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Strong and total Fenchel dualities for robust composed convex optimization problems in locally convex spaces

Ahmed Rikouane, Mohamed Laghdir and M'hamed Mabrouk

Abstract

In this paper, we consider the composed convex optimization problem which consists in minimizing the sum of a convex function and a convex composite function in locally convex Hausdorff topological vector spaces. By using the properties of the epigraph of the conjugate functions, we present some new robust-type constraint qualifications, which completely characterize the strong duality and the stable strong duality. Moreover, some sufficient and/or necessary conditions for the total duality are obtained. We also treat some special cases, rediscovering older results in the literature. At last, the obtained results in this paper are applied to an optimization problem with cone constraints.

Keywords: Robust composed convex optimization problem, constraint qualifications, stable strong duality, total duality, converse duality.

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1 Introduction

Robust optimization is a recent methodology that can be used to treat an optimization problem affected by data uncertainty both in the objective and constraints (see [1, 2, 3, 4] and other references therein). Robust optimization was first introduced by Soyster [1], and was later developed by Ben-Tal et al. [2]. The study of convex programming problems that are affected by data uncertainty has attracted the attention of many researchers in the past years, the interested reader can consult for instance [2, 5, 6, 7, 9] and the references therein. Ben-Tal et al. [2] studied a conic-quadratic optimization problem with uncertain data in the so-called "$\cap$-ellipsoid" case. Li et al. [6] presented a robust conjugate duality theory for convex optimization problems under uncertainty with application to data classification. Sun et al. [7], by using the properties of the subdifferential sum formulae, introduced a robust-type subdifferential constraint qualification, and obtained some completely characterizations of the robust optimal solution of an uncertain convex optimization problem.

Recently, in [9] the authors have presented some strong and total Fenchel dualities for convex programming problems with data uncertainty within the framework of robust optimization in locally convex Hausdorff vector spaces. By applying the properties of the epigraph of the conjugate functions, they gave some new constraint qualifications which completely characterize the strong dualities and the stable strong dualities, and they also obtained some sufficient and/or necessary conditions for the total duality and converse duality for the following uncertain convex programming problems (see [9])

$$(P_A) \qquad \sup_{(u_1,u_2)\in U_1\times U_2} \inf_{x\in X} \{f_{u_1}(x) + g_{u_2}(Ax)\}$$

and

$$(\widetilde{P}_A) \qquad \inf_{x\in X} \sup_{(u_1,u_2)\in U_1\times U_2} \{f_{u_1}(x) + g_{u_2}(Ax)\},$$

where $f_{u_1} : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $u_1 \in U_1$, $g_{u_2} : Y \to \overline{\mathbb{R}}$, $u_2 \in U_2$ are proper and convex functions not necessarily lower semicontinuous (lsc in brief), and $A : X \to Y$ a linear continuous operator such that $\bigcap_{(u_1,u_2)\in U_1\times U_2} [A(\mathrm{dom} f_{u_1}) \cap (\mathrm{dom} g_{u_2})] \neq \emptyset$, and U_1,$U_2 \subseteq Z$, where Z is a locally convex space.

The corresponding dual problem, associated with (P_A), is defined by

$$(D_A) \qquad \sup_{(u_1,u_2)\in U_1\times U_2} \sup_{y^*\in Y^*} \{-f^*_{u_1}(-A^*y^*) - g^*_{u_2}(y^*)\},$$

where $f^*_{u_1}$ and $g^*_{u_2}$ are the conjugate functions of f_{u_1} and g_{u_2}, respectively, and $A^* : Y^* \to X^*$ is the adjoint operator of A.

In the present work, we extend the results of [9] by substituting for the linear operator $A : X \to Y$ the proper and Y_+-convex mapping $h : X \to Y^\bullet := Y \cup \{+\infty_Y\}$, with Y_+ is a nonempty subset closed convex cone of Y. In the absence of data uncertainty, the classical form of composed convex programming problem is (see, for example, [10, 11, 12, 14, 16, 17])

$$(\mathrm{P}) \qquad \inf_{x\in X} \{f(x) + g \circ h(x)\},$$

where X and Y are locally convex Hausdorff topological vector spaces, $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$ are proper convex functions and g is nondecreasing, and $h : X \to Y^\bullet$ is a proper and Y_+-convex mapping such that $\mathrm{dom} f \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g) \neq \emptyset$.
Following [17], we define the two dual problems of the problem (P) by

$$(\mathrm{D}) \qquad \sup_{y^*\in Y^*_+} \{-g^*(y^*) - (f + y^* \circ h)^*(0)\},$$

and

$$(\overline{\mathrm{D}}) \qquad \sup_{\substack{x^*\in X^* \\ y^*\in Y^*_+}} \{-g^*(y^*) - f^*(x^*) - (y^* \circ h)^*(-x^*)\}.$$

where f^*, g^*, $(f + y^* \circ h)^*$ and $(y^* \circ h)^*$ are the conjugate functions of f, g, $(f + y^* \circ h)$ and $(y^* \circ h)$, $y^* \in Y^*_+$, respectively, where Y^*_+ denotes the dual cone of Y_+.

The optimal values for these problems $\vartheta(\mathrm{P})$, $\vartheta(\mathrm{D})$ and $\vartheta(\overline{\mathrm{D}})$, respectively, satisfy the so-called weak Fenchel duality, i.e., $\vartheta(\mathrm{P}) \geq \vartheta(\mathrm{D})$ and $\vartheta(\mathrm{P}) \geq \vartheta(\overline{\mathrm{D}})$.

Given a locally convex Hausdorff topological vector space Z. The convex composed programming problem (P) in the face of data uncertainty in the objective function can be captured by the problem

$$(\mathrm{P_u}) \qquad \sup_{(u_1,u_2)\in U_1\times U_2} \inf_{x\in X} \{f_{u_1}(x) + g_{u_2} \circ h(x)\},$$

where u_1, u_2 are uncertain parameters, U_1, U_2 are nonempty uncertainty subsets of Z, $f_{u_1} : X \to \overline{\mathbb{R}}, u_1 \in U_1$, $g_{u_2} : Y \to \overline{\mathbb{R}}, u_2 \in U_2$ are two proper convex functions (not necessarily l.s.c), and $h : X \to Y^\bullet$ is proper and Y_+-convex (not necessarily Y_+-epi-closed).

To study the uncertainty problem $(\mathrm{P_u})$, we associate with it the uncertain converse optimization problem

$$(\widetilde{\mathrm{P}}_\mathrm{u}) \qquad \inf_{x\in X} \sup_{(u_1,u_2)\in U_1\times U_2} \{f_{u_1}(x) + g_{u_2} \circ h(x)\}.$$

The corresponding dual problems, associated with $(\mathrm{P_u})$, are defined by

$$(\mathrm{D_u}) \qquad \sup_{(u_1,u_2)\in U_1\times U_2} \sup_{y^*\in Y^*_+} \{-g^*_{u_2}(y^*) - (f_{u_1} + y^* \circ h)^*(0)\},$$

and

$$(\overline{\mathrm{D}}_{\mathrm{u}}) \qquad \sup_{(u_1,u_2)\in U_1\times U_2} \sup_{\substack{x^*\in X^*\\ y^*\in Y_+^*}} \{-g_{u_2}^*(y^*) - f_{u_1}^*(x^*) - (y^*\circ h)^*(-x^*)\}.$$

In particular, in the case when U_1 and U_2 are singletons, problems $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\mathrm{P}_{\mathrm{u}})$ coincide with the problem (P).

We denote by $\vartheta(\mathrm{P}_{\mathrm{u}})$, $\vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}})$, $\vartheta(\mathrm{D}_{\mathrm{u}})$ and $\vartheta(\overline{\mathrm{D}}_{\mathrm{u}})$ the optimal values of the optimization problems $(\mathrm{P}_{\mathrm{u}})$, $(\widetilde{\mathrm{P}}_{\mathrm{u}})$, $(\mathrm{D}_{\mathrm{u}})$ and $(\overline{\mathrm{D}}_{\mathrm{u}})$, respectively. Obviously, we have $\vartheta(\mathrm{D}_{\mathrm{u}}) \le \vartheta(\mathrm{P}_{\mathrm{u}}) \le \vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}})$ (resp. $\vartheta(\overline{\mathrm{D}}_{\mathrm{u}}) \le \vartheta(\mathrm{P}_{\mathrm{u}}) \le \vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}})$), that is, the weak dualities hold between $(\mathrm{P}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$ and between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$ (resp. between $(\mathrm{P}_{\mathrm{u}})$ and $(\overline{\mathrm{D}}_{\mathrm{u}})$ and between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\overline{\mathrm{D}}_{\mathrm{u}})$).

The purpose of this paper is to use the properties of the epigraph of the conjugate functions to introduce some new regularity conditions, which completely characterize the strong dualities and the stable strong dualities between $(\mathrm{P}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$ and between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$ (resp. between $(\mathrm{P}_{\mathrm{u}})$ and $(\overline{\mathrm{D}}_{\mathrm{u}})$ and between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\overline{\mathrm{D}}_{\mathrm{u}})$), and establish the sufficient and/or necessary conditions for the total duality between $(\mathrm{P}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$ (resp. between $(\mathrm{P}_{\mathrm{u}})$ and $(\overline{\mathrm{D}}_{\mathrm{u}})$). This choice allows us to find as a particular case the results obtained in [9], that is if $h(x) = Ax$ where A is a linear operator, the problem (P) is reduced to the classic optimization problem

$$(P) \qquad \inf_{x\in X}\{f(x) + g(Ax)\},$$

and the dual problems $(\mathrm{D}_{\mathrm{u}})$ and $(\overline{\mathrm{D}}_{\mathrm{u}})$ become the same dual problem (D_A) (see Section 5).

The paper is organized as follows: In Section 2, we recall some fundamental definitions and preliminary results relating essentially to convex analysis which are used later. In Sections 3 and 4, we give some new constraint qualifications which completely characterize the strong duality, the stable strong duality, and the total duality. In Section 5, we show that our results extend and improve some existing ones in the literature. In Section 6, as an application of the previous results, the strong duality, the stable strong duality, and the total duality are obtained for the conical optimization problem

$$(R) \min_{h(x)\in -Y_+} f(x).$$

2 Notations, definitions and preliminary

Throughout this paper, let X and Y be two real locally convex Hausdorff topological vector spaces, and their continuous dual spaces X^* and Y^*, endowed with the weak*-topology $\omega(X^*,X)$ and $\omega(Y^*,Y)$, respectively. By $\langle y^*, y\rangle$ we denote the value of the functional $y^*\in Y^*$ at $y \in Y$, i.e., $\langle y^*, y\rangle = y^*(y)$. Let C be a nonempty subset of X, the interior, closure and the convex hull of C are denoted by int C, cl C and co C, respectively. If $C \subseteq X^*$, then cl C denotes the weak*-closure of C. For the whole paper, we endow $X^*\times\mathbb{R}$ with the product topology of $\omega(X^*,X)$ and the usual Euclidean topology. Let $Y_+ \subseteq Y$ be a nonempty convex cone. The dual cone Y_+^* of Y_+ is given by

$$Y_+^* := \{y^*\in Y^* : \langle y^*, y\rangle \ge 0,\ \forall y \in Y_+\}.$$

Denote by $\le_{Y_+}$ the partial order on Y induced by Y_+,

$$y_1, y_2 \in Y,\ y_1 \le_{Y_+} y_2 \Longleftrightarrow y_2 - y_1 \in Y_+.$$

Moreover, we attach to Y a greatest element with respect to $\le_{Y_+}$ denoted by $+\infty_Y$ which does not belong to Y. Then for any $y\in Y^\bullet$ one has $y \le_{Y_+} +\infty_Y$ and we consider the following operations on $Y^\bullet$:

$$y + (+\infty_Y) = (+\infty_Y) + y = +\infty_Y,\ \ \alpha.(+\infty_Y) = +\infty_Y,\ \forall y\in Y,\ \forall\alpha \ge 0.$$

The scalar indicator function of a nonempty subset $C \subset X$, denoted by δ_C, is defined as $\delta_C :\ X \longrightarrow \overline{\mathbb{R}}$

$$\delta_C(x) := \begin{cases} 0, & \text{if } x\in C, \\ +\infty, & \text{otherwise.}\end{cases}$$

A mapping $h : X \to Y^\bullet$ is said to be
- proper if its effective domain

$$\text{dom } h := \{x \in X : \ h(x) \in Y\} \neq \emptyset.$$

- Y_+-convex, if for every $\lambda \in [0,1]$ and $x_1, x_2 \in X$

$$h(\lambda x_1 + (1-\lambda)x_2) \leq_{Y_+} \lambda h(x_1) + (1-\lambda)h(x_2).$$

- Y_+-epi-closed, if its epigraph

$$\text{epi } h := \{(x,y) \in X \times Y : \ h(x) \leq_{Y_+} y\} \text{ is closed.}$$

