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On power maps over periodic rings

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Abstract

A ring R is called *weakly periodic* if every $x \in R$ can be written in the form $x = a + b$, where a is nilpotent and $b^m = b$ for some integer $m > 1$. The aim of this note is to consider when a nonzero nilpotent element r is the period of some power map $f(x) = x^n$, in the sense that $f(x + r) = f(x)$ for all $x \in R$, and how this relates to the structure of weakly periodic rings.

In particular, we provide a new, elementary proof of the fact that weakly periodic rings with central and torsion nilpotent elements are periodic commutative torsion rings. We also prove that x^n is periodic over such rings whenever n is not coprime with any of the additive orders of the nilpotent elements. These are in fact the only periodic power maps over finite commutative rings with unity. Finally, we describe and enumerate the distinct power maps over Corbas (p, k, ϕ)-rings, Galois rings, $\mathbb{Z}/n\mathbb{Z}$, endomorphism rings of finite abelian groups, and matrix rings over finite fields.

Keywords: nilpotent, potent, power maps, torsion, weakly periodic rings.

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1 Introduction

Throughout, R is a ring (not necessarily commutative or unital) and R^+ is the additive group of R. By order of a ring element we will always mean its group-theoretic order as a member of R^+. The set of nilpotent elements of R will be denoted by $\text{Nil}(R)$. The index of a nilpotent element $x \in R$ is the smallest positive integer n such that $x^n = 0$, and the index of $\text{Nil}(R)$ is the largest index, if extant, among all elements of $\text{Nil}(R)$. Any additional terminology and notation not explicitly defined herein are standard in the literature.

A ring R is called *periodic* if for each $x \in R$ the set $\{x^n : n \in \mathbb{N}\}$ is finite. Equivalently, for each $x \in R$, there are positive integers $m(x)$ and $n(x)$ such that $x^{m(x)+n(x)} = x^{m(x)}$. If the aforementioned $n(x)$ can be taken to be constant across all x, then the smallest such constant permissible is called the *exponential period* of R, which we will denote by $\mu_1 := \mu_1(R)$, and the *onset of exponential periodicity* of R, which we will denote by $\mu_0 := \mu_0(R)$, is given by

$$\mu_0 = \max_{x \in R} \min\{m \in \mathbb{N} : x^{m+\mu_1} = x^m\},$$

provided this maximum exists. Otherwise, we say that μ_1 and μ_0 are infinite. Notice that μ_0 is no smaller than the index of $\text{Nil}(R)$ and that, for rings with unity, μ_1 is no smaller than the exponent of the group of units $R^\times$.

An element $x \in R$ is called potent if $x^m = x$ for some integer $m > 1$. Let $\mathrm{Pot}(R)$ denote the set of potent elements of R. A ring R is called *weakly periodic* if $R = \mathrm{Nil}(R) + \mathrm{Pot}(R)$. We remark that if $R = \mathrm{Nil}(R) \cup \mathrm{Pot}(R)$, then μ_0 is clearly the index of $\mathrm{Nil}(R)$. If $R = \mathrm{Pot}(R)$, then R is called a *J-ring*. It is well known that every J-ring is commutative [23].

Some obvious examples of periodic rings are finite rings, Boolean rings, and nil rings. In fact, finite fields and Boolean rings are J-rings. Bell [6] proved that periodic rings are weakly periodic, but the status of the converse is still unresolved. Periodicity has been established for several special classes of weakly periodic rings though (e.g. [1], [2], [7], [8], [9], [14], [17], [28]).

