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On the size of sets avoiding a general structure

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Abstract

Given a finite abelian group G and a subset $S \subseteq G$, we let $N_{G,\ S}$ be the smallest integer N such that for any subset $A \subseteq G$ with N elements, we have $g + S \subseteq A$ for some $g \in G$. Using the probabilistic method, we prove that

$$\frac{|H_G(S)|-1}{|H_G(S)|}|G| + \left\lceil \left(\frac{|G|}{|H_G(S)|}\right)^{1-|H_G(S)|/|S|} \right\rceil \leq N_{G,\ S} \leq \left\lfloor \frac{|S|-1}{|S|}|G| \right\rfloor + 1,$$

where $H_G(S)$ is the stabilizer of S.

Keywords: Finite abelian group, avoiding a structure, probabilistic method.

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1 Introduction and Main Result

Problems about avoiding structures, especially avoiding arithmetic progressions, are well-known and have been extensively studied. For example, the famous Roth's theorem, which is about avoiding three-term arithmetic progressions, was proved in [4] and has been refined in [2, 3, 5, 6]. In this succinct paper, we take the avoided structure to be a general set.

For a finite abelian group $(G, +)$, an element $g \in G$, and a subset $S \subseteq G$, we define $g + S$ to be $\{g + s : s \in S\}$, and define the *stabilizer* of S to be

$$H_G(S) = \{g' \in G : \ g' + S = S\}.$$

It is easy to check that $H_G(S)$ is a subgroup of G, and S is the union of some cosets of $H_G(S)$.

Given a finite abelian group G and a subset $S \subseteq G$, we let $N_{G,\ S}$ denote the smallest integer $N \geq |S|$ such that for any subset $A \subseteq G$ with N elements, we have $g + S \subseteq A$ for some $g \in G$. Thus, for any $M \leq N_{G,\ S} - 1$, there exists a subset $B \subseteq G$ with M elements, such that $g + S \nsubseteq B$ for any $g \in G$. Roughly speaking, this means the additive structure of S is avoided in B.

Firstly we prove the following bounds on $N_{G,\ S}$, and the lower bound will be improved later.

Theorem 1.1. *We have*

$$\frac{|H_G(S)|-1}{|H_G(S)|}|G| + 1 \leq N_{G,\ S} \leq \left\lfloor \frac{|S|-1}{|S|}|G| \right\rfloor + 1.$$

Proof. For the lower bound, we can construct a subset $B \subseteq G$ with $\frac{|H_G(S)|-1}{|H_G(S)|}|G|$ elements by excluding one element from each coset of $H_G(S)$, then we will have $g+S \nsubseteq B$ for any $g \in G$.

For the upper bound, let us assume for some subset $A \subseteq G$ with $\left\lfloor \frac{|S|-1}{|S|}|G| \right\rfloor + 1$ elements, we have $g+S \nsubseteq A$ for any $g \in G$, which means $(g+S) \cap (G \setminus A) \neq \emptyset$ for any $g \in G$. For each $\alpha \in G \setminus A$, we have

$$|\{g \in G : \ \alpha \in g+S\}| = |\{g \in G : \ g \in \alpha - S\}| = |\alpha - S| = |S|,$$

which implies

$$\begin{aligned} |\{g \in G : \ (g+S) \cap (G \setminus A) \neq \emptyset\}| &\leq \sum_{\alpha \in G \setminus A} |\{g \in G : \ \alpha \in g+S\}| \\ &= |G \setminus A||S| \\ &= \left(|G| - \left(\left\lfloor \frac{|S|-1}{|S|}|G| \right\rfloor + 1\right)\right)|S| \\ &< |G|, \end{aligned}$$

contradicting the assumption that $(g+S) \cap (G \setminus A) \neq \emptyset$ for any $g \in G$. So for any subset $A \subseteq G$ with $\left\lfloor \frac{|S|-1}{|S|}|G| \right\rfloor + 1$ elements, we can find $g+S$ in A for some $g \in G$, and thus $N_{G,\ S} \leq \left\lfloor \frac{|S|-1}{|S|}|G| \right\rfloor + 1$. □

We have a direct corollary.

Corollary 1.2. *If S is a coset of some subgroup of G, then*

$$N_{G,\ S} = \frac{|S|-1}{|S|}|G| + 1.$$

Proof. If S is a coset of a subgroup, then $|H_G(S)| = |S|$, and the equalities in Theorem 1.1 will be attained. □

2 The Lower Bound

The lower bound in Theorem 1.1 is not always good. For example, if S is *aperiodic*, which means $|H_G(S)| = 1$, then the lower bound will be trivial.

