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A fast and efficient iterative method for solving some class of Toeplitz linear equations systems

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Abstract

We present an innovative and straightforward algorithm designed to solving a class of linear equations systems $A\mathrm{x} = \mathrm{y}$ where A is an $n \times n$ Toeplitz matrix. Whenever the symbol function of the Toeplitz matrix A remains away from zero, our method enables us to efficiently approximate the system's solution, utilizing only $O(n \log n)$ arithmetic operations. This iterative approach involves expanding the original matrix by incorporating an associated circulant matrix and exploiting the properties of the symbol function to compute its inverse.

Keywords: Linear system, Toeplitz matrix, circulant matrix, iterative method.

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1 Introduction

A *Toeplitz system* is a system of linear equations whose coefficients matrices are Toeplitz matrices. A Toeplitz matrix is a special type of matrix that is constant along its diagonals. This structural characteristic allows efficient algorithms and methods to be developed for solving Toeplitz systems. This is a fundamental problem in numerical linear algebra, with a wide range of practical applications in optimization theory, signal processing, and many other fields.

Numerical solutions of Toeplitz systems are essential across various application domains; e.g. [12, 14, 3, 4]. These equations can represent a variety of real-world problems, such as signal processing, image reconstruction, and time series analysis. The challenge lies in devising techniques that take advantage of the specific structure of Toeplitz matrices to reduce the computational complexity and improve the efficiency of solving such systems. Solving Toeplitz systems is made efficiently by the Levinson algorithm [13], with an $O(n^2)$ time complexity. Furthermore, variants of this algorithm have been shown to be weakly stable.

This paper aims to introduce an algorithm for efficiently solving a specific class of Toeplitz systems with reduced computational complexity. It is known that any Toeplitz matrix can be assigned a corresponding symbol function [1, 10]. We have previously applied this concept for other ends [2, 1]. This time, using the Banach contraction principle, we are leveraging this feature to solve special classes of Toeplitz systems. More precisely, for our proposed iteration method to converge, the generator function of the Toeplitz matrix of the system should be away from zero.

Consider an $n \times n$ linear equations system $A\mathrm{x} = \mathrm{y}$ where $A = T_n(a)$, i.e. a is the symbol or generating function of Toeplitz matrix A. In this approach by extending the Toeplitz matrix $T_n(a)$ to a circulant matrix $C_m(a)$ with $m = 2n$; we address the system $C_m(a)\begin{bmatrix}\mathrm{x}\\ \mathrm{x}\end{bmatrix} = \begin{bmatrix}\mathrm{y} + E_n(a)\mathrm{x}\\ \mathrm{y} + E_n(a)\mathrm{x}\end{bmatrix}$ where the matrix $E_n(a)$ is another Toeplitz matrix with zero entries on main diagonal derived from A; see (3.1). It is known that $C_m(a)^{-1} = C_m(1/a)$; see [9]. This property helps us bypass the direct computation of the inverse of $C_m(a)$ and, instead, compute the entries of $C_m(1/a)$. These entries can be computed using the FFT algorithm, which entails $O(m \log m)$ arithmetic operations [6, 11]. Since $m = 2n$, the algorithm can be executed with a computational complexity of $O(n \log n)$. Finally, we have a *fixed point problem* that can be solved via the standard Banach contraction principle. The remaining steps of the algorithm do not demand more than $O(n \log n)$ flops and the convergence rate can be extremely fast, provided that the diagonal entry of A is sufficiently large.

The paper is organized as follows: Section 2 provides essential definitions and theoretical results. Section 3 introduces the algorithm and states a theorem on its convergence. Section 4 discusses some numerical examples.

2 Some classical results

A Toeplitz matrix is an $n \times n$ matrix in which entries along their diagonals are constant, i.e., a matrix of the form

$$T_n = \begin{bmatrix} a_0 & a_{-1} & \cdots & & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & \\ \vdots & & \ddots & & \vdots \\ & & \ddots & & a_{-1} \\ a_{n-1} & & \cdots & a_1 & a_0 \end{bmatrix} \tag{2.1}$$

where $\{a_k\}_{|k|<n} \subset \mathbb{C}$. A *circulant* matrix is a Toeplitz matrix such that each row of the matrix is a right cyclic shift of the row above it, i.e., $a_k = a_{-(n-k)}$ for $k = 1, 2, \cdots, n-1$. Hence, a circulant matrix has the following form

$$C_n = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-(n-1)} \\ a_{-(n-1)} & a_0 & a_{-1} & & \\ a_{-(n-2)} & a_{-(n-1)} & a_0 & & \vdots \\ \vdots & & & & \ddots \\ a_{-1} & a_{-2} & & \cdots & a_0 \end{bmatrix}. \tag{2.2}$$