The power maps over a ring are the functions of the form $f(x) = x^n$ for some fixed $n \in \mathbb{N}$. Despite the simplicity of the concept, power maps in periodic rings are rarely if ever discussed in introductory courses in abstract algebra. In this paper, we assess what power map "oscillations" may inform us about the arithmetical structure of weakly periodic rings in a manner that is accessible to undergraduates. In particular, we will show that if the nilpotent elements of R are additively torsion and multiplicatively central, then each nilpotent is a period of some power map. This leads to an alternative, elementary proof of the fact that weakly periodic rings with this property are periodic commutative torsion rings. A generalization of this was proved by Bell and Tominaga in [8], who relied on Pierce decompositions and Chacron's periodicity criterion [11]. We will also prove that x^n is periodic over such rings whenever n is not coprime with any of the orders of the nilpotent elements. This will allow us to derive tight lower and upper bounds for the number of periodic power maps over commutative rings when μ_1 and μ_0 are both finite.

Assorted examples are peppered throughout to help illustrate the results. For ease of naviagation, Table 1 below catalogs the most instructive and concrete examples featured in this paper. The formulas listed below have been symbolically verified in Maple with the aid of Bruno Salvy's `equivalent` package.

Tab. 1: Periodic power map enumerations for some rings

Ring R	$\mu_1(R)$	$\mu_0(R)$	*# of periodic power maps*
Corbas (p,k,ϕ)-ring (pp. 7, 12)	$p(p^k-1)$	2	p^k-1 if ϕ is the identity, 0 otherwise
$R_1 \times R_2$, R_1 reduced, R_2 nonzero nil (p. 8)	$\mu_1(R_1)$	$\mu_0(R_2) =$ index of R_2	at least $\mu_1(R_1)$, at most $\mu_1(R_1)+\mu_0(R_2)-2$
$\mathrm{GR}(p^k,d)$, $d \geq 2$ (pp. 12–13)	$p^{k-1}(p^d-1)$	k	$p^{k-2}(p^d-1) + \left\lfloor \frac{k-1}{p} \right\rfloor$ if $k \geq 2$; 0 if $k=1$
$\mathbb{Z}/n\mathbb{Z}$ (p. 13)	Carmichael function $\lambda(n)$	$E(n) :=$ maximal exponent in prime factorization of n	$\sum_{\substack{d \mid \mathrm{rad}(n/\mathrm{rad}(n)),\\ d \neq 1}} (-1)^{\omega(d)+1} \left\lfloor \frac{\lambda(n)+E(n)-1}{d} \right\rfloor$
$\mathcal{M}_n(\mathbb{F}_q)$ (pp. 9, 14)	the exponent of $\mathrm{GL}(n,\mathbb{F}_q)$	n	0

2 Periodic Power Maps

If G is an additive group and X is a set, then a function $\alpha : G \to X$ is periodic if there exists a nonidentity group element h such that $\alpha(g+h) = \alpha(g)$ for all $g \in G$. Such an element h is called a *period* of α. It is easy to see that the periods of a function α together with 0 form a subgroup of G. We will call this subgroup $\mathrm{Per}(\alpha)$. Clearly, a power map f over a ring R is periodic as a function on R^+ only if $\mathrm{Per}(\alpha) \subseteq \mathrm{Nil}(R)$. On the other hand, nilpotence alone does not guarantee that an element is the period of some power map. Here are a couple of counterexamples.

Example 2.1. Let $R = \prod_{i\in\mathbb{N}} \mathbb{Z}/p_i^2\mathbb{Z}$, where p_i is the i^{th} prime number. The sequence $(p_i)_{i\in\mathbb{N}}$ is a nilpotent element of index 2. Therefore, if $(x_i)_{i\in\mathbb{N}} \in R$, then

$$((x_i + p_i)^n)_{i\in\mathbb{N}} = \left(x_i^n + np_ix_i^{n-1}\right)_{i\in\mathbb{N}} \tag{2.1}$$

for all $n \in \mathbb{N}$. Since $(p_i)_{i\in\mathbb{N}}$ has infinite order, there is no integer n such that $x_i^n + np_ix_i^{n-1} = x_i^n$ for every $i \in \mathbb{N}$.

