\psi'\left(\phi(t)\right)}\right]\Big|_{t=\bar{F}_1(x)}\\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the decreasing property of $tln'\left[\frac{\psi'[(n-1)\phi(t)]-\ \psi'[n\phi(t)]}{\psi'(\phi(t))}\right]$ with respect to $t\in[0,1]$. ☐

**Remark 5.2.** The condition "$t\ \ln'\left[\frac{\psi'[(n-1)\phi(t)]-\ \psi'[n\phi(t)]}{\psi'(\phi(t))}\right]$ is decreasing" in Theorem 5.1 is general and is satisfied for many Archimedean copulas. For example, we observe the following:

(i) If $\psi(t)=(\theta t+1)^{-\frac{1}{\theta}}$, $\theta\in\mathbb{R}_+$, for $n=2$, we have

$$t\ \ln'\left[\frac{\psi'\left[\phi(t)\right]-\ \psi'\left[2\phi(t)\right]}{\psi'\left(\phi(t)\right)}\right]=-\frac{(1+\theta)t^\theta}{(2-t^\theta)\left\{(2-t^\theta)^{\frac{1}{\theta}+1}-1\right\}}$$

to be decreasing in $t\in(0,1)$;

(ii) If $\psi(t)=e^{-t}$, for $n=3$, we have

$$t\ \ln'\left[\frac{\psi'\left[2\phi(t)\right]-\ \psi'\left[3\phi(t)\right]}{\psi'\left(\phi(t)\right)}\right]=\frac{1-2t}{1-t}$$

to be decreasing in $t\in(0,1)$;

(iii) If $\psi(t)=\frac{1}{\sqrt{t+1}}$, for $n=2$, we have

$$t\ \ln'\left[\frac{\psi'\left[\phi(t)\right]-\ \psi'\left[2\phi(t)\right]}{\psi'\left(\phi(t)\right)}\right]=\frac{2\left(1-\sqrt{2}\right)t}{\left\{\sqrt{2}\left(1-t\right)+t\right\}\left\{\sqrt{2}\left(\sqrt{2}\left(1-t\right)+t\right)^2-1\right\}}$$

to be decreasing in $t\in(0,1)$;

(iv) If $\psi(t)=e^{-t^\theta}$, $\theta\in(0,0.5]$ for $n=4$, we have

$$t\ \ln'\left[\frac{\psi'\left[3\phi(t)\right]-\ \psi'\left[4\phi(t)\right]}{\psi'\left(\phi(t)\right)}\right]=\frac{\left(3^{\theta-1}\left(3^\theta-1\right)t^{\left(3^\theta-1\right)}-4^{\theta-1}\left(4^\theta-1\right)t^{\left(4^\theta-1\right)}\right)}{3^{\theta-1}t^{(3^\theta-1)}-4^{\theta-1}t^{(4^\theta-1)}}$$

to be decreasing in $t\in(0,1)$.

## 6 Concluding Remarks

In this paper, we have compared the lifetimes of two fail-safe systems consisting of dependent components with joint distribution as an Archimedean copula by means of usual stochastic, hazard rate and likelihood ratio orderings. We have also presented some numerical examples to illustrate all the established results. It will be of natural interest to extend these results to the case of general $k$-out-of-$n$ systems. This is a challenging problem due to the complicated form of the distribution function of the lifetime of a $k$-out-of-$n$ system in the case when the components are dependent. We are currently working on this problem and hope to report the findings in a future paper.

## References

[1] Balakrishnan, N., Haidari, A., Barmalzan, G. (2015). Improved ordering results for fail-safe systems with exponential components, *Communications in Statistics-Theory and Methods*, **44**, 2010-2023.

[2] Balakrishnan, N., Rao, C.R. (1998a). *Handbook of Statistics, Vol. 16: Order Statistics: Theory and Methods.* Amsterdam: Elsevier.

[3] Balakrishnan, N., Rao, C.R. (1998b). *Handbook of Statistics, Vol. 17: Order Statistics: Applications.* Amsterdam: Elsevier.

[4] Li, X., Fang, R. (2015). Ordering properties of order statistics from random variables of Archimedean copulas with applications. *Journal of Multivariate Analysis,* **133**, 304-320.

[5] Marshall, A.W., Olkin, I., Arnold, B.C. (2011). *Inequalities: Theory of Majorization and its Applications*, 2nd ed. New York: Springer.

[6] Marshall, A.W., Olkin, I. (2007). *Life Distributions.* New York: Springer.

[7] Müller, A., Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks.* Hoboken, New Jersey: John Wiley & Sons.

[8] Nelsen, R.B. (2006). *An Introduction to Copulas.* New York: Springer.

[9] Shaked, M., Shanthikumar, J.G. (2007) *Stochastic Orders.* New York: Springer.

[10] Zhang, Y., Amini-Seresht, E., Zhao, P. (2019). On fail-safe systems under random shocks. *Applied Stochastic Models in Business and Industry*, 3**5**, 591-602.

Ghobad Saadat Kia

Department of Basic Science, Kermanshah
University of Technology, Kermanshah, Iran.

*E-mail address*: gh.saadatkia@kut.ac.ir

Ali Akbar Hosseinzadeh

Department of Mathematics,
University of Zabol, Sistan and Baluchestan, Iran.

*E-mail address*: hosseinzadeh@uoz.ac.ir

Narayanaswamy Balakrishnan

Department of Mathematics and Statistics,
McMaster University, Hamilton, Canada.

*E-mail address*: bala@mcmaster.ca