In several applications, we come across the system

$$T_n\mathrm{x} = \begin{bmatrix} a_0 & a_{-1} & \cdots & & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & \\ \vdots & & \ddots & & \vdots \\ & & \ddots & & a_{-1} \\ a_{n-1} & & \cdots & a_1 & a_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \mathrm{y} \tag{2.3}$$

of linear equations.

Before describing our method, it is useful to recall some preliminary definitions and results. Consider the unit circle in the complex numbers plane, $\mathbb{T} = \{e^{i\theta} | \theta \in \mathbb{R}\}$, equipped with the Lebesgue measure. The Fourier coefficients of any $a \in L^\infty(\mathbb{T})$ are given by a sequence $\{a_k\}_{k\in\mathbb{Z}}$ where

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-ik\theta}d\theta, \quad (k \in \mathbb{Z}).$$

Moreover, the sequence $\{a_k\}$ determines the function a via [1]

$$a(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \quad (0 \leqslant \theta \leqslant 2\pi).$$

Any function $a \in L^\infty(\mathbb{T})$ generates a unique *Toeplitz operator* $T(a)$ acts on $L^2(\mathbb{T})$; see [5]. The Toeplitz matrix T_n (2.1), hereafter denoted by $T_n(a)$, is the $n\times n$ truncated matrix of $T(a)$. Intuitively, the sequence $\{T_n(a)\}$ approaches $T(a)$ as n goes to infinity, but in this paper, we confine ourselves to the finite case. Furthermore, we have

$$\|T_n(a)\| \le \|T(a)\| \le \|a\|_\infty. \tag{2.4}$$

The rest of this section is mainly adapted from [9].

Definition 2.1. A function $a \in L^\infty(\mathbb{T})$ is considered to be of *Wiener class* if the Fourier coefficients $\{a_k\}$ of its Fourier series are absolutely summable, i.e., $\sum_{k\in\mathbb{Z}} |a_k| < \infty$. A sequence of Toeplitz matrices, $\{T_n(a)\}$ is of Wiener class if its symbol function a is of Wiener class.

Notice that the symbol function associated with the system (2.3) is of Wiener class as a trigonometric polynomial. In general, assuming a be in Wiener class and fixed $m \ge 1$, one can define

$$c_k^{(m)} = \frac{1}{m}\sum_{j=0}^{m-1} a\left(\frac{2\pi j}{m}\right) e^{2\pi ijk/m}. \tag{2.5}$$

We can now construct the circulant matrix $C_m(a)$ with the top row $\left(c_0^{(m)}, \cdots, c_{m-1}^{(m)}\right)$, i.e.

$$C_m(a) = \begin{bmatrix} c_0^{(m)} & \cdots & & c_{m-1}^{(m)} \\ c_{m-1}^{(m)} & c_0^{(m)} & \cdots & \\ \vdots & & \ddots & \vdots \\ c_1^{(m)} & \cdots & c_{m-1}^{(m)} & c_0^{(m)} \end{bmatrix}. \tag{2.6}$$

We have the following extremely useful lemma.

Lemma 2.2 ([9])**.** *Assume $a \in L^\infty(\mathbb{T})$ be of Wiener class and its circulant matrix $C_m(a)$ with top row* (2.5)*. Then*

$$C_m(a)^{-1} = C_m(\frac{1}{a}) \tag{2.7}$$

for any $m \ge 1$ provided that the generating function is away from zero, i.e. $\operatorname{ess\,inf}\{|a(\theta)|; \theta \in [0, 2\pi]\} > 0$.

In fact, we apply a special case of this process. First, let us recall another basic notion in matrix theory.