Example 2.2. If p is a prime number, k is a positive integer, and ϕ is an automorphism of the Galois field $\mathbb{F}_{p^k}$, then the Corbas (p, k, ϕ)-ring [13] is the ring R in which $R^+ = \mathbb{F}_{p^k} \oplus \mathbb{F}_{p^k}$ and the ring multiplication $\cdot$ is defined by

$$(a, b) \cdot (c, d) = (ac, ad + b\phi(c)).$$

It is straightforward to confirm that R satisfies the following three properties:

- For all $n \in \mathbb{N}$,

  $$(a, b)^n = \left(a^n, b\phi\left(a^{n-1}\right)\sum_{j=0}^{n-1} a^j\phi\left(a^{-j}\right)\right), \tag{2.2}$$

  If ϕ is not the identity automorphism and $a \neq 0$, then this can be simplified to

  $$(a, b)^n = \left(a^n, b\phi\left(a^{n-1}\right)\frac{a^n\phi\left(a^{-n}\right) - 1}{a\phi\left(a^{-1}\right) - 1}\right) \tag{2.3}$$

  where $\frac{x}{y}$ is defined to be xy^{-1}.
- $\text{Nil}(R) = 0 \times \mathbb{F}_{p^k}$ has index 2,
- R is commutative if and only if ϕ is the identity.

Hence, for each $a, b, y \in \mathbb{F}_{p^k}$ and $n \in \mathbb{N}$,

$$((a, b) + (0, y))^n = \left(a^n, (b + y)\sum_{j=0}^{n-1} a^j\phi(a^{n-j-1})\right). \tag{2.4}$$

If ϕ is not the identity, then 0 is the only value of y for which (2.4) simplifies to $(a, b)^n$. In this case, R has no periodic power maps whatsoever.

Unboundedness of order and noncommutativity are responsible for the pathologies of Examples 2.1 and 2.2, respectively. However, as long as $\text{Nil}(R)$ circumvents these traits, we can promise that each nonzero nilpotent element is a period of some power map. We will refer to such rings as *nilperiod*.

Theorem 2.3 below offers a sufficient condition for a ring to be nilperiod. It along with its corollary are specific instances of a recent improvement to Jacobson's Theorem concerning periodic rings due to Anderson and Danchev [5].

Theorem 2.3. *If R is a ring in which every nilpotent element is central and torsion, then R is nilperiod.*

Proof. Let $r \in \text{Nil}(R)$, and let i and j be the index and order, respectively, of r. If we set $n = (i-1)!j$, then for all $x \in R$,

$$(x + r)^n = x^n + \sum_{k=1}^{i-1} \binom{n}{k} r^k x^{n-k}. \tag{2.5}$$

Since the binomial coefficient $\binom{n}{k}$ is a multiple of $n/\gcd(n, k)$ for each integer k (cf Problem B2 of the 2000 Putnam competition [25]) and $n/\gcd(n, k)$ is a multiple of j for each $k \in \{1, 2, \ldots, i-1\}$, the summation on the right side of (2.5) vanishes. Therefore $(x+r)^n = x^n$. □

Corollary 2.4. *If R is a weakly periodic ring in which every nilpotent element is central and torsion, then R is a periodic commutative torsion ring.*

Proof. Let $x \in \text{Pot}(R)$ and $y \in \text{Nil}(R)$ be given. By Theorem 2.3, there is an integer n such that $(x+y)^n = x^n$. Since there is also an integer $m > 1$ such that $x^m = x$, we can see that

$$(x+y)^{nm} = x^{nm} = x^n = (x+y)^n. \tag{2.6}$$

Hence R is periodic, and because all nilpotent elements are central, we can further conclude that R is commutative (see [18]). Consequently, $\text{Nil}(R)$ is an ideal.

To see that R is torsion, notice that $R/\text{Nil}(R)$ is a J-ring. As Jacobson explained in the proof of his classic "$a^n = a$" theorem [23], the additive group of a J-ring is torsion. This implies that for each $x \in \text{Pot}(R)$, there is a positive integer j such that $jx \in \text{Nil}(R)$. Since every nilpotent element of R is itself torsion, it follows that R is torsion as a whole. □

It may be interesting to figure out the extent to which the conditions of Theorem 2.3 can be loosened. Here is an example demonstrating that either of the hypotheses in Theorem 2.3 can be soundly defied.