A function $g : Y \to \overline{\mathbb{R}}$ is said to be Y_+-nondecreasing function, if for each $y_1, y_2 \in Y$ we have

$$y_1 \leq_{Y_+} y_2 \Longrightarrow g(y_1) \leq g(y_2).$$

The composite function $g \circ h : X \longrightarrow \overline{\mathbb{R}}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), & \text{if } \ x \in \text{dom } h, \\ \\ +\infty, & \text{otherwise,} \end{cases} ,$$

and its effective domain is given by $\text{dom}(g \circ h) := h^{-1}(\text{dom} g) \cap \text{dom} h$.
Let $f : X \to \overline{\mathbb{R}}$ be a given function. The conjugate function of f is defined by

$$\begin{aligned} f^* : X^* &\longrightarrow \overline{\mathbb{R}} \\ x^* &\longrightarrow f^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}. \end{aligned}$$

By definition, the Young-Fenchel inequality holds:

$$f(x) + f^*(x^*) \geq \langle x^*, x\rangle, \quad \text{for each } (x, x^*) \in X \times X^*. \tag{2.1}$$

The Isc hull and the lsc convex hull of f, denoted respectively by $\text{cl} f$ and $\text{cl}(\text{co} f)$, are defined by

$$\text{epi}(\text{cl} f) = \text{cl}(\text{epi} f) \text{ and } \text{epi}(\text{cl}(\text{co} f)) = \text{cl}(\text{co}(\text{epi} f)).$$

The subdifferential of f at $\overline{x} \in \text{dom} f$ is defined by

$$\partial f(\bar{x}) := \{x^* \in X^* : f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle, \forall x \in X\}.$$

Then, for each $x \in \text{dom} f$,

$$0 \in \partial f(x) \Leftrightarrow x \text{ is a minimiser of } f, \tag{2.2}$$

and

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle \Leftrightarrow (x^*, \langle x^*, x\rangle - f(x)) \in \text{epi} f^*. \tag{2.3}$$

In particular, let $p \in X^*$. Define a function on X that $p(x) := \langle p, x\rangle$ for each $x \in X$. Then for any $a \in \mathbb{R}$ and for any function $\ell : X \to \overline{\mathbb{R}}$,

$$\begin{gathered} (\ell + p + a)^*(x^*) = \ell^*(x^* - p) - a, \quad \text{for each } x^* \in X^*, \\ \text{epi}(\ell + p + a)^* = \text{epi}\ell^* + (p, -a). \end{gathered} \tag{2.4}$$

Given two proper functions $\ell_1, \ell_2 : X \to \overline{\mathbb{R}}$, we define the infimal convolution of ℓ_1 and ℓ_2 as the function $\ell_1 \square \ell_2 : X \to \overline{\mathbb{R}}$ given by

$$(\ell_1 \square \ell_2)(a) := \inf_{x \in X} \{\ell_1(x) + \ell_2(a - x)\}.$$

If there is an $x \in X$ such that $(\ell_1 \square \ell_2)(a) = \ell_1(x) + \ell_2(a - x)$ we say that the infimal convolution is exact at a.

We recall the following important lemmas, which will be used throughout this paper.

Lemma 2.1 (cf.[16]). *Let $f_1, f_2 : X \longrightarrow \overline{\mathbb{R}}$ be proper convex functions such that* $\mathrm{dom} f_1 \cap \mathrm{dom} f_2 \neq \emptyset$.

(i) If f_1 and f_2 are lower semicontinuous, then

$$\mathrm{epi}(f_1 + f_2)^* = \mathrm{cl}(\mathrm{epi}(f_1^* \square f_2^*)) = \mathrm{cl}(\mathrm{epi} f_1^* + \mathrm{epi} f_2^*). \tag{2.5}$$

(ii) If one of f_1 and f_2 is continuous at some $\bar{x} \in \mathrm{dom} f_1 \cap \mathrm{dom} f_2$, then

$$\mathrm{epi}(f_1 + f_2)^* = \mathrm{epi} f_1^* + \mathrm{epi} f_2^*. \tag{2.6}$$

Lemma 2.2 (cf. [10], Lemma 5.1). *The scalar indicator $\delta_{-Y_+} : Y \to \overline{\mathbb{R}}$ is convex, proper, lower semicontinuous and Y_+-nondecreasing.*

Finally, we conclude this section with a lemma that can be found in ([13]).

Lemma 2.3 (cf. [13], Lemma 2.5). *Let I be an index set and let $\{f_i : i \in I\}$ be a family of proper convex lower semicontinuous functions on X with $\sup\limits_{i \in I} f_i(\overline{x}) < +\infty$ for some $\overline{x} \in X$. Then*

$$\mathrm{epi}(\sup_{i \in I} f_i)^* = \mathrm{cl}(\mathrm{co} \bigcup_{i \in I} \mathrm{epi} f_i^*),$$

where $\sup\limits_{i \in I} f_i : X \longrightarrow \overline{\mathbb{R}}$ is defined by $(\sup\limits_{i \in I} f_i)(x) := \sup\limits_{i \in I} f_i(x)$ for all $x \in X$.

3 Robust stable Fenchel duality

Let X, Y and Z be real locally convex Hausdorff topological vector spaces, $U_1, U_2 \subseteq Z$. Let $f_{u_1} : X \longrightarrow \overline{\mathbb{R}}, u_1 \in U_1$, $g_{u_2} : Y \longrightarrow \overline{\mathbb{R}}, u_2 \in U_2$ are two proper convex functions, and let $h : X \longrightarrow Y^\bullet$ be a proper and Y_+-convex mapping such that $\bigcap\limits_{(u_1,u_2) \in U_1 \times U_2} (\mathrm{dom} f_{u_1} \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g_{u_2})) \neq \emptyset$.
For simplicity, we denote

$$u := (u_1, u_2) \text{ and } U := U_1 \times U_2.$$

Given $p \in X^*$, consider the following robust optimization problems with a linear perturbation

$$(\mathrm{P_p}) \quad \sup_{u \in U} \inf_{x \in X} \{f_{u_1}(x) + g_{u_2} \circ h(x) - \langle p, x \rangle\} \tag{3.1}$$

and

$$(\widetilde{\mathrm{P}}_\mathrm{p}) \quad \inf_{x \in X} \sup_{u \in U} \{f_{u_1}(x) + g_{u_2} \circ h(x) - \langle p, x \rangle\}. \tag{3.2}$$

The corresponding dual problems, associated with $(\mathrm{P_p})$, are defined by

$$(\mathrm{D_p}) \quad \sup_{u \in U} \sup_{y^* \in Y_+^*} \{-g_{u_2}^*(y^*) - (f_{u_1} + y^* \circ h)^*(p)\}, \tag{3.3}$$

$$(\overline{D}_p) \quad \sup_{u \in U} \sup_{\substack{x^* \in X^* \\ y^* \in Y_+^*}} \{-g_{u_2}^*(y^*) - f_{u_1}^*(x^*) - (y^* \circ h)^*(p - x^*)\}. \tag{3.4}$$

As usual, we denote by $\vartheta(\mathrm{P_p}), \vartheta(\widetilde{\mathrm{P}}_\mathrm{p}), \vartheta(\mathrm{D_p})$ and $\vartheta(\overline{\mathrm{D}}_\mathrm{p})$ the optimal values of the problems $(\mathrm{P_p}), (\widetilde{\mathrm{P}}_\mathrm{p}), (\mathrm{D_p})$ and $(\overline{\mathrm{D}}_\mathrm{p})$, respectively, i.e.,

$$\vartheta(\mathrm{P_p}) := \sup_{u \in U} \inf_{x \in X} \{f_{u_1}(x) + g_{u_2} \circ h(x) - \langle p, x \rangle\},$$

$$\vartheta(\widetilde{\mathrm{P}}_\mathrm{p}) := \inf_{x \in X} \sup_{u \in U} \{f_{u_1}(x) + g_{u_2} \circ h(x) - \langle p, x \rangle\},$$

$$\vartheta(\mathrm{D_p}) := \sup_{u \in U} \sup_{y^* \in Y_+^*} \{-g_{u_2}^*(y^*) - (f_{u_1} + y^* \circ h)^*(p)\},$$

and

$$\vartheta(\overline{\mathrm{D}}_{\mathrm{p}}) := \sup_{u\in U} \sup_{\substack{x^*\in X^*\\ y^*\in Y^*_+}} \{-g^*_{u_2}(y^*) - f^*_{u_1}(x^*) - (y^*\circ h)^*(p-x^*)\}.$$

By definition, for each $p \in X^*$, we can get the inequalities

$$\vartheta(\mathrm{D}_{\mathrm{p}}) \leq \vartheta(\mathrm{P}_{\mathrm{p}}) \leq \vartheta(\widetilde{\mathrm{P}}_{\mathrm{p}}), \tag{3.5}$$

$$\vartheta(\overline{\mathrm{D}}_{\mathrm{p}}) \leq \vartheta(\mathrm{P}_{\mathrm{p}}) \leq \vartheta(\widetilde{\mathrm{P}}_{\mathrm{p}}). \tag{3.6}$$

Let us consider the following auxiliary functions

$$\begin{array}{rrcl} F_{u_1}: & X\times Y & \longrightarrow & \overline{\mathbb{R}} \\ & (x,y) & \longrightarrow & F_{u_1}(x,y) := f_{u_1}(x) + \delta_{\mathrm{epih}}(x,y), \end{array}$$

and

$$\begin{array}{rrcl} G_{u_2}: & X\times Y & \longrightarrow & \overline{\mathbb{R}} \\ & (x,y) & \longrightarrow & G_{u_2}(x,y) := g_{u_2}(y), \end{array}$$

where $\delta_{\mathrm{epih}} : X\times Y \longrightarrow \overline{\mathbb{R}}$ stands for the indicator function defined for any $(x,y)\in X\times Y$ by

$$\delta_{\mathrm{epih}}(x,y) := \begin{cases} 0, & \text{if } (x,y)\in \mathrm{epih}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Such as $g_{u_2} : Y \longrightarrow \overline{\mathbb{R}}$ is Y_+-nondecreasing, one has for each $(x,y)\in X\times Y$

$$(f_{u_1} + g_{u_2}\circ h)(x) \leq f_{u_1}(x) + g_{u_2}(y) + \delta_{\mathrm{epih}}(x,y),$$

it follows that, for each $x\in X$

$$\begin{aligned} (f_{u_1} + g_{u_2}\circ h)(x) &= \inf_{y\in Y}\{f_{u_1}(x) + g_{u_2}(y) + \delta_{\mathrm{epih}}(x,y)\} \\ &= \inf_{y\in Y}\{F_{u_1}(x,y) + G_{u_2}(x,y)\}. \end{aligned}$$

The conjugate functions of F_{u_1} and G_{u_2} are defined on $X^*\times Y^*$ as follows (see [13])

$$F^*_{u_1}(a,b) := \begin{cases} (f_{u_1} + (-b\circ h))^*(a), & \text{if } b\in -Y^*_+, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$G^*_{u_2}(a,b) := \begin{cases} g^*_{u_2}(b), & \text{if } a = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Lemma 3.1 ([13])**.** *Let $F_{u_1}, G_{u_2} : X\times Y \longrightarrow \overline{\mathbb{R}}, u\in U$ be defined by $F_{u_1}(x,y) := f_{u_1}(x) + \delta_{\mathrm{epih}}(x,y)$ and $G_{u_2}(x,y) := g_{u_2}(y)$, for $(x,y)\in X\times Y$.*

(i) F_{u_1} and G_{u_2} are proper and convex functions and $\mathrm{dom}F_{u_1}\cap \mathrm{dom}G_{u_2} \neq \emptyset$.

(ii) For $(p,r)\in X^*\times\mathbb{R} : (p,r)\in \mathrm{epi}(f_{u_1} + g_{u_2}\circ h)^* \Longleftrightarrow (p,0,r)\in \mathrm{epi}(F_{u_1} + G_{u_2})^*$.

(iii) $\mathrm{epi}G^*_{u_2} = \{0\}\times \mathrm{epi}g^*_{u_2}$ *and* $\mathrm{epi}F^*_{u_1} = \bigcup_{y^*\in Y^*_+} \{(a,-y^*,r) : (a,r)\in \mathrm{epi}(f_{u_1} + y^*\circ h)^*\}$.

Remark 3.2. (1) $\mathrm{dom}F_{u_1} = (\mathrm{dom}f_{u_1}\times Y)\cap(\mathrm{epi}h)$ and $\mathrm{dom}G_{u_2} = X\times \mathrm{dom}g_{u_2}$.

(2) $\bigcap_{u\in U}(\mathrm{dom}f_{u_1}\cap \mathrm{dom}h\cap h^{-1}(\mathrm{dom}g_{u_2})) \neq \emptyset \Leftrightarrow \bigcap_{u\in U}(\mathrm{dom}F_{u_1}\cap \mathrm{dom}G_{u_2}) \neq \emptyset$.