Definition 2.3. Consider an $n \times n$ matrix $A = [a_{k,j}]$. Let $\lambda_1, \cdots, \lambda_n \ge 0$ be the eigenvalues of positive semi-definite matrix A^*A, where A^* is the adjoint matrix of A. The *weak* norm of A, also known as the Hilbert-Schmidt norm, is defined as

$$|A| = \left(\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} |a_{k,j}|^2\right)^{\frac{1}{2}} = (\frac{1}{n}\sum_{k=0}^{n-1}\lambda_k)^{\frac{1}{2}} = \frac{1}{\sqrt{n}}\|A\|_{\mathrm{F}} \tag{2.8}$$

where $\|A\|_{\mathrm{F}} = (\mathrm{tr}(A^*A))^{\frac{1}{2}}$ is the Frobenius or Euclidean norm of A.

[1] To be mathematically rigor, one may write $a(e^{i\theta})$ instead of $a(\theta)$. Nonetheless, here the notation does not cause any ambiguity.

Recall the so-called *strong* or 2-norm of A

$$\|A\|_2^2 = \max\{x^* A^* A x \,|\, x \in \mathbb{C}^n \text{ and } x^* x = 1\}.$$

Here is the relationship between these two norms

$$\|A\|_2^2 = \max_k \lambda_k \geq \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k = |A|^2 \geq \frac{1}{n} \|A\|_2^2.$$

Finally, we have the following technical relation.

Lemma 2.4 ([9]). *Given two $n \times n$ matrices A and B, then* $|AB| \leq \|A\||B|$.

3 Main results

In this section, we present an algorithm that calculates the solution of equation (2.3) by employing extending a Toeplitz matrix into a circulant matrix. Consider the equation (2.3) and assume that $\operatorname{ess\,inf} |a| > 0$.

First, in order to apply the powerful relation (2.7), we extend the Toeplitz matrix $T_n(a)$ in (2.1) to a circulant matrix $C_m(a)$ (2.6). In this setting, let $m = 2n$, and $C_m(a)$ be a $m \times m$ circulant matrix with the top row

$$c_k^{(m)} = \begin{cases} a_{-k} & k = 0, \cdots, n-1 \\ 0 & k = n \\ a_{m-k} & k = n+1, \cdots, m-1 \end{cases}.$$

We can write

$$C_m(a) = \begin{bmatrix} T_n(a) & E_n(a) \\ E_n(a) & T_n(a) \end{bmatrix},$$

where

$$E_n(a) = \begin{bmatrix} 0 & a_{n-1} & \cdots & a_1 \\ a_{-n+1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1} \\ a_{-1} & \cdots & a_{-n+1} & 0 \end{bmatrix}. \tag{3.1}$$

Now, consider the following two elementary matrices: $m \times n$ matrix $L_n = \begin{bmatrix} I_n \\ I_n \end{bmatrix}$ and $n \times m$ matrix $S_n = [I_n \quad Z_n]$ where I_n and Z_n are the $n \times n$ identity and zero matrices, respectively. Note that

$$S_n L_n = I_n. \tag{3.2}$$

Using the equation (2.3), we get

$$\begin{aligned} C_m(a) L_n \mathrm{x} &= \begin{bmatrix} T_n(a) & E_n(a) \\ E_n(a) & T_n(a) \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{x} \end{bmatrix} \\ &= L_n(\mathrm{y} + E_n(a)\mathrm{x}). \end{aligned}$$

Using (2.7) and (3.2), we get

$$\mathrm{x} = S_n C_m(\frac{1}{a}) L_n(\mathrm{y} + E_n(a)\mathrm{x}).$$

This is a *fixed point* equation and it suggests using the standard Banach contraction principle in a suitable setting; see e.g., [8]. More precisely, we propose the following iteration method

$$\mathrm{x}_k = S_n C_m(\frac{1}{a}) L_n(\mathrm{y} + E_n(a)\mathrm{x}_{k-1}), \quad \text{for } k = 1, 2, \cdots, \tag{3.3}$$

with an arbitrary initial matrix x_0. Recall that the circulant matrix $C_m(1/a)$ can be calculated via (2.5). In fact, $C_m(1/a)$ is a circulant matrix with the top row $\left(c_0^{(m)}, \cdots, c_{m-1}^{(m)}\right)$ where

$$c_k^{(m)} = \frac{1}{m}\sum_{j=0}^{m-1} \frac{e^{2\pi ijk/m}}{a(2\pi j/m)}, \quad k = 0, \cdots, m-1.$$

In order to investigate the convergence of the iteration method (3.3), we have to show that the norm of the matrix $S_nC_m(1/a)L_nE_n(a)$ is *strictly less than* 1; see [7] or see [8] for a more general version. Since $\|L_n\|_2 = \|S_n\|_2 = 1$, we get

$$\|S_nC_m(1/a)L_nE_n(a)\|_2 \leq \|C_m(1/a)\|_2\, \|E_n(a)\|_2$$

which can be exactly equal to 1. However, the requirement can be achieved by the weak norm; see Definition 2.3. Recall that in any finite dimensional vector spaces all norms are equivalent so the space of all $n \times n$ complex matrices is Banach space in all norms.