Example 2.5. Consider the direct product $R = R_1 \times R_2$, where R_1 is a reduced ring and R_2 is a nil ring of bounded index, say n. Then $\text{Nil}(R) = 0 \times R_2$ has index n as well. Furthermore,

$$((a,b) + (0,y))^n = (a^n, 0) = (a,b)^n \tag{2.7}$$

for all $a \in R_1, b, y \in R_2$, and integers $m \geq n$.

Take R_2 to be a torsion-free abelian group equipped with the zero multiplication to see that nilperiod rings can be entirely devoid of torsion nilpotent elements. Further still, the existence of noncommutative nil rings rules out the necessity of central nilpotent elements. However, nilperiod rings seem constrained enough to compel $\text{Nil}(R)$ to be an ideal.

We call R an NI-ring if $\text{Nil}(R)$ is an ideal. Equivalently, $\text{Nil}(R)$ coincides with the upper nilradical $\text{Nil}^*(R)$ (i.e. the sum of all nil ideals of R).

Conjecture 2.6. *Every nilperiod ring is an NI-ring.*

If Conjecture 2.6 is true, then noncommutative weakly periodic nilperiod rings are "almost" commutative in the sense that their commutator ideals are nil. Indeed, for if R is is a weakly periodic NI-ring, then $R/\text{Nil}(R)$ is commutative due to being a J-ring. Moreover, because the Jacobson radical $\text{J}(R)$ of a weakly periodic ring is nil, we could report that $\text{J}(R) = \text{Nil}(R) = \text{Nil}^*(R)$ and that R is ultimately periodic, courtesy of Lemma 1 in [17].

Of course in noncommutative rings, $\text{Nil}^*(R)$ typically differs from the lower nilradical $\text{Nil}_*(R)$, that is, the intersection of all the prime ideals of R. A ring R is called *2-primal* if $\text{Nil}_*(R) = \text{Nil}(R)$. Marks [26] provided a thorough list of conditions on noncommutative rings that enforce 2-primality together with their interdependencies, but none involve weak periodicity. Furthermore, being 2-primal is a necessary and sufficient condition for a ring with bounded nilpotency index to be an NI-ring [21, Proposition 1.4]. We should therefore suspect that a weakly periodic NI-ring may fail to be 2-primal provided its nilpotency index is unbounded.

Example 2.7. Let S be a finite 2-primal ring, $n \in \mathbb{N}$, and R_n be the $2^n \times 2^n$ upper triangular matrix ring over S. Each R_n is finite and thus periodic. Proposition 2.5 of [7] further implies that each R_n is 2-primal, and so they are NI-rings. Now embed each R_n into R_{n+1} via the monomorphism σ defined by

$$\sigma(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

Then $\mathcal{D} = \langle R_n, \sigma_{nm} \rangle$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $\mathbb{N}$. Set $R = \varinjlim R_n$, the direct limit of $\mathcal{D}$. Since $R = \bigcup_{n=1}^{\infty} R_n$ is the union of periodic rings, R is itself periodic. However, [21, Proposition 1.1 and Example 1.2] explains why R fails to be 2-primal, even though the direct limit of a direct system of NI-rings is itself an NI-ring.

We can at least verify Conjecture 2.6 for polynomial identity algebras, and, as corollaries, periodic algebras and finite rings. The proof adapts techniques employed by Herstein in [18] and [19]. First, we will need the following lemma.

Lemma 2.8. *If D is a division ring, then there are no periodic power maps over $\mathcal{M}_n(D)$, the ring of $n \times n$ matrices with entries in D.*

Proof. First extend D to a division ring L, if needed, with the same center as D in which every matrix over L has left and right eigenvalues in L. (This is always possible per Corollary 8.3.7 of [12].) According to Theorem 8.3.6 of [12], every matrix in $\mathcal{M}_n(L)$ thus has a Jordan canonical form.