It easy to see that

$$\vartheta(\mathrm{P_p}) = \sup_{u\in U}[-(f_{u_1} + g_{u_2} \circ h)^*(p)] = -\inf_{u\in U}(f_{u_1} + g_{u_2} \circ h)^*(p), \tag{3.7}$$

$$\vartheta(\mathrm{D_p}) = \sup_{u\in U}[-(F^*_{u_1}\Box G^*_{u_2})(p,0)] = -\inf_{u\in U}(F^*_{u_1}\Box G^*_{u_2})(p,0), \tag{3.8}$$

and

$$\begin{aligned}\vartheta(\overline{\mathrm{D}}_\mathrm{p}) &= \sup_{u\in U}\sup_{\substack{x^*\in X^*\\ y^*\in Y^*_+}}[-g^*_{u_2}(y^*) - f^*_{u_2}(x^*) - (y^*\circ h)^*(p-x^*)]\\ &= -\inf_{u\in U}\inf_{\substack{x^*\in X^*\\ y^*\in Y^*_+}}[g^*_{u_2}(y^*) + f^*_{u_2}(x^*) + (y^*\circ h)^*(p-x^*)].\end{aligned} \tag{3.9}$$

In particular, when $p = 0$, the problems $(\mathrm{P_p})$, $(\widetilde{\mathrm{P}}_\mathrm{p})$, $(\mathrm{D_p})$ and $(\overline{\mathrm{D}}_\mathrm{p})$ reduce to the problems $(\mathrm{P_u})$, $(\widetilde{\mathrm{P}}_\mathrm{u})$, $(\mathrm{D_u})$ and $(\overline{\mathrm{D}}_\mathrm{u})$, respectively. Furthermore, by (3.7), (3.8) and (3.9), we get that

$$\vartheta(\mathrm{P_u}) = -\inf_{u\in U}(f_{u_1} + g_{u_2}\circ h)^*(0), \tag{3.10}$$

$$\vartheta(\mathrm{D_u}) = -\inf_{u\in U}(F^*_{u_1}\Box G^*_{u_2})(0,0), \tag{3.11}$$

and

$$\vartheta(\overline{\mathrm{D}}_\mathrm{u}) = -\inf_{u\in U}\inf_{\substack{x^*\in X^*\\ y^*\in Y^*_+}}[g^*_{u_2}(y^*) + f^*_{u_2}(x^*) + (y^*\circ h)^*(-x^*)]. \tag{3.12}$$

This section is dedicated to the study of the strong and stable strong dualities between $(\mathrm{P_u})$ and $(\mathrm{D_u})$ and between $(\widetilde{\mathrm{P}}_\mathrm{u})$ and $(\mathrm{D_u})$ (resp. between $(\mathrm{P_u})$ and $(\overline{\mathrm{D}}_\mathrm{u})$ and between $(\widetilde{\mathrm{P}}_\mathrm{u})$ and $(\overline{\mathrm{D}}_\mathrm{u})$), which are defined as follows.

Definition 3.3. It is said that

(i) the strong duality holds between $(\mathrm{P_u})$ and $(\mathrm{D_u})$ (resp. between $(\widetilde{\mathrm{P}}_\mathrm{u})$ and $(\mathrm{D_u})$) if $\vartheta(\mathrm{P_u}) = \vartheta(\mathrm{D_u})$ (resp. $\vartheta(\widetilde{\mathrm{P}}_\mathrm{u}) = \vartheta(\mathrm{D_u})$) and $(\mathrm{D_u})$ has an optimal solution.

(ii) the stable strong duality holds between $(\mathrm{P_u})$ and $(\mathrm{D_u})$ (resp. between $(\widetilde{\mathrm{P}}_\mathrm{u})$ and $(\mathrm{D_u})$) if for each $p \in X^*$, the strong duality holds between $(\mathrm{P_p})$ and $(\mathrm{D_p})$ (resp. between $(\widetilde{\mathrm{P}}_\mathrm{p})$ and $(\mathrm{D_p})$).

Definition 3.4. It is said that

(i) the strong duality holds between $(\mathrm{P_u})$ and $(\overline{\mathrm{D}}_\mathrm{u})$ (resp. between $(\widetilde{\mathrm{P}}_\mathrm{u})$ and $(\overline{\mathrm{D}}_\mathrm{u})$) if $\vartheta(\mathrm{P_u}) = \vartheta(\overline{\mathrm{D}}_\mathrm{u})$ (resp. $\vartheta(\widetilde{\mathrm{P}}_\mathrm{u}) = \vartheta(\overline{\mathrm{D}}_\mathrm{u})$) and $(\overline{\mathrm{D}}_\mathrm{u})$ has an optimal solution.

(ii) the stable strong duality holds between $(\mathrm{P_u})$ and $(\overline{\mathrm{D}}_\mathrm{u})$ (resp. between $(\widetilde{\mathrm{P}}_\mathrm{u})$ and $(\overline{\mathrm{D}}_\mathrm{u})$) if for each $p \in X^*$, the strong duality holds between $(\mathrm{P_p})$ and $(\overline{\mathrm{D}}_\mathrm{p})$ (resp. between $(\widetilde{\mathrm{P}}_\mathrm{p})$ and $(\overline{\mathrm{D}}_\mathrm{p})$).

Remark 3.5. If the strong duality holds between $(\widetilde{\mathrm{P}}_\mathrm{p})$ and $(\mathrm{D_p})$ (resp. between $(\widetilde{\mathrm{P}}_\mathrm{p})$ and $(\overline{\mathrm{D}}_\mathrm{p})$), i.e. $\vartheta(\widetilde{\mathrm{P}}_\mathrm{p}) = \vartheta(\mathrm{D_p})$ and $(\mathrm{D_p})$ has an optimal solution (resp. $\vartheta(\widetilde{\mathrm{P}}_\mathrm{p}) = \vartheta(\overline{\mathrm{D}}_\mathrm{p})$ and $\vartheta(\overline{\mathrm{D}}_\mathrm{p})$ has an optimal solution), then by (3.5) (resp. (3.6)), we can conclude that $\vartheta(\mathrm{P_p}) = \vartheta(\mathrm{D_p})$ and $(\mathrm{D_p})$ has an optimal solution (resp. $\vartheta(\mathrm{P_p}) = \vartheta(\overline{\mathrm{D}}_\mathrm{p})$ and $\vartheta(\overline{\mathrm{D}}_\mathrm{p})$ has an optimal solution), by Definition 3.3 (resp. Definition 3.4), the strong duality holds between $(\mathrm{P_p})$ and $(\mathrm{D_p})$ (resp. between $(\mathrm{P_p})$ and $(\overline{\mathrm{D}}_\mathrm{p})$). However, the opposite is not true.

Definition 3.6. The family (f_{u_1}, g_{u_2}, h, U) is said to satisfy

(a) the strong further regularity condition (SFRC) if

$$\begin{aligned}&\mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\Big\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2}\circ h)^*\Big\}\right] \sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times \mathrm{epi} g^*_{u_2} +\\ &\bigcup_{y^*\in Y^*_+}\Big\{(a,-y^*,r) : (a,r)\in \mathrm{epi}(f_{u_1} + y^*\circ h)^*\Big\}\Big\} \cap (\{0\}\times\{0\}\times\mathbb{R});\end{aligned} \tag{3.13}$$

(b) the asymptotic further regularity condition (AFRC) if

$$\begin{aligned}\operatorname{cl}\left[\bigcup_{u\in U}\left\{(0,0,r):(0,r)\in\operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\right\}\right]\sqsubseteq\bigcup_{u\in U}\Bigg\{\{0_{X^*}\}\times\operatorname{epi}g^*_{u_2}+\\ \bigcup_{y^*\in Y^*_+}\left\{(a,-y^*,r):(a,r)\in\operatorname{epi}(f_{u_1}+y^*\circ h)^*\right\}\Bigg\}\cap(\{0\}\times\{0\}\times\mathbb{R});\end{aligned}\tag{3.14}$$

(c) the further regularity condition (FRC) if

$$\begin{aligned}\bigcup_{u\in U}\left\{(0,0,r):(0,r)\in\operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\right\}\sqsubseteq\bigcup_{u\in U}\Bigg\{\{0_{X^*}\}\times\operatorname{epi}g^*_{u_2}+\\ \bigcup_{y^*\in Y^*_+}\left\{(a,-y^*,r):(a,r)\in\operatorname{epi}(f_{u_1}+y^*\circ h)^*\right\}\Bigg\}\cap(\{0\}\times\{0\}\times\mathbb{R});\end{aligned}\tag{3.15}$$

(d) the strong closure condition (SCC) if

$$\begin{aligned}\operatorname{cl}\left[\operatorname{co}\bigcup_{u\in U}\left\{(a,0,r):(a,r)\in\operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\right\}\right]\sqsubseteq\bigcup_{u\in U}\Bigg\{\{0_{X^*}\}\times\operatorname{epi}g^*_{u_2}+\\ \bigcup_{y^*\in Y^*_+}\left\{(a,-y^*,r):(a,r)\in\operatorname{epi}(f_{u_1}+y^*\circ h)^*\right\}\Bigg\}\cap(X^*\times\{0\}\times\mathbb{R});\end{aligned}\tag{3.16}$$

(e) the asymptotic closure condition (ACC) if

$$\begin{aligned}\operatorname{cl}\left[\bigcup_{u\in U}\left\{(a,0,r):(a,r)\in\operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\right\}\right]\sqsubseteq\bigcup_{u\in U}\Bigg\{\{0_{X^*}\}\times\operatorname{epi}g^*_{u_2}+\\ \bigcup_{y^*\in Y^*_+}\left\{(a,-y^*,r):(a,r)\in\operatorname{epi}(f_{u_1}+y^*\circ h)^*\right\}\Bigg\}\cap(X^*\times\{0\}\times\mathbb{R});\end{aligned}\tag{3.17}$$

(f) the closure condition (CC) if

$$\begin{aligned}\bigcup_{u\in U}\left\{(a,0,r):(a,r)\in\operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\right\}\sqsubseteq\bigcup_{u\in U}\Bigg\{\{0_{X^*}\}\times\operatorname{epi}g^*_{u_2}+\\ \bigcup_{y^*\in Y^*_+}\left\{(a,-y^*,r):(a,r)\in\operatorname{epi}(f_{u_1}+y^*\circ h)^*\right\}\Bigg\}\cap(X^*\times\{0\}\times\mathbb{R}).\end{aligned}\tag{3.18}$$

Definition 3.7. The family (f_{u_1}, g_{u_2}, h, U) is said to satisfy

(a) the strong further regularity condition ($\overline{\text{SFRC}}$) if

$$\begin{aligned}\operatorname{cl}\left[\operatorname{co}\bigcup_{u\in U}\left\{(0,0,r):(0,r)\in\operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\right\}\right]\sqsubseteq\bigcup_{u\in U}\Bigg\{\{0_{X^*}\}\times\operatorname{epi}g^*_{u_2}+\\ \{(p,0,r):(p,r)\in\operatorname{epi}f^*_{u_1}\}+\bigcup_{y^*\in Y^*_+}\left\{(p,-y^*,r):(p,r)\in\operatorname{epi}(y^*\circ h)^*\right\}\Bigg\}\cap(\{0\}\times\{0\}\times\mathbb{R});\end{aligned}$$

(b) the asymptotic further regularity condition ($\overline{\text{AFRC}}$) if

$$\begin{aligned}\operatorname{cl}\left[\bigcup_{u\in U}\left\{(0,0,r):(0,r)\in\operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\right\}\right]\sqsubseteq\bigcup_{u\in U}\Bigg\{\{0_{X^*}\}\times\operatorname{epi}g^*_{u_2}+\\ \{(p,0,r):(p,r)\in\operatorname{epi}f^*_{u_1}\}+\bigcup_{y^*\in Y^*_+}\left\{(p,-y^*,r):(p,r)\in\operatorname{epi}(y^*\circ h)^*\right\}\Bigg\}\cap(\{0\}\times\{0\}\times\mathbb{R});\end{aligned}$$

(c) the further regularity condition ($\overline{\text{FRC}}$) if

$$\bigcup_{u\in U}\Big\{(0,0,r):(0,r)\in \operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\Big\}\sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times \operatorname{epi}g^*_{u_2}+$$
$$\{(p,0,r):(p,r)\in \operatorname{epi}f^*_{u_1}\}+\bigcup_{y^*\in Y^*_+}\Big\{(p,-y^*,r):(p,r)\in \operatorname{epi}(y^*\circ h)^*\Big\}\Big\}\cap(\{0\}\times\{0\}\times\mathbb{R});$$

