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Continuous images of a topological space

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Abstract

It is not customary in general topology textbooks to characterize the continuous images of a given topological space up to homeomorphism. In this note, we discuss a novel characterization of such spaces and possible ways to integrate our result into the curriculum.

Keywords: continuous image, quotient space, decomposition space, weak topology, strong topology.

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1 Introduction

One of the most important results that the student sees in an introduction to group theory characterizes the homomorphic images of a given group. It has two parts. First, If $(G,+)$ is a group and N is a normal subgroup, then G/N, the set of cosets of N, can be made into a group called a *quotient group* because $(g_1+N)\oplus(g_2+N) := (g_1+g_2)+N$ is well-defined. Further, $g \mapsto g+N$ is a surjective homomorphism from G to G/N. Conversely, if the group $(H,\widetilde{+})$ is the image of $(G,+)$ under a homomorphism f, then the kernel of f which we denote by $\ker(f)$ is a normal subgroup of G and $G/\ker(f)$ is isomorphic to H. In short, the homomorphic images of $(G,+)$ are (up to isomorphism) quotient groups of $(G,+)$ [3, pp. 56-60].

One would think that an analogous result would be a feature of general topology, where homomorphic images of a group $(G,+)$ are replaced by continuous images of a topological space (X,τ). and where the image space is asserted to be homeomorphic to some quotient of the domain. There is nothing explicit in this direction in any standard text. The purpose of this note is to supply this missing piece to the curriculum, describing two different ways to obtain it. We apply it to quickly obtain this known characterization of the continuous images of (X,τ): they are the bijective continuous images of quotients of the domain space.

2 Preliminaries

Let X be a nonempty set and let (Y,σ) be a topological space and suppose $f : X \to Y$. Then

$$\tau_{f,\sigma}^{weak} := \{f^{-1}(V) : V \in \sigma\}$$

is the weakest topology on X making f continuous. We call this the *weak topology* on X determined by f and σ. On the other hand, if (X,τ) is a topological space and Y is a nonempty set, then the family $\{V \subseteq Y : f^{-1}(V) \in \tau\}$, which we denote by $\sigma_{f,\tau}^{strong}$ is the strongest topology on Y making f continuous. We of course call this the *strong topology* on Y determined by f and τ.

Lemma 2.1. *Let (Y, σ) be a topological space and let X be a nonempty set. Suppose $f : X \to Y$ is onto and X is equipped with $\tau_{f,\sigma}^{weak}$. Then σ is the strong topology on Y determined by f and $\tau_{f,\sigma}^{weak}$ on X.*

Proof. Let E belong to the strong topology. This means $f^{-1}(E)$ belongs to $\tau_{f,\sigma}^{weak}$, that is, $f^{-1}(E) = f^{-1}(V)$ for some $V \in \sigma$. But then by surjectivity,

$$E = f(f^{-1}(E)) = f(f^{-1}(V)) = V \in \sigma.$$

This shows that the strong topology on Y determined by f and $\tau_{f,\sigma}^{weak}$ is a subfamily of σ. The reverse inclusion is obvious. □

Corollary 2.2. *Let X be a nonempty set and let (Y, σ) be a topological space. Suppose $f : X \to Y$ is surjective. Then (Y, σ) satisfies the T_1-separation property if and only if for each $y \in Y$, $f^{-1}(\{y\})$ is weakly closed, that is, $X \backslash f^{-1}(\{y\})$ belongs to $\tau_{f,\sigma}^{weak}$.*

Proof. By Lemma 2.1, a subset of Y is σ-closed if and only if its preimage under f is weakly closed. □

Suppose $\{A_i : i \in I\}$ is a partition of a nonempty set X by nonempty subsets. We define the *natural map* $q : X \to \{A_i : i \in I\}$ by letting $q(x)$ be the unique block of the partition containing x. Suppose now that τ is a topology on X. We equip $\{A_i : i \in I\}$ with a topology μ_τ where $\{A_i : i \in I_0\}$ for $I_0 \subseteq I$ is placed in μ_τ provided $\bigcup_{i \in I_0} A_i$ belongs to τ. The topological space $(\{A_i : i \in I\}, \mu_\tau)$ is called a *quotient space* [1, 2, 5, 6] or a *decomposition space* [4, 5, 6, 7] of (X, τ). Not only is q continuous, but μ_τ is also the strong topology on $\{A_i : i \in I\}$ determined by q and τ, because for $I_0 \subseteq I$, $\{A_i : i \in I_0\} \in \mu_\tau$ if and only if $q^{-1}(\{A_i : i \in I_0\}) = \bigcup_{i \in I_0} A_i \in \tau$ [6, p. 59].