Let $T_G(S)$ be a transversal of $G/H_G(S)$, which means $T_G(S)$ contains exactly one element from each coset of $H_G(S)$, so $|T_G(S)| = \frac{|G|}{|H_G(S)|}$. For a subset $A \subseteq G$, it is easy to see that the following two statements are equivalent.

- There exists $g \in G$, such that $g+S \subseteq A$.
- There exists $g' \in T_G(S)$, such that $g'+S \subseteq A$.

Using a classic probabilistic method that can be found in multiple places throughout Alon and Spencer's book [1], we prove another lower bound on $N_{G,\ S}$. We will use this result as a lemma to prove a better lower bound in Theorem 2.2, which is our final goal.

Lemma 2.1. *We have*

$$N_{G,\ S} \geq |T_G(S)|^{-1/|S|}|G| = |H_G(S)|^{1/|S|}|G|^{1-1/|S|}$$

Proof. Suppose $N \geq |S|$ is an integer such that for any subset $A \subseteq G$ with N elements, there exists $g \in G$ such that $g + S \subseteq A$. We randomly choose a set X from all N-element subsets of G, then

$$\begin{aligned}\mathbb{P}(\exists\, g \in G \text{ s.t. } g + S \subseteq X) &= \mathbb{P}(\exists\, g' \in T_G(S) \text{ s.t. } g' + S \subseteq X) \\ &\leq \sum_{g' \in T_G(S)} \mathbb{P}(g' + S \subseteq X) \\ &= |T_G(S)| \frac{\binom{|G|-|S|}{N-|S|}}{\binom{|G|}{N}} \\ &= |T_G(S)| \frac{N!(|G|-|S|)!}{|G|!(N-|S|)!} \\ &= |T_G(S)| \frac{N(N-1)...(N-|S|+1)}{|G|(|G|-1)...(|G|-|S|+1)} \\ &\leq |T_G(S)| \left(\frac{N}{|G|}\right)^{|S|}.\end{aligned}$$

If $N < |T_G(S)|^{-1/|S|}|G|$, then $\mathbb{P}(\exists\, g \in G \text{ s.t. } g + S \subseteq X) < 1$, which means there is some N-element set $A \subseteq G$ such that $g + S \nsubseteq A$ for any $g \in G$, contradiction. So $N \geq |T_G(S)|^{-1/|S|}|G|$, and thus $N_{G,\,S} \geq |T_G(S)|^{-1/|S|}|G| = |H_G(S)|^{1/|S|}|G|^{1-1/|S|}$. □

Combining the ideas in Theorem 1.1 and Lemma 2.1, we prove the following result. In the proof, we take $G' := G/H_G(S)$ and $S' := S/H_G(S)$, then $H_{G'}(S')$ will be trivial, and by Lemma 2.1, we have $N_{G',\,S'} \geq |G'|^{1-1/|S'|}$. And because $N_{G',\,S'}$ is an integer, we know $N_{G',\,S'} \geq \left\lceil |G'|^{1-1/|S'|} \right\rceil$.

Theorem 2.2. *We have*

$$N_{G,\,S} \geq \frac{|H_G(S)|-1}{|H_G(S)|}|G| + \left\lceil \left(\frac{|G|}{|H_G(S)|}\right)^{1-|H_G(S)|/|S|} \right\rceil.$$

Note that in this bound, if $|H_G(S)|$ is large, then $\frac{|H_G(S)|-1}{|H_G(S)|}|G|$ will be large; if $|H_G(S)|$ is small, then $\left\lceil \left(\frac{|G|}{|H_G(S)|}\right)^{1-|H_G(S)|/|S|} \right\rceil$ will be large. So we always have a nontrivial lower bound.

Proof. We shall construct a subset $B \subseteq G$ with $\frac{|H_G(S)|-1}{|H_G(S)|}|G| + \left\lceil \left(\frac{|G|}{|H_G(S)|}\right)^{1-|H_G(S)|/|S|} \right\rceil - 1$ elements, and show that $g + S \nsubseteq B$ for any $g \in G$.