(d) the strong closure condition ($\overline{\text{SCC}}$) if

$$\operatorname{cl}\left[\operatorname{co}\bigcup_{u\in U}\Big\{(a,0,r):(a,r)\in \operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\Big\}\right]\sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times \operatorname{epi}g^*_{u_2}+$$
$$\{(p,0,r):(p,r)\in \operatorname{epi}f^*_{u_1}\}+\bigcup_{y^*\in Y^*_+}\Big\{(p,-y^*,r):(p,r)\in \operatorname{epi}(y^*\circ h)^*\Big\}\Big\}\cap(X^*\times\{0\}\times\mathbb{R});$$

(e) the asymptotic closure condition ($\overline{\text{ACC}}$) if

$$\operatorname{cl}\left[\bigcup_{u\in U}\Big\{(a,0,r):(a,r)\in \operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\Big\}\right]\sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times \operatorname{epi}g^*_{u_2}+$$
$$\{(p,0,r):(p,r)\in \operatorname{epi}f^*_{u_1}\}+\bigcup_{y^*\in Y^*_+}\Big\{(p,-y^*,r):(p,r)\in \operatorname{epi}(y^*\circ h)^*\Big\}\Big\}\cap(X^*\times\{0\}\times\mathbb{R});$$

(f) the closure condition ($\overline{\text{CC}}$) if

$$\bigcup_{u\in U}\Big\{(a,0,r):(a,r)\in \operatorname{epi}(f_{u_1}+g_{u_2}\circ h)^*\Big\}\sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times \operatorname{epi}g^*_{u_2}+$$
$$\{(p,0,r):(p,r)\in \operatorname{epi}f^*_{u_1}\}+\bigcup_{y^*\in Y^*_+}\Big\{(p,-y^*,r):(p,r)\in \operatorname{epi}(y^*\circ h)^*\Big\}\Big\}\cap(X^*\times\{0\}\times\mathbb{R}).$$

The following propositions describe the relationship between the (SCC) (resp. the $(\overline{SCC})$, the (ACC), the $(\overline{ACC})$, the (CC), the $(\overline{CC})$) and the $(SFRC)$ (resp. the $(\overline{SFRC})$, the $(AFRC)$, the $(\overline{AFRC})$, the (FRC), the $(\overline{FRC})$).

Proposition 3.8. *The family (f_{u_1},g_{u_2},h,U) satisfies the* (SCC) (*resp. the* (ACC), *the* (CC)) *if and only if for each $p\in X^*$, the family $(f_{u_1}-p,g_{u_2},h,U)$ satisfies the* (SFRC) (*resp. the* (AFRC), *the* (FRC)).

Proof. Let $p\in X^*$ and let $K_1(p)$, $K_2(p)$ be defined by

$$K_1(p):\;=\;\bigcup_{u\in U}\Big\{(a,0,r):(a,r)\in \operatorname{epi}(f_{u_1}-p+g_{u_2}\circ h)^*\Big\}$$
$$K_2(p):\;=\;\bigcup_{u\in U}\Big\{\{0_{X^*}\}\times \operatorname{epi}g^*_{u_2}+\bigcup_{y^*\in Y^*_+}\Big\{(a,-y^*,r):(a,r)\in \operatorname{epi}(f_{u_1}-p+y^*\circ h)^*\Big\}\Big\},$$

respectively. Then, by (2.4), we obtain that $K_1(p)=K_1(0)+(-p,0,0)$ and $K_2(p)=K_2(0)+(-p,0,0)$. Hence, we have

$$K_1(p)\cap(\{0\}\times\{0\}\times\mathbb{R})=K_1(0)\cap(\{p\}\times\{0\}\times\mathbb{R})+(-p,0,0),$$

and

$$K_2(p)\cap(\{0\}\times\{0\}\times\mathbb{R})=K_2(0)\cap(\{p\}\times\{0\}\times\mathbb{R})+(-p,0,0).$$

Let us prove, for example, that the family (f_{u_1},g_{u_2},h,U) satisfies the (CC) if and only if for any $p\in X^*$ the family $(f_{u_1}-p,g_{u_2},h,U)$ satisfies the (FRC). Thus, if the family (f_{u_1},g_{u_2},h,U) satisfies the (CC), we have

$$K_1(0)\sqsubseteq K_2(0).$$

Then,

$$K_1(0) \cap (\{p\} \times \{0\} \times \mathbb{R}) + (-p, 0, 0) \quad \sqsubseteq \quad K_2(0) \cap (\{p\} \times \{0\} \times \mathbb{R}) + (-p, 0, 0).$$

Hence,

$$K_1(p) \cap (\{0\} \times \{0\} \times \mathbb{R}) \sqsubseteq K_2(p) \cap (\{0\} \times \{0\} \times \mathbb{R}).$$

Thus, the family $(f_{u_1} - p, g_{u_2}, h, U)$ satisfies the (FRC).
The other equivalences are obtained similarly, and the proof is complete. □

Analogously one can show the following statement.

Proposition 3.9. *The family (f_{u_1}, g_{u_2}, h, U) satisfies the $(\overline{\text{SCC}})$ (resp. the $(\overline{\text{ACC}})$, the $(\overline{\text{CC}})$) if and only if for each $p \in X^*$, the family $(f_{u_1} - p, g_{u_2}, h, U)$ satisfies the $(\overline{\text{SFRC}})$ (resp. the $(\overline{\text{AFRC}})$, the $(\overline{\text{FRC}})$).*

Lemma 3.10. *Let $r \in \mathbb{R}$. Then the following assertions are hold.*

(i) $(p, 0, r) \in \bigcup_{u \in U} \{\{0_{X^*}\} \times \text{epi} g^*_{u_2} + \bigcup_{y^* \in Y^*_+} \{(a, -y^*, r) : (a, r) \in \text{epi}(f_{u_1} + y^* \circ h)^*\}\}$ *if and only if there exist* $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ *and* $\overline{y}^* \in Y^*_+$ *satisfying* $g^*_{\overline{u}_2}(\overline{y}^*) + (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(p) \le r$.

(ii) Suppose that $f_{u_1} : X \longrightarrow \overline{\mathbb{R}}, u_1 \in U_1$ *and* $g_{u_2} : Y \longrightarrow \overline{\mathbb{R}}, u_2 \in U_2$ *are lower semicontinuous functions, and* $h : X \longrightarrow Y^\bullet$ *is* Y_+*-epi-closed. Then* $(p, 0, r) \in \text{cl}\Big[\text{co} \bigcup_{u \in U} \{(a, 0, r) : (a, r) \in \text{epi}(f_{u_1} + g_{u_2} \circ h)^*\}\Big]$ *if and only if* $\vartheta(\widetilde{P}_p) \ge -r$.

Proof. (i) Suppose that $(p, 0, r) \in \bigcup_{u \in U} \Big\{\{0_{X^*}\} \times \text{epi} g^*_{u_2} + \bigcup_{y^* \in Y^*_+} \Big\{(a, -y^*, r) : \ (a, r) \in \text{epi}(f_{u_1} + y^* \circ h)^*\Big\}\Big\}$, then there exist $\overline{u} = (\overline{u}_1, \overline{u}_2)$, $\overline{y}^* \in Y^*_+$, (p_1, q_1, r_1) and (p_2, q_2, r_2) verifying

$$p_1 + p_2 = p, \quad q_1 + q_2 = 0, \quad r_1 + r_2 = r,$$

such that $(p_1, q_1, r_1) \in \{0_{X^*}\} \times \text{epi} g^*_{\overline{u}_2}$, $(p_2, q_2, r_2) \in \Big\{(a, -\overline{y}^*, r) : \ (a, r) \in \text{epi}(f_{\overline{u}_1} + \overline{y}^* \circ h)^*\Big\}$, then

$$\begin{cases} p_1 = 0, & q_1 = -q_2 = \overline{y}^*, \\ p_2 = p, & r_1 + r_2 = r, \end{cases}$$

with

$$g^*_{\bar{u}_2}(q_1) = g^*_{\bar{u}_2}(\overline{y}^*) \le r_1 \text{ and } (f_{\bar{u}_1} + \overline{y}^* \circ h)^*(p) \le r_2. \tag{3.19}$$

By summing, we obtain

$$g^*_{\overline{u}_2}(\overline{y}^*) + (f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p) \le r.$$

Conversely, assume that there exist $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{y}^* \in Y^*_+$ satisfying

$$g^*_{\overline{u}_2}(\overline{y}^*) + (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(p) \le r.$$

Set $x^*_1 := 0$, $x^*_2 := p$, $r_1 := G^*_{\overline{u}_2}(x^*_1, \overline{y}^*)$, $r_2 := (f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p)$ and $r := r_1 + r_2$, it results that $(x^*_1, y^*, r_1) \in \text{epi} G^*_{\overline{u}_2}$, i.e. $g^*_{\overline{u}_2}(\overline{y}^*) \le r_1$ and $(p, r_2) \in \text{epi}(f_{u_1} + \overline{y}^* \circ h)^*$, then $(x^*_2, -\overline{y}^*, r_2) \in \text{epi} F^*_{\overline{u}_1}$. Thus,

$$\begin{aligned}
(p, 0, r) &= (0, \overline{y}^*, r_1) + (p, -\overline{y}^*, r_2) \\
&= (x^*_1, \overline{y}^*, r_1) + (x^*_2, -\overline{y}^*, r_2) \\
&\in \text{epi} G^*_{\overline{u}_2} + \text{epi} F^*_{\overline{u}_1} \\
&\in \bigcup_{u \in U} \Big\{\text{epi} G^*_{u_2} + \text{epi} F^*_{u_1}\Big\} \\
&\in \bigcup_{u \in U} \Big\{\{0_{X^*}\} \times \text{epi} g^*_{u_2} + \bigcup_{y^* \in Y^*_+} \Big\{(a, -y^*, r) : (a, r) \in \text{epi}(f_{u_1} + y^* \circ h)^*\Big\}\Big\}.
\end{aligned}$$

(ii) Define the function $l : X \to \overline{\mathbb{R}}$ by
$$l(x) := \sup_{u\in U}\{f_{u_1}(x) + g_{u_2}\circ h(x)\},\ \forall x \in X.$$

Then, by definition,
$$\begin{aligned} l^*(p) &= \sup_{x\in X}\{\langle p, x\rangle - l(x)\} \\ &= \sup_{x\in X}\{\langle p, x\rangle - \sup_{u\in U}\{f_{u_1}(x) + g_{u_2}\circ h(x)\}\} \\ &= -\inf_{x\in X}\sup_{u\in U}\{f_{u_1}(x) + g_{u_2}\circ h(x) - \langle p, x\rangle\} \\ &= -\vartheta(\widetilde{\mathrm{P}}_{\mathrm{p}}). \end{aligned}$$

Then,
$$\vartheta(\widetilde{\mathrm{P}}_{\mathrm{p}}) \geq -r \Leftrightarrow l^*(p) \leq r \Leftrightarrow (p, r) \in \mathrm{epi} l^*. \tag{3.20}$$

Note, by Lemma 2.3, that
$$\mathrm{epi} l^* = \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\mathrm{epi}(f_{u_1} + g_{u_2}\circ h)^*\right].$$

Therefore, by (3.20), we get
$$(p, r) \in \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\mathrm{epi}(f_{u_1} + g_{u_2}\circ h)^*\right].$$

Hence,
$$(p, 0, r) \in \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\Big\{(a, 0, r) :\ (a, r) \in \mathrm{epi}(f_{u_1} + g_{u_2}\circ h)^*\Big\}\right].$$

Thus,
$$\vartheta(\widetilde{\mathrm{P}}_{\mathrm{p}}) \geq -r \Leftrightarrow (p, 0, r) \in \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\Big\{(a, 0, r) :\ (a, r) \in \mathrm{epi}(f_{u_1} + g_{u_2}\circ h)^*\Big\}\right].$$

This completes the proof.

□

Similarly, we can prove the following lemma.

Lemma 3.11. *Let $r \in \mathbb{R}$. Then the following statements hold.*

(i) $(p, 0, r) \in \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times \mathrm{epi} g^*_{u_2} + \{(x^*, 0, r) : (x^*, r) \in \mathrm{epi} f^*_{u_1}\} + \bigcup_{y^*\in Y^*_+}\Big\{(a, -y^*, r) : (a, r) \in \mathrm{epi}(y^*\circ h)^*\Big\}\Big\}$ *if and only if there exist* $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$, $\overline{x}^* \in X^*$ *and* $\overline{y}^* \in Y^*_+$ *satisfying* $g^*_{\overline{u}_2}(\overline{y}^*) + f^*_{\overline{u}_2}(\overline{x}^*) + (\overline{y}^*\circ h)^*(p - \overline{x}^*) \leq r$.