Theorem 3.1. *The iteration method* (3.3) *converges to the unique solution of the system* (2.3), *provided that* $\operatorname{ess\,inf}\{|a(\theta)|;\ \theta \in \mathbb{T}\} > 0$ *and*

$$\alpha := \frac{\sqrt{\frac{1}{n}\sum\limits_{1\leq|k|\leq n-1} |k||\, a_k|^2}}{\operatorname{ess\,inf}\{|a(\theta)|;\ \theta \in \mathbb{T}\}} < 1. \tag{3.4}$$

Moreover,

$$\|\mathrm{x}_j - \mathrm{x}\| \leq \frac{\alpha^j}{1-\alpha}\|\mathrm{x}_1 - \mathrm{x}_0\|, \quad j = 1, 2, \cdots$$

where x *and* x_0 *are the exact solution and the initial matrix, respectively.*

Proof. Assume $\operatorname{ess\,inf}\{|a(\theta)|;\ \theta \in \mathbb{T}\} > 0$. It is enough to show that

$$|S_nC_m\left(\frac{1}{a}\right)L_nE_n(a)| < 1.$$

Using (2.4), (2.8), (3.1) and Lemma 2.4, we have

$$\begin{aligned}
|S_nC_m\left(\frac{1}{a}\right)L_nE_n(a)| &\leq \|S_nC_m(\frac{1}{a})L_n\|_2\, |E_n(a)| \\
&\leq \|S_n\|_2\|C_m\left(\frac{1}{a}\right)\|_2\, \|L_n\||E_n(a)| \\
&= \|C_m\left(\frac{1}{a}\right)\|_2\, |E_n(a)| \\
&\leq \operatorname{ess\,sup}\left\{\left|\frac{1}{a(\theta)}\right|;\ \theta \in \mathbb{T}\right\} \cdot \sqrt{\frac{1}{n}\sum_{1\leq|k|\leq n-1} |k||\, a_k|^2} \\
&= \alpha.
\end{aligned}$$

□

As we saw, the condition (3.4) is vital for convergence of the proposed method. Several examples align with the conditions outlined in the preceding case. Here, we highlight a few noteworthy instances. Let us to begin with some subclass of diagonal dominated matrices.

Example 3.2. Consider the system (2.3). We have $\alpha < 1$, provided that

$$\sqrt{n-1}\max_{1\leq|k|\leq n-1}|a_k| + \sum_{1\leq|k|\leq n-1}|a_k| < |a_0|. \tag{3.5}$$

Proof. Let $M := \max_{1\le|k|\le n-1} |a_k|$. First, note that

$$\sqrt{\frac{1}{n}\sum_{1\le|k|\le n-1} |k||\,a_k|^2} \le \sqrt{\frac{1}{n}\sum_{1\le|k|\le n-1} |k|\,M^2} = \sqrt{n-1}\,M.$$

Also, for each $\theta \in \mathbb{T}$, we have

$$|a(\theta)| = |\sum_{|k|\le n-1} a_k e^{ik\theta}| \ge |a_0| - |\sum_{1\le|k|\le n-1} a_k e^{ik\theta}| \ge |a_0| - \sum_{1\le|k|\le n-1} |a_k| > 0.$$

Hence,

$$\alpha \le \frac{\sqrt{n-1}\,M}{|a_0| - \sum\limits_{1\le|k|\le n-1} |a_k|} < 1$$

□

Obviously, the condition (3.5), implies a stronger constraint than diagonal dominance. Assuming $a_0 = 1$, without loss of generality, if A is diagonally dominant it can be represented as $A = I - E$ with $\|E\| < 1$. Consequently, the iterative method $x_{k+1} = Ex_k + y$ converges, demonstrating a computational cost of $O(n\log n)$ per step. In this case, the solution is given by the series $x = y + E(y) + E(Ey) + E(E(Ey)) + \cdots$, and stopping after k terms only requires $(k-1)O(n\log n)$ operations. But there are many other class of systems which can be applied in our methods.