The central result about decomposition spaces is now described in the most qualitative way possible (see, e.g., [7, p. 61]), as this best serves our purposes.

Theorem 2.3. *Let (X, τ) and (Y, σ) be topological spaces. Then there exists a surjection $f : X \to Y$ such that $\sigma = \sigma_{f,\tau}^{strong}$ if and only if (Y, σ) is homeomorphic to a decomposition space of (X, τ).*

Functions f as described in Theorem 2.3 are called either *quotient maps* [2, 7] or *identifications* [1, 4]. For the proof of necessity, the particular partition of X that is used is $\{f^{-1}(y) : y \in Y\}$.

3 The main result

Initially, we give a self-contained, naive proof of our main result without any reference to strong topologies/quotient maps/identifications. We do so to allow the instructor to present the result rather early in the course, without having to discuss some version of Theorem 2.3 at all. In the interest of time, the instructor might not be willing cover Theorem 2.3, given the choices he/she/they have with respect to topics that might be included at the end of the course, e.g., nets, uniformities or complete metrizability.

This will be followed by an economical proof based on Theorem 2.3 for the benefit of those instructors willing to include better coverage of decomposition spaces in their course.

Theorem 3.1. *Let (X, τ) and (Y, σ) be topological spaces. Then (Y, σ) is the continuous image of (X, τ) if and only if (Y, σ) is homeomorphic to some decomposition space of (X, τ^*) where τ^* is some topology on X coarser than τ.*

Proof. First, suppose $(\{A_i : i \in I\}, \mu_{\tau^*})$ is a decomposition space of (X, τ^*) where $\tau^* \subseteq \tau$. We claim that the natural map $q : (X, \tau) \to (\{A_i : i \in I\}, \mu_{\tau^*})$ is continuous. If $\{A_i : i \in I_0\} \in \mu_{\tau^*}$ where $I_0 \subseteq I$, then by definition of the decomposition space topology,

$$q^{-1}(\{A_i : i \in I_0\}) = \bigcup_{i \in I_0} A_i \in \tau^* \subseteq \tau,$$

and so q is continuous on (X, τ). As a result, if $h : (\{A_i : i \in I\}, \mu_{\tau^*}) \to (Y, \sigma)$ is a homeomorphism, then (Y, σ) is the continuous image of (X, τ) under $h \circ q$.

Conversely, suppose (Y, σ) is the continuous image of (X, τ). By surjectivity, $\{f^{-1}(\{y\}) : y \in Y\}$ is a partition of X by nonempty subsets. Equip X with $\tau^* := \tau^{weak}_{f,\sigma} \subseteq \tau$ and consider the decomposition space topology it determines on $\{f^{-1}(\{y\}) : y \in Y\}$, which we denote simply by μ to cut down on symbol shock. We intend to show $(\{f^{-1}(\{y\}) : y \in Y\}, \mu)$ is homeomorphic to (Y, σ).

We work with the bijection $h : \{f^{-1}(\{y\}) : y \in Y\} \to Y$ that assigns y to $f^{-1}(\{y\})$. To show h maps open sets to open sets, let $\{f^{-1}(\{y\}) : y \in E\}$ belong to μ where $E \subseteq Y$. By the definition of μ,

$$f^{-1}(E) = \bigcup_{y \in E} f^{-1}(\{y\}) \in \tau^{weak}_{f,\sigma}.$$

By the definition of $\tau^{weak}_{f,\sigma}$, we conclude that for some $V \in \sigma$, we have $f^{-1}(E) = f^{-1}(V)$ and since f is onto we get $E = V$. Finally,

$$h(\{f^{-1}(\{y\}) : y \in E\}) = \{y : y \in E\} = V,$$

and thus h maps $\{f^{-1}(\{y\}) : y \in E\}$ to a member of σ.