(ii) Suppose that $f_{u_1} : X \longrightarrow \overline{\mathbb{R}}, u_1 \in U_1$ *and* $g_{u_2} : Y \longrightarrow \overline{\mathbb{R}}, u_2 \in U_2$ *are lower semicontinuous functions, and* $h : X \longrightarrow Y^\bullet$ *is* Y_+*-epi-closed. Then* $(p, 0, r) \in \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\Big\{(a, 0, r) : (a, r) \in \mathrm{epi}(f_{u_1} + g_{u_2}\circ h)^*\Big\}\right]$ *if and only if* $\vartheta(\widetilde{P}_p) \geq -r$.

The following theorems give a characterization of the strong duality between $(\mathrm{P_u})$ and $(\mathrm{D_u})$ (resp. between $(\mathrm{P_u})$ and $(\overline{\mathrm{D}}_{\mathrm{u}})$) in terms of the (AFRC) and (FRC) (resp. the $(\overline{\mathrm{AFRC}})$ and $(\overline{\mathrm{FRC}})$).

Theorem 3.12. *Consider the following assertions.*

(i) The family (f_{u_1}, g_{u_2}, h, U) *satisfies the* (AFRC).

(ii) The strong duality holds between $(\mathrm{P_u})$ *and* $(\mathrm{D_u})$.

(iii) The family (f_{u_1}, g_{u_2}, h, U) *satisfies the* (FRC).

Then, (i) ⇒ (ii) ⇒ (iii). Furthermore, if $\bigcup_{u\in U}\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\}$ *is closed, then (i) ⇔ (ii) ⇔ (iii).*

Proof. (i) ⇒ (ii) Assume that the family (f_{u_1}, g_{u_2}, h, U) satisfies the (AFRC). If $\vartheta(\mathrm{P_u}) = -\infty$, then, by the weak duality between $(\mathrm{P_u})$ and $(\mathrm{D_u})$, we have $\vartheta(\mathrm{D_u}) = \vartheta(\mathrm{P_u}) = -\infty$, and the conclusion holds. Otherwise, we assume that $-r := \vartheta(\mathrm{P_u}) \in \mathbb{R}$. Then, by applying (3.10), we have

$$-\vartheta(\mathrm{P_u}) = \inf_{u\in U}(f_{u_1} + g_{u_2} \circ h)^*(0) = r.$$

Thus, for all $\varepsilon > 0$, there exist $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ such that $(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)^*(0) \le r + \varepsilon$, this implies that $(0, r + \varepsilon) \in \mathrm{epi}(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)^*$. Then,

$$\begin{aligned}(0,0,r+\varepsilon) &\in \Big\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)^*\Big\}\\ &\in \bigcup_{u\in U}\Big\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\Big\}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$(0,0,r) \in \mathrm{cl}\left[\bigcup_{u\in U}\Big\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\Big\}\right] \cap (\{0\} \times \{0\} \times \mathbb{R}).$$

Since the (AFRC) condition is satisfied by hypothesis, then we have

$$(0,0,r) \in \bigcup_{u\in U}\left\{\{0_{X^*}\} \times \mathrm{epi} g^*_{u_2} + \bigcup_{y^*\in Y^*_+}\{(a,-y^*,r) : (a,r) \in \mathrm{epi}(f_{u_1} + y^* \circ h)^*\}\right\} \cap (\{0\} \times \{0\} \times \mathbb{R}).$$

Applying Lemma 3.10 (i), there exists $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{y}^* \in Y^*_+$ satisfying

$$g^*_{\overline{u}_2}(\overline{y}^*) + (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(0) \le r.$$

Thus, by definition of $\vartheta(\mathrm{D_u})$, we have

$$\vartheta(\mathrm{D_u}) \ge -g^*_{\overline{u}_2}(\overline{y}^*) - (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(0) \ge -r = \vartheta(\mathrm{P_u}).$$

Therefore, by the weak duality, we can conclude that $\vartheta(\mathrm{P_u}) = \vartheta(\mathrm{D_u})$ and $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{y}^* \in Y^*_+$ are the optimal solution of $(\mathrm{D_u})$.
(ii) ⇒ (iii) Assume that the strong duality holds between $(\mathrm{P_u})$ and $(\mathrm{D_u})$. Let $(0,0,r) \in \{0\}\times\{0\}\times\mathbb{R}$ such that

$$(0,0,r) \in \bigcup_{u\in U}\Big\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\Big\}.$$

Then, there exist $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ such that $(0, r) \in \mathrm{epi}(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)^*$. Thus,

$$-r \le -(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)^*(0) \le -\inf_{u\in U}(f_{u_1} + g_{u_2} \circ h)^*(0) = \vartheta(P_u). \tag{3.21}$$

By the strong duality between $(\mathrm{P_u})$ and $(\mathrm{D_u})$, it results that there exist $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{y}^* \in Y^*_+$ such that $\vartheta(\mathrm{P_u}) = \vartheta(\mathrm{D_u}) = -g^*_{\overline{u}_2}(\overline{y}^*) - (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(0)$. Hence, by (3.21) we have $g^*_{\overline{u}_2}(\overline{y}^*) + (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(0) \le r$. According to Lemma 3.10 (i), we obtain that

$$(0,0,r) \in \bigcup_{u\in U}\left\{\{0_{X^*}\} \times \mathrm{epi} g^*_{u_2} + \bigcup_{y^*\in Y^*_+}\{(a,-y^*,r) : (a,r) \in \mathrm{epi}(f_{u_1} + y^* \circ h)^*\}\right\} \cap (\{0\} \times \{0\} \times \mathbb{R}).$$

Then, the implication $(ii) \Rightarrow (iii)$ is proved.
Now, we suppose that $\bigcup_{u\in U} \left\{(0,0,r) : (0,r) \in \operatorname{epi}(f_{u_1} + g_{u_2} \circ h)^*\right\}$ is closed. Then

$$(\text{FRC}) \Leftrightarrow (\text{AFRC}).$$

Thus, $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$, and the proof is complete. □

Analogously one can obtain the following theorem.

Theorem 3.13. *Consider the following statements.*

(i) The family (f_{u_1}, g_{u_2}, h, U) *satisfies the* $(\overline{\text{AFRC}})$.

(ii) The strong duality holds between (P_u) *and* $(\overline{\text{D}}_\text{u})$.

(iii) The family (f_{u_1}, g_{u_2}, h, U) *satisfies the* $(\overline{\text{FRC}})$.

Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$. *Furthermore, if* $\bigcup_{u\in U} \{(0,0,r) : (0,r) \in \operatorname{epi}(f_{u_1} + g_{u_2} \circ h)^*\}$ *is closed, then* $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Through Theorem 3.12 and Proposition 3.8 one can show the following statement.

Theorem 3.14. *Consider the following statements.*

(i) The family (f_{u_1}, g_{u_2}, h, U) *satisfies the* (ACC).

(ii) The stable strong duality exists between (P_u) *and* (D_u).

(iii) The family (f_{u_1}, g_{u_2}, h, U) *satisfies the* (CC).

Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$. *Furthermore, if* $\bigcup_{u\in U} \{(a,0,r) : (a,r) \in \operatorname{epi}(f_{u_1} + g_{u_2} \circ h)^*\}$ *is closed, then* $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Analogously, by Theorem 3.13 and Proposition 3.9, we obtain the following theorem.

Theorem 3.15. *Consider the following assertions.*

(i) The family (f_{u_1}, g_{u_2}, h, U) *satisfies the* $(\overline{\text{ACC}})$.

(ii) The stable strong duality holds between (P_u) *and* $(\overline{\text{D}}_\text{u})$.

(iii) The family (f_{u_1}, g_{u_2}, h, U) *satisfies the* $(\overline{\text{CC}})$.

Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$. *Furthermore, if* $\bigcup_{u\in U} \{(a,0,r) : (a,r) \in \operatorname{epi}(f_{u_1} + g_{u_2} \circ h)^*\}$ *is closed, then* $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

The following theorems provide a characterization for the strong duality and the stable strong duality between $(\widetilde{\text{P}}_\text{u})$ and (D_u) (resp. between $(\widetilde{\text{P}}_\text{u})$ and $(\overline{\text{D}}_\text{u})$) in terms of the (SFRC) and (SCC) (resp. the $(\overline{\text{SFRC}})$ and $(\overline{\text{SCC}})$).

Theorem 3.16. *Suppose that* $f_{u_1} : X \to \overline{\mathbb{R}}, u_1 \in U_1$ *and* $g_{u_2} : Y \to \overline{\mathbb{R}}, u_2 \in U_2$ *are lower semicontinuous, and* $h : X \to Y^\bullet$ *is* Y_+*-epi-closed mapping. Then the following assertions are hold.*

(i) The strong duality holds between $(\widetilde{\text{P}}_\text{u})$ *and* (D_u) *if and only if the family* (f_{u_1}, g_{u_2}, h, U) *satisfies the* $(SFRC)$.

(ii) The stable strong duality holds between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ *and* $(\mathrm{D}_{\mathrm{u}})$ *if and only if the family* (f_{u_1}, g_{u_2}, h, U) *satisfies the* (SCC).

Proof. (i) Assume that the strong duality hold between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$. Let $(0,0,r) \in X^* \times Y^* \times \mathbb{R}$ such that

$$(0,0,r) \in \mathrm{cl}\left[\mathrm{co} \bigcup_{u\in U} \left\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\right\}\right].$$

By applying Lemma 3.10 (ii), we have $\vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}}) \geq -r$. Since the strong duality exists between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$, then there exist $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{y}^* \in Y_+^*$ such that

$$\vartheta(\mathrm{D}_{\mathrm{u}}) = \vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}}) \geq -r \quad \text{and} \quad -\vartheta(\mathrm{D}_{\mathrm{u}}) = g^*_{\overline{u}_2}(\overline{y}^*) + (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(0). \tag{3.22}$$

Then, we obtain $g^*_{\overline{u}_2}(\overline{y}^*) + (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(0) \leq r$. This together with Lemma 3.10 (i), implies that

$$(0,0,r) \in \bigcup_{u\in U} \left\{\{0_{X^*}\} \times \mathrm{epi} g^*_{u_2} + \bigcup_{y^*\in Y_+^*} \left\{(a, -y^*, r) : (a,r) \in \mathrm{epi}(f_{u_1} + y^* \circ h)^*\right\}\right\} \cap (\{0\} \times \{0\} \times \mathbb{R}).$$

Therefore, the inclusion (3.13) holds.
Conversely, assume that the family (f_{u_1}, g_{u_2}, h, U) satisfies the (SFRC). If $\vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}}) = -\infty$, then by the weak duality, we have that $\vartheta(\mathrm{D}_{\mathrm{u}}) = -\infty$. Hence, the conclusion follows automatically. Otherwise, let $r \in \mathbb{R}$ such that $-r := \vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}})$. Then, by Lemma 3.10 (ii), we have

$$(0,0,r) \in \mathrm{cl}\left[\mathrm{co} \bigcup_{u\in U} \left\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\right\}\right],$$

and because of the (SFRC), one has

$$(0,0,r) \in \bigcup_{u\in U} \left\{\{0_{X^*}\} \times \mathrm{epi} g^*_{u_2} + \bigcup_{y^*\in Y_+^*} \{(a, -y^*, r) : (a,r) \in \mathrm{epi}(f_{u_1} + y^* \circ h)^*\}\right\} \cap (\{0\} \times \{0\} \times \mathbb{R}).$$

Then, by applying Lemma 3.10 (i), there exist $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{y}^* \in Y_+^*$ such that

$$g^*_{\overline{u}_2}(\overline{y}^*) + (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(0) \leq r.$$

It follows that

$$\vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}}) = -r \leq -g^*_{\overline{u}_2}(\overline{y}^*) - (f_{\overline{u}_2} + (\overline{y}^* \circ h))^*(0) \leq \vartheta(\mathrm{D}_{\mathrm{u}}).$$

Thus, by the weak duality, we see that $\vartheta(\widetilde{\mathrm{P}}_{\mathrm{u}}) = \vartheta(\mathrm{D}_{\mathrm{u}})$ and $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$, $\overline{y}^* \in Y_+^*$ are the optimal solutions of $(\mathrm{D}_{\mathrm{u}})$. Therefore, the strong duality exists between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$.
For (ii), by applying Proposition 3.8 and using (i), we find our results, and the proof is complete. □

Following an approach similar to Theorem 3.16, we can obtain the following theorem.

Theorem 3.17. *Suppose that* $f_{u_1} : X \to \overline{\mathbb{R}}, u_1 \in U_1$ *and* $g_{u_2} : Y \to \overline{\mathbb{R}}, u_2 \in U_2$ *are lower semicontinuous, and* $h : X \to Y^\bullet$ *is* Y_+*-epi-closed mapping. Then the following assertions hold.*

(i) The strong duality exists between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ *and* $(\overline{\mathrm{D}}_{\mathrm{u}})$ *if and only if the family* (f_{u_1}, g_{u_2}, h, U) *satisfies the* $(\overline{\mathrm{SFRC}})$.