Now consider a nonzero nilpotent matrix $A \in \mathcal{M}_n(D) \subseteq \mathcal{M}_n(L)$. The Jordan canonical form J of A is a strictly lower triangular matrix in $\mathcal{M}_n(L)$ with its unity entries lying on the subdiagonal and all other entries equal to zero. Let $C \in \mathcal{M}_n(D)$ be the companion matrix of the polynomial $p(t) = t^n - 1$. The only nonzero entries of C are at entry $(1, n)$, which is 1, and the subdiagonal completely populated by 1s, and so $C - J$ is a $(0, 1)$-matrix. Because L and D share the same center, we see that J and, consequently, $C - J$ are matrices in $\mathcal{M}_n(D)$. Furthermore, $C - J$ is singular due to having rows comprised entirely of zeros.

Let $S \in \mathcal{M}_n(D)$ be the change of basis matrix for which $A = SJS^{-1}$. Then for every $m \in \mathbb{N}$,

$$\left(SCS^{-1} - A\right)^m = S(C - J)^m S^{-1} \tag{2.8}$$

is singular whereas $(SCS^{-1})^m = SC^m S^{-1}$ is not, thus precluding the periodicity of $f(x) = x^m$. □

Theorem 2.9. *If R is a nilperiod PI-algebra, then R is an NI-ring.*

Proof. Let $a, b \in \mathrm{Nil}(R)$. Then there are positive integers ℓ and m such that $(x + a)^m = x^m$ and $(x + b)^\ell = x^\ell$ for all $x \in R$. Consequently, $(a + b)^{\ell m} = 0$, and so $a + b$ is nilpotent.

Next, suppose $c \in \mathrm{Nil}(R)$, and let $r \in R$ be arbitrary. Let S be the subalgebra of R generated by c and r. Since S is a finitely-generated PI-algebra, $\mathrm{J}(S)$ is nil due to Amitsur's Nullstellensatz [4], [10].

Now assume, to the contrary, that $c \notin \mathrm{J}(S)$. Then the coset $\bar{c} = (c + \mathrm{J}(S)) \in S/\mathrm{J}(S)$ is a nonzero nilpotent element of index, say, j. Because $S/\mathrm{J}(S)$, as a semiprimitve ring, is a subdirect product of primitive rings S_i, each of which is a homomorphic image of $S/\mathrm{J}(S)$, the coset $\bar{c}$ projects to a nilpotent element ν_i within each factor S_i. Note that not all of these ν_i can be 0 , otherwise $\bar{c} = 0$, which would indicate that $c \in \mathrm{J}(S)$.

Let S_i be a factor in which $\nu_i \neq 0$. Since c is the period of some power map $f(x) = x^n$, the factor S_i inherits the periodicity of the same power map, but with period ν_i. Furthermore, S_i cannot be a division ring as it contains the nonzero nilpotent ν_i. Due to Jacobson's density theorem, S_i is thus isomorphic to a dense subring of $\mathrm{End}(V_i)$ for some vector space V_i over a division ring D_i. Since S_i is itself a PI-algebra, we may assume that it is finite-dimensional (cf [24, Lemma 5 and Theorem 1]), and so $S_i \cong \mathcal{M}_{s_i}(D_i)$. But by Lemma 2.8 the power map $f(x) = x^n$ cannot be periodic, which is at odds with our original assumptions.