To show h is continuous, let $V \in \sigma$ be arbitrary. We compute

$$h^{-1}(V) = \{h^{-1}(y) : y \in V\} = \{f^{-1}(\{y\}) : y \in V\} \in \mu$$

because $\bigcup_{y \in V} f^{-1}(\{y\}) = f^{-1}(V) \in \tau^{weak}_{f,\sigma}$. □

Images of continuous maps can be unexpected, e.g., consider the space filling curves $f : [0,1] \to [0,1] \times [0,1]$. We say this simply to remark that the square is a decomposition space for some weird topology on the interval, coarser than the usual metric topology on $[0,1]$, which is for us remarkable.

As announced at the beginning of this section, we indicate a second approach to the proof of necessity in our main result that the instructor willing to cover some version of Theorem 2.3 probably would prefer. Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous surjection. By Lemma 2.1, σ is the strong topology on Y determined by f and $\tau^{weak}_{f,\sigma}$. Apply Theorem 2.3 where τ is replaced by $\tau^{weak}_{f,\sigma}$, noting that by continuity of f, $\tau^{weak}_{f,\sigma} \subseteq \tau$.

Let $f : (X, \tau) \to (Y, \sigma)$ be continuous and onto. Theorem 2.3 says that we can take τ^* to be τ in the statement of Theorem 3.1 if and only if σ is the strong topology on Y determined by f and τ. In particular, this is so if f is either open or closed in addition to being continuous [7, Theorem 9.2]. We find it useful to appreciate in concrete terms how proper subtopologies of τ can also do the job even when τ does.

Lemma 3.2. *Let (X, τ) be a disconnected topological space where τ has at least 5 members. Suppose $\{A_1, A_2\}$ is a nontrivial separation of the space by members of τ. Let $\tau^* := \{\emptyset, A_1, A_2, X\}$. Then $(\{A_1, A_2\}, \mu_\tau) = (\{A_1, A_2\}, \mu_{\tau^*})$ while τ^* is properly coarser that τ.*

Proof. Both μ_τ and μ_{τ^*} are the discrete topology on the blocks of the partition. The topology τ^* is properly coarser than τ because τ^* contains less than 5 members. □

Example 3.3. Consider $X := (-\infty, 0) \cup (0, \infty)$ equipped with the relative topology τ that X inherits from the real line. Let $Y := \{0, 1\}$ equipped with the discrete topology σ. Let $f : X \to Y$ be the characteristic function of $(0, \infty)$. Obviously, f is open and continuous, and so (Y, σ) is homeomorphic to a quotient space of (X, τ). At the same time, by our main result, (Y, σ) is homeomorphic to a quotient space of X equipped with the weak topology determined by f and σ, i.e, by $\{\emptyset, (-\infty, 0), (0, \infty), X\}$.

4 A second characterization of continuous images

There is a different well-known way to describe the continuous images of a topological space (X, τ) that is more in concert with Theorem 2.3: they are the continuous bijective images of some decomposition space of (X, τ) (see [2, p. 124] and [7, Problem 9F]). We end our note by deriving this alternative characterization from our main result.

Since each decomposition space of (X, τ) is the continuous image of the base space, each bijective continuous image of the decomposition space will in turn be a continuous image of (X, τ). Conversely, if (Y, σ) is the continuous image of (X, τ), then by Theorem 3.1, (Y, σ) is the continuous bijective image of $(\{f^{-1}(y) : y \in Y\}, \mu)$ where μ is coarser than μ_τ. Using the identity map, $(\{f^{-1}(y) : y \in Y\}, \mu)$ is the continuous bijective image of $(\{f^{-1}(y) : y \in Y\}, \mu_\tau)$, and taking a composition, it follows that (Y, σ) is the continuous bijective image of $(\{f^{-1}(y) : y \in Y\}, \mu_\tau)$.

We prefer our first characterization to the second, and not only because continuous images are described homeomorphically by it. In Theorem 3.1, our condition for surjective continuity for some function from X to Y just seems so much stronger and more descriptive. In either case, proof of sufficiency is immediate.

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