(ii) The stable strong duality exists between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ *and* $(\overline{\mathrm{D}}_{\mathrm{u}})$ *if and only if the family* (f_{u_1}, g_{u_2}, h, U) *satisfies the* $(\overline{\mathrm{SCC}})$.

The corollary follows directly from Theorem 3.16, Theorem 3.12 and Definition 3.6, it provides the friendships between the strong duality of $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$ and the strong duality of $(\mathrm{P}_{\mathrm{u}})$ and $(\mathrm{D}_{\mathrm{u}})$.

Corollary 3.18. *Suppose that $f_{u_1}: X \to \overline{\mathbb{R}}, u_1 \in U_1$ and $g_{u_2}: Y \to \overline{\mathbb{R}}, u_2 \in U_2$ are lower semicontinuous, and $h: X \to Y^\bullet$ is Y_+-epi-closed mapping. Then the following assertions are equivalent.*

(i) The strong duality holds between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ *and* $(\mathrm{D}_{\mathrm{u}})$.

(ii) The strong duality holds between $(\mathrm{P}_{\mathrm{u}})$ *and* $(\mathrm{D}_{\mathrm{u}})$ *and* $\bigcup_{u\in U}\Big\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\Big\}$ *is closed and convex.*

By Theorem 3.17, Theorem 3.13 and Definition 3.7, we can get the following corollary.

Corollary 3.19. *Suppose that $f_{u_1}: X \to \overline{\mathbb{R}}, u_1 \in U_1$ and $g_{u_2}: Y \to \overline{\mathbb{R}}, u_2 \in U_2$ are lower semicontinuous, and $h: X \to Y^\bullet$ is Y_+-epi-closed mapping. Then the following statements are equivalent.*

(i) The strong duality holds between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ *and* $(\overline{\mathrm{D}}_{\mathrm{u}})$.

(ii) The strong duality holds between $(\mathrm{P}_{\mathrm{u}})$ *and* $(\overline{\mathrm{D}}_{\mathrm{u}})$ *and* $\bigcup_{u\in U}\Big\{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\Big\}$ *is closed and convex.*

By combining Corollary 3.18 and Proposition 3.8, we have the next corollary.

Corollary 3.20. *Suppose that $f_{u_1}: X \to \overline{\mathbb{R}}, u_1 \in U_1$ and $g_{u_2}: Y \to \overline{\mathbb{R}}, u_2 \in U_2$ are lower semicontinuous, and $h: X \to Y^\bullet$ is Y_+-epi-closed mapping. Then the following assertions are equivalent.*

(i) The stable strong duality exists between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ *and* $(\mathrm{D}_{\mathrm{u}})$.

(ii) The stable strong duality exists between $(\mathrm{P}_{\mathrm{u}})$ *and* $(\mathrm{D}_{\mathrm{u}})$ *and* $\bigcup_{u\in U}\Big\{(a,0,r) : (a,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\Big\}$ *is closed and convex.*

By combining Corollary 3.19 and Proposition 3.9, we get the following corollary.

Corollary 3.21. *Suppose that $f_{u_1}: X \to \overline{\mathbb{R}}, u_1 \in U_1$ and $g_{u_2}: Y \to \overline{\mathbb{R}}, u_2 \in U_2$ are lower semicontinuous, and $h: X \to Y^\bullet$ is Y_+-epi-closed mapping. Then the following statements are equivalent.*

(i) The stable strong duality holds between $(\widetilde{\mathrm{P}}_{\mathrm{u}})$ *and* $(\overline{\mathrm{D}}_{\mathrm{u}})$.

(ii) The stable strong duality holds between $(\mathrm{P}_{\mathrm{u}})$ *and* $(\overline{\mathrm{D}}_{\mathrm{u}})$ *and* $\bigcup_{u\in U}\Big\{(a,0,r) : (a,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ h)^*\Big\}$ *is closed and convex.*

4 Total duality

Recall that the problem $(\mathrm{P}_{\mathrm{p}})$ and the corresponding dual problems $(\mathrm{D}_{\mathrm{p}})$ and $(\overline{\mathrm{D}}_{\mathrm{p}})$ are defined by (3.1), (3.3) and (3.4), respectively. Let $p \in X^*$ and $u = (u_1, u_2) \in U$, we define the subproblem of the problem $(\mathrm{P}_{\mathrm{p}})$ by

$$(\mathrm{P}^{\mathrm{u}}_{\mathrm{p}}) \qquad \inf_{x\in X}\{f_{u_1}(x) + g_{u_2} \circ h(x) - \langle p, x\rangle\},$$

and use $\vartheta(\mathrm{P}^{\mathrm{u}}_{\mathrm{p}})$ to denote the optimal value of the problem $(\mathrm{P}^{\mathrm{u}}_{\mathrm{p}})$.

Definition 4.1. Let $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{x} \in \bigcap_{u\in U}(\mathrm{dom} f_{u_1} \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g_{u_2}))$. The family (f_{u_1}, g_{u_2}, h, U) is said to satisfy the Moreau-Reckafellar formula (MRF) at $(\overline{u}_1, \overline{u}_2, \overline{x})$ if

$$\partial(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)(\overline{x}) = \bigcup_{y^*\in\partial g_{\overline{u}_2}(h(\overline{x}))} \partial(f_{\overline{u}_1} + y^* \circ h)(\overline{x}). \tag{4.1}$$

Definition 4.2. Let $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{x} \in \bigcap\limits_{u\in U}(\mathrm{dom} f_{u_1} \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g_{u_2}))$. The family (f_{u_1}, g_{u_2}, h, U) is said to satisfy the Moreau-Reckafellar formula $(\overline{\mathrm{MRF}})$ at $(\overline{u}_1, \overline{u}_2, \overline{x})$ if

$$\partial(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)(\overline{x}) = \partial f_{\overline{u}_1}(\overline{x}) + \bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} (y^* \circ h)(\overline{x}). \tag{4.2}$$

Remark 4.3. (1) Let $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{x} \in \bigcap\limits_{u\in U}(\mathrm{dom} f_{u_1} \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g_{u_2}))$, it easy to prove that

$$\bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} \partial(f_{\overline{u}_1} + y^* \circ h)(\overline{x}) \sqsubseteq \partial(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)(\overline{x}).$$

Hence, the equality (4.1) can be replaced by

$$\partial(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)(\overline{x}) \sqsubseteq \bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} \partial(f_{\overline{u}_1} + y^* \circ h)(\overline{x}). \tag{4.3}$$

(2) Let $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{x} \in \bigcap\limits_{u\in U}(\mathrm{dom} f_{u_1} \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g_{u_2}))$, it easy to prove that

$$\partial f_{\overline{u}_1}(\overline{x}) + \bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} (y^* \circ h)(\overline{x}) \sqsubseteq \partial(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)(\overline{x}).$$

Hence, the equality (4.2) can be replaced by

$$\partial(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)(\overline{x}) \sqsubseteq \partial f_{\overline{u}_1}(\overline{x}) + \bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} (y^* \circ h)(\overline{x}). \tag{4.4}$$

(3) Let $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$ and $\overline{x} \in \mathrm{dom} f_{u_1} \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g_{u_2})$, it easy to see that

$$\partial f_{\overline{u}_1}(\overline{x}) + \bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} (y^* \circ h)(\overline{x}) \sqsubseteq \bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} \partial(f_{\overline{u}_1} + y^* \circ h)(\overline{x}). \tag{4.5}$$

Then, by (4.5), the validity of $(\overline{\mathrm{MRF}})$ implies the validity of (MRF).

This section is devoted to the study of characterizing the total duality. For this purpose, let $p \in X^*$ and $S(\mathrm{P_p})$ denote the optimal solution set of $(\mathrm{P_p})$, i.e.

$$(\overline{u}_1, \overline{u}_2, \overline{x}) \in S(\mathrm{P_p}) \Leftrightarrow f_{\overline{u}_1}(\overline{x}) + g_{\overline{u}_2} \circ h(\overline{x}) - \langle p, \overline{x}\rangle \ = \vartheta(\mathrm{P_p}) \tag{4.6}$$

and for each $u = (u_1, u_2) \in U$. Let $\mathrm{S}(\mathrm{P_p^u})$ denote the optimal solution set of $(\mathrm{P_p^u})$, i.e.

$$\overline{x} \in S(\mathrm{P_p^u}) \Leftrightarrow f_{u_1}(\overline{x}) + g_{u_2} \circ h(\overline{x}) - \langle p, \overline{x}\rangle \ = \vartheta(\mathrm{P_p^u}). \tag{4.7}$$

Theorem 4.4. *Let* $\overline{x} \in \bigcap\limits_{u\in U}(\mathrm{dom} f_{u_1} \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g_{u_2}))$ *and* $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$*. If the Moreau-Reckafellar formula* (MRF) *holds at* $(\overline{u}_1, \overline{u}_2, \overline{x})$*, then for each* $p \in X^*$ *satisfying* $(\overline{u}_1, \overline{u}_2, \overline{x}) \in S(\mathrm{P_p})$,

$$\vartheta(\mathrm{P_p}) = \max_{y^* \in Y_+^*} [-g^*_{\overline{u}_2}(y^*) - (f_{\overline{u}_1} + y^* \circ h)^*(p)]. \tag{4.8}$$

Conversely, if (4.8) *holds for each* $p \in X^*$ *satisfying* $\overline{x} \in S(\mathrm{P_p^{\overline{u}}})$ *then, the Moreau-Reckafellar formula holds at* $(\overline{u}_1, \overline{u}_2, \overline{x})$.

Proof. Suppose that the Moreau-Reckafellar formula holds at $(\overline{u}_1, \overline{u}_2, \overline{x})$. Let $p \in X^*$ such that $(\overline{u}_1, \overline{u}_2, \overline{x}) \in S(\mathrm{P_p})$, then we have

$$f_{\overline{u}_1}(\overline{x}) + g_{\overline{u}_2} \circ h(\overline{x}) - \langle p, \overline{x}\rangle \ = \vartheta(\mathrm{P_p}). \tag{4.9}$$

It follows by (2.2) that $0 \in \partial(f_{\overline{u}_1} - p + g_{\overline{u}_2} \circ h)(\overline{x})$, which is equivalent to $p \in \partial(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)(\overline{x})$. This together which (4.1) implies that $p \in \bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} \partial(f_{\overline{u}_1} + y^* \circ h)(\overline{x})$. Therefore, there exist $\overline{y}^* \in \partial g_{\overline{u}_2}(h(\overline{x}))$ such that $p \in \partial(f_{\overline{u}_1} + \overline{y}^* \circ h)(\overline{x})$. By applying the Young equality (2.3), one has

$$(f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p) + (f_{\overline{u}_1} + \overline{y}^* \circ h)(\overline{x}) = \langle p, \overline{x} \rangle$$

and

$$g^*_{\overline{u}_2}(\overline{y}^*) + g_{\overline{u}_2} \circ h(\overline{x}) = \langle \overline{y}^*, h(\overline{x}) \rangle.$$

By adding the equalities above, we obtain

$$(f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p) + g^*_{\overline{u}_2}(\overline{y}^*) + f_{\overline{u}_1}(\overline{x}) + g_{\overline{u}_2} \circ h(\overline{x}) = \langle p, \overline{x} \rangle.$$

Hence, by (4.9), we get

$$-(f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p) - g^*_{\overline{u}_2}(\overline{y}^*) = f_{\overline{u}_1}(\overline{x}) + g_{\overline{u}_2} \circ h(\overline{x}) - \langle p, \overline{x} \rangle = \vartheta(\mathrm{P_p}).$$

While, by the definition of $\vartheta(\mathrm{D_p})$, one has that

$$\vartheta(\mathrm{D_p}) \geq -(f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p) - g^*_{\overline{u}_2}(\overline{y}^*) = \vartheta(\mathrm{P_p}).$$

By combining this result with the weak duality, we obtain.

$$\vartheta(\mathrm{D_p}) = -(f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p) - g^*_{\overline{u}_2}(\overline{y}^*) = \vartheta(\mathrm{P_p}).$$

Thus, (4.8) holds.
Conversely, let $p \in \partial(f_{\overline{u}_1} + g_{\overline{u}_2} \circ h)(\overline{x})$. Then $0 \in \partial(f_{\overline{u}_1} - p + g_{\overline{u}_2} \circ h)(\overline{x})$ and by (2.2), we have

$$f_{\overline{u}_1}(\overline{x}) + g_{\overline{u}_2} \circ h(\overline{x}) - \langle p, \overline{x} \rangle \leq f_{\overline{u}_1}(x) + g_{\overline{u}_2} \circ h(x) - \langle p, x \rangle, \quad \forall x \in X.$$