Hence, $\bar{c} = 0$. It follows that $c \in \mathrm{J}(S)$, and so cr and rc are in $\mathrm{J}(S)$ as well. Since $\mathrm{J}(S)$ is nil, both cr and rc are nilpotent, as required. □

Corollary 2.10. *If R is a periodic nilperiod algebra with finite μ_1 and μ_0, then R is an NI-algebra.*

Proof. Since $x^{\mu_0+\mu_1} = x^{\mu_0}$ for all $x \in R$, it follows that R is a PI-algebra, and so Theorem 2.9 applies. $\square$

The logistics of Theorem 2.9 apply to any ring where the $\mathrm{J}(S)$ and S_i so constructed in the proof are nil and finite-dimensional, respectively. To that end, we can see that finite nilperiod rings are NI-rings. Indeed, for S is Artinian, and so $\mathrm{J}(S)$ is nilpotent.

3 The Number of Distinct Power Maps

We now restrict our attention to periodic rings with finite μ_1 and μ_0, for which there are $\mu_0 + \mu_1 - 1$ distinct power maps over R. [1] Let $\mu_P := \mu_P(R)$ denote the number of distinct periodic power maps over R. Pinpointing the precise ring properties that determine the value of μ_P is currently beyond our grasp. However, a lower bound for μ_P is quite tenable, provided that R is commutative and $\mathrm{Nil}(R)^+$ is a torsion group with finite exponent. In essence, nilpotent elements of simultaneously prime order and index 2 are the "fundamental" periods of the easiest-to-distinguish periodic power maps. First, we borrow some notation from number theory. The squarefree radical of a positive integer n, denoted $\mathrm{rad}(n)$, is the product of the distinct prime factors of n, and $\omega(n)$ is the number of distinct prime factors of n. Despite the threat of notational confusion, we will let $\mu(n)$ be the Möbius function (because one can never get enough of the Greek letter mu).

Theorem 3.1. *Let R be a commutative ring in which the exponent of* $\mathrm{Nil}(R)^+$ *is finite. If N is the least common multiple of the orders of the nilpotent elements, then $f(x) = x^n$ is periodic for every integer $n \in \mathbb{N}$ that is not coprime with N. Accordingly,*

$$\mu_P \geq -\sum_{\substack{d\,|\,N,\\ d\neq 1}} \mu(d)\left\lfloor\frac{\mu_0+\mu_1-1}{d}\right\rfloor = \sum_{\substack{d\,|\,\mathrm{rad}(N),\\ d\neq 1}} (-1)^{\omega(d)+1}\left\lfloor\frac{\mu_0+\mu_1-1}{d}\right\rfloor$$

if μ_1 and μ_0 are finite. This lower bound is attained if R is also finite and unital.

Proof. By Cauchy's theorem for abelian groups, if p is a prime number that divides N, then $\mathrm{Nil}(R)$ contains a necessarily nonzero element r of order p. If the index of r is k, then the index of r^{k-1} is 2 , and since $r \mid r^{k-1}$, the order of r^{k-1} is also p. We can thus invoke the "freshman's dream" to see that

$$\left(x + r^{k-1}\right)^{pm} = \left(x^p + r^{(k-1)p}\right)^m = x^{pm} \tag{3.1}$$

for every $x \in R$ and $m \in \mathbb{N}$.

So if $pm \leq \mu_0 + \mu_1 - 1$, then $f(x) = x^{pm}$ is a nonrepetitive periodic power map over R. We can enumerate all of these maps in two different ways. One way is a routine application of the principle of inclusion-exclusion. The other way is to note that the number of integers

[1] The only exception to this is the zero ring. This is because $\mu_0, \mu_1 \geq 1$, but the zero ring only has one mapping on it.

no larger than $\mu_0 + \mu_1 - 1$ which are coprime to N is given by

$$\sum_{\substack{1\le n\le \mu_0+\mu_1-1,\\ \gcd(n,N)=1}} 1 = \sum_{n=1}^{\mu_0+\mu_1-1} \sum_{d\,|\,\gcd(n,N)} \mu(d) \tag{3.2}$$

$$= \sum_{d\,|\,N} \mu(d) \sum_{\substack{1\le n\le \mu_0+\mu_1-1,\\ d\,|\,n}} 1 \tag{3.3}$$