First, let us examine two simple 2×2 matrices.

Example 3.3. Let $A = \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix}$. Then we have $\min\{|a(\theta)|;\ \ \theta \in \mathbb{T}\} = 2\sqrt{95}/5$ and so $\alpha \approx 0.925 < 1$.

Example 3.4. Consider a class of 2×2 matrices in the form of $A = \begin{bmatrix} 0 & ai \\ bi & 0 \end{bmatrix}$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. Let $\lambda = \min\{\frac{a}{b}, \frac{b}{a}\}$, and assume that $0 < \lambda < \sqrt{2} - 1$. Simple calculations show that $\min\{|a(\theta)|;\ \ \theta \in \mathbb{T}\} = 1 - \lambda$ and $\alpha = \sqrt{\frac{1+\lambda^2}{2-2\lambda}} < 1$.

The next example introduces a class of 'large' Toeplitz matrices applicable to proposed method.

Example 3.5. Let A be an $n \times n$ matrices with real entries such that $a_{-k} = -a_k$ for $1 \le |k| \le n-1$, and $a_0 \ne 0$. Therefore $a(\theta) = a_0 + 2i\sum\limits_{k=1}^{n-1} a_k \sin(k\theta)$ and $\min\{|a(\theta)|;\ \ \theta \in \mathbb{T}\} \ge |a_0| > 0$. Now, if a_k is small for larege k then $\alpha < 1$ for sufficiently large n. In fact, this is the case whenever

$$n > \frac{|a_0|}{\sum\limits_{1\le|k|\le n-1} |k||a_k|^2}.$$

For example, consider $n = 9$, and A is a Toeplitz matrix that $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{2}$, $a_{-1} = \frac{1}{2}$ and other entries are zero. The minimum of the symbol function $|a(\theta)|$ equals $\frac{1}{2}$ and then $\alpha < 1$.

The last example belongs to the class of so-called 'banded' Toeplitz matrices. Let A be an $n \times n$ Toeplitz matrix such that $\min\{|a(\theta)|;\ \ \theta \in \mathbb{T}\} > 0$ and $a_k = 0$ for $m < |k| \le n-1$. Definition (3.4) shows that $\alpha < 1$ for sufficiently large n.

Example 3.6. Assume that A represents a banded Toeplitz matrix with elements defined as $a_0 = \frac{7}{5}$, $a_k = \frac{1}{2}$, for $|k| = 1, 2, 3$ and all other entries are zero. Hence its symbol function is $a(\theta) = \frac{7}{5} + \cos(\theta) + \cos(2\theta) + \cos(3\theta)$ and usual calculations show that $\min\{|a(\theta)|;\ \ \theta \in \mathbb{T}\} = \frac{1}{135}(104 - 35\sqrt{7})$. It is easy to see that for $n \ge 462$, the condition $\alpha < 1$ holds.

The aforementioned examples illustrate diverse instances that satisfy condition (3.4). Therefore, this approach can be applied to a broad variety of Toeplitz equations systems.

4 The Algorithm and some numerical experiments

The key operations of the proposed method (3.3) can be outlined as follows:

Algorithm 1 Solving a Toeplitz system.

Input: Toeplitz matrix $T_n(a)$ satisfying (3.4) and vectors y, x_0.
Construct the matices $C_{2n}(a), E_n(a), L_n, S_n$.
Compute the matrix $C_{2n}(\frac{1}{a})$.
Repeat
  $\mathrm{x}_k = S_n C_{2n}(\frac{1}{a}) L_n(\mathrm{y} + E_n(a)\mathrm{x}_{k-1})$
until stopping criterion is satisfied.
Output: x_k.