This means $\overline{x} \in S(\mathrm{P}^{\overline{\mathrm{u}}}_{\mathrm{p}})$. Thus, by (4.8), we have that

$$f_{\overline{u}_1}(\overline{x}) + g_{\overline{u}_2} \circ h(\overline{x}) - \langle p, \overline{x} \rangle \leq \max_{y^* \in Y^*_+} \Big[- g^*_{\overline{u}_2}(y^*) - (f_{\overline{u}_1} + y^* \circ h)^*(p) \Big].$$

This implies that there exist $\overline{y}^* \in Y^*_+$ such that

$$f_{\overline{u}_1}(\overline{x}) + g_{\overline{u}_2} \circ h(\overline{x}) - \langle p, \overline{x} \rangle \leq -g^*_{\overline{u}_2}(\overline{y}^*) - (f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p).$$

Noting above inequality and utilising the definition of the conjugate function, we see that

$$0 \leq g^*_{\overline{u}_2}(\overline{y}^*) + g_{\overline{u}_2} \circ h(\overline{x}) - \langle \overline{y}^*, h(\overline{x}) \rangle \leq -(f_{\overline{u}_1} + \overline{y}^* \circ h)(\overline{x}) - (f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p) + \langle p, \overline{x} \rangle \leq 0.$$

It follows that

$$g^*_{\overline{u}_2}(\overline{y}^*) + g_{\overline{u}_2} \circ h(\overline{x}) - \langle \overline{y}^*, h(\overline{x}) \rangle = 0$$

and

$$(f_{\overline{u}_1} + \overline{y}^* \circ h)(\overline{x}) + (f_{\overline{u}_1} + \overline{y}^* \circ h)^*(p) - \langle p, \overline{x} \rangle = 0.$$

Consequently, $\overline{y}^* \in \partial g_{\overline{u}_2}(h(\overline{x}))$ and $p \in \partial(f_{\overline{u}_1} + \overline{y}^* \circ h)(\overline{x})$, then $p \in \bigcup_{y^* \in \partial g_{\overline{u}_2}(h(\overline{x}))} \partial(f_{\overline{u}_1} + y^* \circ h)(\overline{x})$. The proof is complete because the converse inclusion automatically holds. □

With a similar method of Theorem 4.4, we can get the following theorem.

Theorem 4.5. *Let* $\overline{x} \in \bigcap_{u \in U} (\mathrm{dom} f_{u_1} \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g_{u_2}))$ *and* $\overline{u} = (\overline{u}_1, \overline{u}_2) \in U$. *If the Moreau-Reckafellar formula* $(\overline{\mathrm{MRF}})$ *holds at* $(\overline{u}_1, \overline{u}_2, \overline{x})$, *then for each* $p \in X^*$ *satisfying* $(\overline{u}_1, \overline{u}_2, \overline{x}) \in S(\mathrm{P_p})$,

$$\vartheta(\mathrm{P_p}) = \max_{\substack{x^* \in X^* \\ y^* \in Y^*_+}} [-g^*_{\overline{u}_2}(y^*) - f^*_{\overline{u}_1}(x^*) - (y^* \circ h)^*(p - x^*)]. \tag{4.10}$$

Conversely, if (4.10) *holds for any* $p \in X^*$ *satisfying* $\overline{x} \in S(\mathrm{P}^{\overline{\mathrm{u}}}_{\mathrm{p}})$, *then the Moreau-Reckafellar formula holds at* $(\overline{u}_1, \overline{u}_2, \overline{x})$.

5 A special case

In this section, we apply the approach from the preceding section to a specific case of our general results. Let $h(x) = Ax$ for each $x \in X$, where $A : X \to Y$ is a linear continuous mapping, and let $Y_+ = \{0\}$. Then $Y_+^* = Y^*$ and h is Y_+-convex and Y_+-epi-closed. So, the problems $(\mathrm{P_u})$ and $(\widetilde{\mathrm{P}}_\mathrm{u})$ become

$$(\mathrm{P}^1) \quad \sup_{u\in U} \inf_{x\in X} \{f_{u_1}(x) + g_{u_2}(Ax)\}, \tag{5.1}$$

and

$$(\widetilde{\mathrm{P}}^1) \quad \inf_{x\in X} \sup_{u\in U} \{f_{u_1}(x) + g_{u_2}(Ax)\}. \tag{5.2}$$

The linearly perturbed problems $(\mathrm{P_p})$ and $(\widetilde{\mathrm{P}}_\mathrm{p})$ become

$$(\mathrm{P}^1_\mathrm{p}) \quad \sup_{u\in U} \inf_{x\in X} \{f_{u_1}(x) + g_{u_2}(Ax) - \langle p, x\rangle\}, \tag{5.3}$$

and

$$(\widetilde{\mathrm{P}}^1_\mathrm{p}) \quad \inf_{x\in X} \sup_{u\in U} \{f_{u_1}(x) + g_{u_2}(Ax) - \langle p, x\rangle\}. \tag{5.4}$$

Moreover, since the adjoint operator $A^* : Y^* \to X^*$ of A, is defined by $A^*(y^*) = y^* \circ A$ for all $y^* \in Y^*$. Then,

$$(f_{u_1} + y^* \circ A)^*(p) = f^*_{u_1}(p - A^*y^*),$$

and

$$(y^* \circ A)^*(p) = \begin{cases} 0, & \text{if } A^*y^* = p, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, the dual problems $(\mathrm{D_p})$ and $(\overline{\mathrm{D}}_\mathrm{p})$ become the same dual problem

$$(\mathrm{D}^1_\mathrm{p}) \quad \sup_{u\in U} \sup_{y^*\in Y^*} \{-g^*_{u_2}(y^*) - f^*_{u_1}(p - A^*y^*)\}, \tag{5.5}$$

and the dual problems $(\mathrm{D_u})$ and $(\overline{\mathrm{D}}_\mathrm{u})$ become the same dual problem

$$(\mathrm{D}^1) \quad \sup_{u\in U} \sup_{y^*\in Y^*} \{-g^*_{u_2}(y^*) - f^*_{u_1}(-A^*y^*)\}. \tag{5.6}$$

The strong further regularity condition derived from (SFRC) would be in this case

$$(\mathrm{SFRC})^1 : \quad \mathrm{cl}\left[\mathrm{co} \bigcup_{u\in U} \{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ A)^*\}\right] \subseteq \bigcup_{u\in U} \Big\{\{0_{X^*}\} \times \mathrm{epi} g^*_{u_2} + \bigcup_{y^*\in Y^*} \Big\{(p, -y^*, r) : (p,r) \in \mathrm{epi}(f_{u_1} + y^* \circ A)^*\Big\}\Big\} \cap (\{0\} \times \{0\} \times \mathbb{R}),$$

while $(\overline{\mathrm{SFRC}})$ turns into

$$(\overline{\mathrm{SFRC}})^1 : \quad \mathrm{cl}\left[\mathrm{co} \bigcup_{u\in U} \{(0,0,r) : (0,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ A)^*\}\right] \subseteq \bigcup_{u\in U} \Big\{\{0_{X^*}\} \times \mathrm{epi} g^*_{u_2} + \{(p,0,r) : (p,r) \in \mathrm{epi} f^*_{u_1}\} + \bigcup_{y^*\in Y^*} \Big\{(p, -y^*, r) : (p,r) \in \mathrm{epi}(y^* \circ A)^*\Big\}\Big\} \cap (\{0\} \times \{0\} \times \mathbb{R}),$$

and the strong closure condition derived from (SCC) would be in this case

$$(\mathrm{SCC})^1 : \quad \mathrm{cl}\left[\mathrm{co} \bigcup_{u\in U} \Big\{(p,0,r) : (p,r) \in \mathrm{epi}(f_{u_1} + g_{u_2} \circ A)^*\Big\}\right] \subseteq \bigcup_{u\in U} \Big\{\{0_{X^*}\} \times \mathrm{epi} g^*_{u_2} + \bigcup_{y^*\in Y^*} \Big\{(p, -y^*, r) : (p,r) \in \mathrm{epi}(f_{u_1} + y^* \circ A)^*\Big\}\Big\},$$

while $(\overline{\text{SCC}})$ turns into

$$(\overline{\text{SCC}})^1: \quad \text{cl}\left[\text{co}\bigcup_{u\in U}\Big\{(p,0,r):(p,r)\in \text{epi}(f_{u_1}+g_{u_2}\circ A)^*\Big\}\right]\subseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times \text{epi}g^*_{u_2}+ \{(p,0,r):(p,r)\in \text{epi}f^*_{u_1}\}+\bigcup_{y^*\in Y^*}\Big\{(p,-y^*,r):(p,r)\in \text{epi}(y^*\circ A)^*\Big\}\Big\}.$$

For each $y^*\in Y^*$ and $p\in X^*$. One has

$$\begin{aligned}&\bigcup_{y^*\in Y^*}\Big\{(p,-y^*,r):(p,r)\in \text{epi}(f_{u_1}+y^*\circ A)^*\Big\}\\ &\quad=\bigcup_{y^*\in Y^*}\Big\{(p,-y^*,r):(p-A^*y^*,r)\in \text{epi}f^*_{u_1}\Big\}\\ &\quad=\bigcup_{y^*\in Y^*}\Big\{(p+A^*y^*,-y^*,r):(p,r)\in \text{epi}f^*_{u_1}\Big\}\\ &\quad=\Big\{(p,0,r):(p,r)\in \text{epi}f^*_{u_1}\Big\}+\Big\{(A^*y^*,-y^*,0):y^*\in Y^*\Big\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\bigcup_{y^*\in Y^*}\{(p,-y^*,r):(p,r)\in \text{epi}(y^*\circ A)^*\} &= \bigcup_{y^*\in Y^*}\{(A^*y^*,-y^*,r):0\le r\}\\ &= \{(A^*y^*,-y^*,0):y^*\in Y^*\}+\{(0,0)\}\times\mathbb{R}_+\end{aligned}$$

Therefore,

$$\begin{aligned}&\{0_{X^*}\}\times \text{epi}g^*_{u_2}+\bigcup_{y^*\in Y^*}\{(p,-y^*,r):(p,r)\in \text{epi}(f_{u_1}+y^*\circ A)^*\}\\ &=\{0_{X^*}\}\times \text{epi}g^*_{u_2}+\{(p,0,r):(p,r)\in \text{epi}f^*_{u_1}\}+\bigcup_{y^*\in Y^*}\{(p,-y^*,r):(p,r)\in \text{epi}(y^*\circ A)^*\}.\end{aligned}$$

We are going to consider the identity map $\text{id}_{\mathbb{R}}$ on $\mathbb{R}$, and the image set $(A^*\times \text{id}_{\mathbb{R}})(\text{epi}g^*_{u_2})$ of $\text{epi}g^*_{u_2}$ through the map $A^*\times \text{id}_{\mathbb{R}}: Y^*\times\mathbb{R}\to X^*\times\mathbb{R}$, defined by

$$(x^*,r)\in (A^*\times \text{id}_{\mathbb{R}})(\text{epi}g^*_{u_2})\Leftrightarrow\begin{cases}\exists y^*\in Y^* \text{ such that } (y^*,r)\in \text{epi}g^*_{u_2},\\ \text{and } A^*y^*=x^*.\end{cases}$$

Then, we obtain an equivalence of the strong further regularity conditions $(\text{SFRC})^1$ and $(\overline{\text{SFRC}})^1$ (resp. the strong closure conditions $(\text{SCC})^1$ and $(\overline{\text{SCC}})^1$). They are equivalent to the following strong further regularity condition $(\text{SFRC})_A$ (resp. the strong closure condition $(\text{SCC})_A$).

Definition 5.1. The family (f_{u_1},g_{u_2},A,U) is said to satisfy

(i) the strong further regularity condition $(\text{SFRC})_A$ if

$$\text{cl}\left[\text{co}\bigcup_{u\in U}\text{epi}(f_{u_1}+g_{u_2}\circ A)^*\right]\cap(\{0\}\times\mathbb{R})\subseteq\bigcup_{u\in U}\Big(\text{epi}f^*_{u_1}+(A^*\times \text{id}_{\mathbb{R}})(\text{epi}g^*_{u_2})\Big)\cap(\{0\}\times\mathbb{R});$$

(ii) the strong closure condition $(\text{SCC})_A$ if

$$\text{cl}\left[\text{co}\bigcup_{u\in U}\text{epi}(f_{u_1}+g_{u_2}\circ A)^*\right]\subseteq\bigcup_{u\in U}\Big(\text{epi}f^*_{u_1}+(A^*\times \text{id}_{\mathbb{R}})(\text{epi}g^*_{u_2})\Big).$$

By utilizing the same methods of Section 3, we can characterize the strong duality and the stable strong duality of the problems $(\widetilde{\text{P}}^1)$ and (D^1).