By Lemma 2.1, we know that there is a subset $B' \subseteq G'$ with $\left\lceil |G'|^{1-1/|S'|} \right\rceil - 1$ elements, such that $g' + S' \nsubseteq B'$ for any $g' \in G'$. We let

$$B_1 = \{b \in G : \ b + H_G(S) \in B'\},$$

so

$$|B_1| = \left(\left\lceil |G'|^{1-1/|S'|} \right\rceil - 1\right)|H_G(S)|.$$

Then, there are $|G'|-|B'|$ cosets of $H_G(S)$ which are not in B', we denote these cosets by $H_1,\ H_2,\ ...,\ H_{|G'|-|B'|}$. In each H_i, we randomly pick an element h_i, and let K_i be $H_i \setminus \{h_i\}$. We let B_2 be the union of K_i's, namely

$$B_2 = \bigcup_{i=1}^{|G'|-|B'|} K_i.$$

We have

$$|B_2| = (|G'| - |B'|)(|H_G(S)| - 1) = \Big(|G'| - \Big(\Big\lceil |G'|^{1-1/|S'|}\Big\rceil - 1\Big)\Big)(|H_G(S)| - 1).$$

Now, we take B to be $B_1 \cup B_2$, then

$$\begin{aligned} |B| &= |G'|(|H_G(S)| - 1) + \Big\lceil |G'|^{1-1/|S'|}\Big\rceil - 1 \\ &= \frac{|H_G(S)| - 1}{|H_G(S)|}|G| + \left\lceil \left(\frac{|G|}{|H_G(S)|}\right)^{1-|H_G(S)|/|S|} \right\rceil - 1. \end{aligned}$$

And we need to show that $g + S \nsubseteq B$ for any $g \in G$.

- If for some $g \in G$, we have $g+S \subseteq B_1$, then $g' := g+H_G(S) \in G'$ and $g'+S' \subseteq B'$, contradicting the definition of B'.
- If for some $g \in G$, we have $g+S \subseteq B$ and $(g+S) \cap B_2 \neq \emptyset$, then again we have a contradiction, because $g+S$ is a union of $H_G(S)$ cosets, but B_2 is a union of $H_G(S)$ cosets with punched holes.

So $g + S \nsubseteq B$ for any $g \in G$. □

We need to check the lower bound obtained in Theorem 2.2 is better than the one in Lemma 2.1. Although this is intuitive, we have a formal verification given by the following proposition, where g, h, and s play the roles of $|G|$, $|H_G(S)|$, and $|S|$ respectively.

Proposition 2.3. *Let $g,\ h,\ s \geq 1$ be three real numbers with $g \geq h$, then*

$$\frac{h-1}{h}g + \left(\frac{g}{h}\right)^{1-h/s} \geq h^{1/s}g^{1-1/s}.$$

Proof. We can fix g and h, and take s as a variable. Note that actually we should have $h \leq s \leq g$, but for calculation convenience, let us take $1 \leq s < \infty$. Let $f(s) = \frac{h-1}{h}g + (\frac{g}{h})^{1-h/s} - h^{1/s}g^{1-1/s}$. It turns out $f'(s) \leq 0$, so $f(s)$ is decreasing on $[1,\ \infty)$. And if s is taken to be ∞, then $f(\infty) = 0$. So we always have $f(s) \geq 0$, and thus $\frac{h-1}{h}g + \left(\frac{g}{h}\right)^{1-h/s} \geq h^{1/s}g^{1-1/s}$. □

Remark 2.4. Note that if S is a coset of some subgroup of G, then the lower bound in Theorem 2.2 is the same as the one in Theorem 1.1; if $|H_G(S)| = 1$ or $H_G(S) = S = G$, then the lower bound in Theorem 2.2 is the same as the one in Lemma 2.1.

In this paper, G has been assumed to be an abelian group, but in fact, if G is a finite non-abelian group, and S is a subset of G such that $H_G(S)$ is a normal subgroup, then the same argument still works.

3 An Example

We finish this short paper with an example.

Let us take G to be C_{2024}, the cyclic group of order $2024 = 2^3 \cdot 11 \cdot 23$, and take S to be the union of n cosets of the subgroup of order eight. So $|S| = 8n$, and if we restrict n to be in $[1,\ 10]$, then

$|H_G(S)|$ must be eight. Then by Theorem 1.1, Corollary 1.2, and Theorem 2.2, we have

$$N_{G,\ S} \begin{cases} = 1772 & if\ n = 1, \\ \in [1787,\ 1898] & if\ n = 2, \\ \in [1812,\ 1940] & if\ n = 3, \\ \in [1835,\ 1961] & if\ n = 4, \\ \in [1855,\ 1974] & if\ n = 5, \\ \in [1872,\ 1982] & if\ n = 6, \\ \in [1886,\ 1988] & if\ n = 7, \\ \in [1898,\ 1993] & if\ n = 8, \\ \in [1908,\ 1996] & if\ n = 9, \\ \in [1917,\ 1999] & if\ n = 10. \end{cases}$$

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