The stopping criterion can be of the form $\|\mathrm{x}_{k+1} - \mathrm{x}_k\|_2 < \epsilon$ or $\dfrac{\alpha^k}{1-\alpha}\|\mathrm{x}_1 - \mathrm{x}_0\| \leq \epsilon$ where ϵ is a positive small precision; see Theorem 3.1. On the other hand, since by assumption $\min |a(\theta)| > 0$, utilizing the Fourier coefficient computation for the function $\frac{1}{a}$, we can efficiently calculate the elements of matrix $C_m(\frac{1}{a})$ that appeared in the Algorithm. where $m = 2n$. For this aim, one can use the so-called Fast Fourier Transform (FFT) algorithm [11, 6]. It is known that the arithmetic cost of FFT is $O(m \log m)$, and this is computed only once through the algorithm. Also, note that multiplying an $m \times m$ Toeplitz matrix with a vector costs $O(m \log m)$ operations when using FFT [4]. As the matrix $E_n(a)$ is an $n \times n$ Toeplitz matrix, computing the product $E_n(a)$ involves an operation with a complexity of $O(n \log n)$. Consequently, the computation of $\mathrm{y} + L_n E_n(a)\mathrm{x}$ requires no more than $O(m)$ floating-point operations. On the other hand, $S_n C_m(1/a) L_n(\mathrm{y} + E_n(a)\mathrm{x})$can be computed in $O(m \log m)$ time. Since $m = 2n$, it follows that $O(n \log n)$ arithmetic operations are sufficient for a single run of the algorithm. If we denote the number of necessary iterations as p, the cost of solving an $n \times n$ Toeplitz system requires $p\,O(n \log n)$ floating-point operations.

Now, we have implemented our algorithms for several sizes and we show results in Tables 1 and 2. In Table 1, we generate some random Toeplitz matrices for $n = 3, 6, 9, 12, 15$ such that they satisfy Example 3.2. Moreover, the right-hand vector of the system has been chosen so that the system has the exact solution $x = (1, \cdots 1)^T$. Also, recall in this situation $\alpha \leq \bar{\alpha} := \dfrac{\sqrt{n-1}\,M}{|a_0| - \sum_{1 \leq |k| \leq n-1} |a_k|}$. We denote the values computed by our algorithm by x_c and the number of iterations for their errors is indicated in the table. As we know, one of the most effective algorithms in solving Toeplitz systems is Levinson's algorithm [13] with $O(n^2)$ arithmetic operations. Compare this to our proposed algorithm which is accurate due to the $O(n \log n)$ of operations as well as recall the simplicity of the calculations. In Table 2, we have confined ourselves to $n = 10$ and we have shown the obtained $\bar{\alpha}$ value. The number of iterations and the errors obtained demonstrate that the algorithm has a high convergence rate as well.

n	$\|\mathrm{x}_c - \mathrm{x}\|_2$	$\bar{\alpha}$	p
3	8.9838e-08	0.4612	10
6	4.2014e-08	0.8043	11
9	5.8694e-08	0.4432	11
12	4.7595e-08	0.4215	11
15	2.6942e-08	0.8386	13

Tab. 1: Numerical results for some random generated Toeplitz systems

p	$\|\mathrm{x}_c - \mathrm{x}\|_2$
3	0.0139
6	4.6701e-05
9	1.6108e-07
12	5.8579e-10
15	2.2371e-12

Tab. 2: Error estimations for $n = 10$.

5 Conclusion

Addressing Toeplitz linear equation systems of specialized types poses a significant computational challenge, with the prevailing algorithms incurring a cost of $O(n^2)$ arithmetic operations. Despite the absence of a universal algorithm applicable to all scenarios, our paper introduces a novel approach for linear equation systems in which their coefficients matrices are Toeplitz matrices such that their symbol functions are away from zero. We present some various classes of matrices that satisfy the main condition (3.4). By leveraging insights from the theory of circulant matrices, particularly drawing on results outlined in [9], we recast the problem as a 'fixed point problem'. Then under some reasonable circumstances, one can apply Banach's Contraction Principle to get an iterative method to approximate the solution efficiently. The outcome is an iterative method that efficiently approximates the solution, boasting a remarkable combination of low computational cost and rapid convergence rates. Specifically, our proposed algorithm, facilitated by the identity $C(a)^{-1} = C(1/a)$ for invertible circulant matrices and employing the Fast Fourier Transform, achieves a notable reduction in arithmetic complexity to $O(n \log n)$. Furthermore, the algorithm demonstrates an exponential rate of convergence $c\alpha^n$ where $c > 0$ and $0 < \alpha < 1$ are constants. This compelling combination of efficiency and convergence highlights the efficacy of our method in tackling large classes of Toeplitz linear equation systems.

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