Theorem 5.2. *Suppose that $f_{u_1} : X \longrightarrow \overline{\mathbb{R}}, u_1 \in U_1$ and $g_{u_2} : Y \longrightarrow \overline{\mathbb{R}}, u_2 \in U_2$ are lower semicontinuous, and $A : X \longrightarrow Y$ is continuous. Then the following statements hold.*

(i) The strong duality exists between $(\widetilde{\mathrm{P}}^1)$ and (D^1) if and only if the family (f_{u_1}, g_{u_2}, A, U) satisfies the $(\mathrm{SFRC})_A$.

(ii) The stable strong duality exists between $(\widetilde{\mathrm{P}}^1)$ and (D^1) if and only if the family (f_{u_1}, g_{u_2}, A, U) satisfies the $(\mathrm{SCC})_A$.

Corollary 5.3. *Suppose that $f_{u_1} : X \longrightarrow \overline{\mathbb{R}}, u_1 \in U_1$ and $g_{u_2} : Y \longrightarrow \overline{\mathbb{R}}, u_2 \in U_2$ are lower semicontinuous, and $A : X \longrightarrow Y$ is continuous. Then the following assertions equivalent.*

(i) The strong duality exists between $(\widetilde{\mathrm{P}}^1)$ and (D^1).

(ii) The strong duality exists between (P^1) and (D^1) and $\bigcup_{u\in U} \mathrm{epi}(f_{u_1} + g_{u_2} \circ A)^* \cap (\{0\} \times \mathbb{R})$ is closed and convex.

Corollary 5.4. *Suppose that $f_{u_1} : X \longrightarrow \overline{\mathbb{R}}, u_1 \in U_1$ and $g_{u_2} : Y \longrightarrow \overline{\mathbb{R}}, u_2 \in U_2$ are lower semicontinuous, and $A : X \longrightarrow Y$ is continuous. Then the following assertions equivalent.*

(i) The stable strong duality exists between $(\widetilde{\mathrm{P}}^1)$ and (D^1).

(ii) The stable strong duality exists between (P^1) and (D^1) and $\bigcup_{u\in U} \mathrm{epi}(f_{u_1} + g_{u_2} \circ A)^*$ is closed and convex.

6 Application to classical optimization problem

Consider the cone-constrained optimization problem

$$(R) \min_{h(x)\in -Y_+} f(x).$$

where X, Y and Z are real locally convex Hausdorff topological vector spaces, $f : X \longrightarrow \overline{\mathbb{R}}$ be a proper convex function, and $h : X \longrightarrow Y^\bullet$ be a proper and Y_+-convex mapping such that $\mathrm{dom} f \cap \mathrm{dom} h \cap h^{-1}(-Y_+) \neq \emptyset$, with Y_+ be a nonempty subset closed convex cone of Y. By applying the indicator function $\delta_{-Y_+} : X \longrightarrow \overline{\mathbb{R}}$, the problem (R) may be rewritten equivalently as the unconstrained composed convex problem $(\mathrm{P_h})$

$$(\mathrm{P_h}) \qquad \inf_{x\in X} \{f(x) + g \circ h(x)\},$$

where

$$\begin{array}{rccl} g: & Y & \longrightarrow & \overline{\mathbb{R}} \\ & y & \longrightarrow & g(y) := \delta_{-Y_+}(y). \end{array}$$

By Lemma 2.2, the function g is convex, proper, lower semicontinuous and Y_+-nondecreasing. Let $U \subseteq Z$, $f_u : X \longrightarrow \overline{\mathbb{R}}, u \in U$ be a proper convex function, and $h : X \longrightarrow Y^\bullet$ be a proper and Y_+-convex mapping such that $\mathrm{dom} f_u \cap \mathrm{dom} h \cap h^{-1}(-Y_+) \neq \emptyset$. Then, the problems $(\mathrm{P_u})$ and $(\widetilde{\mathrm{P}}_\mathrm{u})$ become

$$(\mathrm{P}^2_\mathrm{u}) \quad \sup_{u\in U} \inf_{x\in X} \{f_u(x) + g \circ h(x)\},$$

$$(\widetilde{\mathrm{P}}^2_\mathrm{u}) \quad \inf_{x\in X} \sup_{u\in U} \{f_u(x) + g \circ h(x)\}.$$

Furthermore, since $g^*(y^*) = 0$, for any $y^* \in Y^*_+$, then, the corresponding dual problems

$$(\mathrm{D}^2_\mathrm{u}) \quad \sup_{u\in U} \sup_{y^*\in Y^*_+} \{-(f_u + y^* \circ h)^*(0)\},$$

$$(\overline{\mathrm{D}}_{\mathrm{u}}^2) \quad \sup_{u\in U} \sup_{\substack{x^*\in X^*\\ y^*\in Y_+^*}} \{-f_u^*(x^*) - (y^*\circ h)^*(-x^*)\}.$$

The linearly perturbed problems $(\mathrm{P_p})$ and $(\widetilde{\mathrm{P}}_{\mathrm{p}})$ become

$$(\mathrm{P}_{\mathrm{p}}^2) \quad \sup_{u\in U} \inf_{x\in X} \{f_u(x) + g\circ h(x) - \langle p, x\rangle\},$$

$$(\widetilde{\mathrm{P}}_{\mathrm{p}}^2) \quad \inf_{x\in X} \sup_{u\in U} \{f_u(x) + g\circ h(x) - \langle p, x\rangle\}.$$

Moreover, since $\mathrm{epi} g^* = Y_+^* \times \mathbb{R}_+$, for each $y^* \in Y_+^*$. Thus, the strong further regularity condition derived from (SFRC) would be in this case

$$(\mathrm{SFRC})^2: \quad \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\{(0,0,r) : (0,r)\in \mathrm{epi}(f_u + g\circ h)^*\}\right] \sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times Y_+^*\times\mathbb{R}_+ + \bigcup_{y^*\in Y_+^*}\{(a,-y^*,r) : (a,r)\in \mathrm{epi}(f_u + y^*\circ h)^*\}\Big\}\cap(\{0\}\times\{0\}\times\mathbb{R}),$$

while $(\overline{\mathrm{SFRC}})$ becomes

$$(\overline{\mathrm{SFRC}})^2: \quad \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\{(0,0,r) : (0,r)\in \mathrm{epi}(f_u + g\circ h)^*\}\right] \sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times Y_+^*\times\mathbb{R}_+ + \{(p,0,r) : (p,r)\in \mathrm{epi} f_u^*\} + \bigcup_{y^*\in Y_+^*}\{(p,-y^*,r) : (p,r)\in \mathrm{epi}(y^*\circ h)^*\}\Big\}\cap(\{0\}\times\{0\}\times\mathbb{R}),$$

and the strong closure condition derived from (SCC) would then be

$$(\mathrm{SCC})^2: \quad \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\{(a,0,r) : (a,r)\in \mathrm{epi}(f_u + g\circ h)^*\}\right] \sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times Y_+^*\times\mathbb{R}_+ + \bigcup_{y^*\in Y_+^*}\{(a,-y^*,r) : (a,r)\in \mathrm{epi}(f_u + y^*\circ h)^*\}\Big\}\cap(X^*\times\{0\}\times\mathbb{R}),$$

while $(\overline{\mathrm{SCC}})$ becomes

$$(\overline{\mathrm{SCC}})^2: \quad \mathrm{cl}\left[\mathrm{co}\bigcup_{u\in U}\{(a,0,r) : (a,r)\in \mathrm{epi}(f_u + g\circ h)^*\}\right] \sqsubseteq \bigcup_{u\in U}\Big\{\{0_{X^*}\}\times Y_+^*\times\mathbb{R}_+ + \{(p,0,r) : (p,r)\in \mathrm{epi} f_u^*\} + \bigcup_{y^*\in Y_+^*}\{(p,-y^*,r) : (p,r)\in \mathrm{epi}(y^*\circ h)^*\}\Big\}\cap(X^*\times\{0\}\times\mathbb{R}).$$

The following corollary gives a characterization of the strong duality and the stable strong duality between $(\widetilde{\mathrm{P}}_{\mathrm{u}}^2)$ and $(\mathrm{D}_{\mathrm{u}}^2)$ (resp. between $(\widetilde{\mathrm{P}}_{\mathrm{u}}^2)$ and $(\overline{\mathrm{D}}_{\mathrm{u}}^2)$) in terms of the $(\mathrm{SFRC})^2$ and $(\mathrm{SCC})^2$ (resp. the $(\overline{\mathrm{SFRC}})^2$ and $(\overline{\mathrm{SCC}})^2$).

Corollary 6.1. *Suppose that $f_u : X \to \overline{\mathbb{R}}, u \in U$ are lower semicontinuous, and $h : X \to Y^\bullet$ is Y_+-epi-closed mapping. Then the following statements hold.*

(i) The strong duality exists between $(\widetilde{\mathrm{P}}_{\mathrm{u}}^2)$ and $(\mathrm{D}_{\mathrm{u}}^2)$ if and only if the family (f_u, g, h, U) satisfies the $(\mathrm{SFRC})^2$.

(ii) The stable strong duality exists between $(\widetilde{\mathrm{P}}_{\mathrm{u}}^2)$ and $(\mathrm{D}_{\mathrm{u}}^2)$ if and only if the family (f_u, g, h, U) satisfies the $(\mathrm{SCC})^2$.

Corollary 6.2. *Suppose that $f_u : X \to \overline{\mathbb{R}}, u \in U$ are lower semicontinuous, and $h : X \to Y^\bullet$ is Y_+-epi-closed mapping. Then the following statements hold.*

(i) The strong duality exists between $(\widetilde{\mathrm{P}}^2_{\mathrm{u}})$ and $(\overline{\mathrm{D}}^2_{\mathrm{u}})$ if and only if the family (f_u, g, h, U) satisfies the $(\overline{\mathrm{SFRC}})^2$.

(ii) The stable strong duality exists between $(\widetilde{\mathrm{P}}^2_{\mathrm{u}})$ and $(\overline{\mathrm{D}}^2_{\mathrm{u}})$ if and only if the family (f_u, g, h, U) satisfies the $(\overline{\mathrm{SCC}})^2$.

Next, we give some sufficient and necessary conditions for the total duality. Let $p \in X^*$ and $u \in U$, we define the subproblem of the problem $(\mathrm{P}^2_{\mathrm{p}})$ by

$$(\mathrm{P}^{\mathrm{u}}_{\mathrm{p}})^2 \qquad \inf_{x\in X}\{f_u(x) + g \circ h(x) - \langle p, x\rangle\}.$$

Corollary 6.3. *Let $\overline{x} \in \bigcap_{u\in U}(\mathrm{dom} f_u \cap \mathrm{dom} h \cap h^{-1}(-Y_+))$ and $\overline{u} \in U$. If the Moreau-Reckafellar formula* (MRF) *holds at $(\overline{u}, \overline{x})$, then, for each $p \in X^*$ satisfying $(\overline{u}, \overline{x}) \in \mathrm{S}(\mathrm{P}^2_{\mathrm{p}})$,*

$$\vartheta(\mathrm{P}^2_{\mathrm{p}}) = \max_{y^*\in Y^*_+}[-(f_{\overline{u}} + y^* \circ h)^*(p)]. \tag{6.1}$$

Conversely, if (6.1) *holds for each $p \in X^*$ satisfying $\overline{x} \in \mathrm{S}((\mathrm{P}^{\overline{\mathrm{u}}}_{\mathrm{p}})^2)$, then the Moreau-Reckafellar formula holds at $(\overline{u}, \overline{x})$.*

Corollary 6.4. *Let $\overline{x} \in \bigcap_{u\in U}(\mathrm{dom} f_u \cap \mathrm{dom} h \cap h^{-1}(-Y_+))$ and $\overline{u} \in U$. If the Moreau-Reckafellar formula* $(\overline{\mathrm{MRF}})$ *holds at $(\overline{u}, \overline{x})$, then, for each $p \in X^*$ satisfying $(\overline{u}, \overline{x}) \in \mathrm{S}(\mathrm{P}^2_{\mathrm{p}})$,*

$$\vartheta(\mathrm{P}^2_{\mathrm{p}}) = \max_{\substack{x^*\in X^*\\ y^*\in Y^*_+}}[-f^*_{\overline{u}}(x^*) - (y^* \circ h)^*(p - x^*)]. \tag{6.2}$$

Conversely, if (6.2) *holds for each $p \in X^*$ satisfying $\overline{x} \in \mathrm{S}((\mathrm{P}^{\overline{\mathrm{u}}}_{\mathrm{p}})^2)$, then the Moreau-Reckafellar formula holds at $(\overline{u}, \overline{x})$.*

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A. Rikouane

Department of Mathematics, Faculty of Sciences
B.P. 8106, Agadir, Morocco.

E-mail address: rikouaneah@gmail.com

M. Laghdir and M. Mabrouk

Departement of Mathematics, Faculty of Sciences,
B.P. 20, El-Jadida, Morocco.