$$= \sum_{d\,|\,N} \mu(d) \left\lfloor \frac{\mu_0+\mu_1-1}{d} \right\rfloor = \mu_0+\mu_1-1 + \sum_{\substack{d\,|N,\\ d\neq 1}} \mu(d) \left\lfloor \frac{\mu_0+\mu_1-1}{d} \right\rfloor \tag{3.4}$$

$$= \mu_0+\mu_1-1 + \sum_{\substack{d\,|\,\mathrm{rad}(N),\\ d\neq 1}} (-1)^{\omega(d)} \left\lfloor \frac{\mu_0+\mu_1-1}{d} \right\rfloor \tag{3.5}$$

Lastly, let us see that these are the only periodic power maps if R is a finite commutative ring with unity. Since finite rings are trivially Artinian, $R \cong \prod_i R_i$, where the R_i are finite commutative local rings, and for each i we have a natural surjective homomorphism $\pi_i : R \to R_i$. Now suppose that $g(x) = x^n$ is periodic and $s \in \mathrm{Per}(g)\backslash\{0\}$. Then $\pi_i s \neq 0_{R_i}$ for some i. Furthermore,

$$(1_{R_i} + \pi_i s)^n = 1_{R_i} = 1_{R_i} + \sum_{k=1}^{n} \binom{n}{k} \pi_i s^k \tag{3.6}$$

and so

$$\sum_{k=1}^{n} \binom{n}{k} \pi_i s^k = \pi_i s \left(n1_{R_i} + \sum_{k=2}^{n} \binom{n}{k} \pi_i s^{k-1} \right) = 0_{R_i} \tag{3.7}$$

Therefore $n1_{R_i} + \sum_{k=2}^{n} \binom{n}{k} \pi_i s^{k-1} = 0_{R_i}$ or is a zero divisor. Either way, $n1_{R_i}$ must be a non-unit due to the locality of R_i. It follows that n is not coprime with $\mathrm{char}(R_i)$, the characteristic of R_i, otherwise Bézout's identity could be used to express the unit 1_{R_i} as a $\mathbb{Z}$-linear combination of the non-units $n1_{R_i}$ and $\mathrm{char}(R_i)1_{R_i}$. However, $\mathrm{char}(R_i)$ must be a prime power p^t, and so p is a common divisor of n and $\mathrm{char}(R_i)$. Since the order of s is a multiple of $\mathrm{char}(R_i)$, we conclude that n and the order of s are not coprime. □

Observe that $\mu_0 = 1$ for J-rings and $\mu_1 = 1$ for nil rings. This quickly leads to the two following corollaries.

Corollary 3.2. *Let R be a J-ring with finite μ_1. Then R has μ_1 distinct power maps, none of which are periodic.*

Corollary 3.3. *If R is a nil ring of finite index μ_0, then R has μ_0 distinct power maps. If, in addition, R is also commutative and of bounded torsion, then*

$$\mu_P \geq -\sum_{\substack{d\,|\,N,\\ d\neq 1}} \mu(d) \left\lfloor \frac{\mu_0}{d} \right\rfloor = \sum_{\substack{d\,|\,\mathrm{rad}(N),\\ d\neq 1}} (-1)^{\omega(d)+1} \left\lfloor \frac{\mu_0}{d} \right\rfloor$$

where the notation of Theorem 3.1 has been reprised.

4 Miscellaneous Examples

4.1 Weakly periodic rings that annihilate Nil(R)

Theorem 2.3 remains true if we swap out the torsionality of $\mathrm{Nil}(R)$ for the condition that $xr = 0 = rx$ for all $x \in R$ and $r \in \mathrm{Nil}(R)$. (The nilpotent elements are still central here.)

In this case, $(x+r)^n = x^n + r^n$. So $\alpha_n(x) = x^n$ is periodic for every integer $n \geq 2$, and the periods of α_n are the nilpotent elements with index at most n. It follows that a weakly periodic ring in which $R \cdot \mathrm{Nil}(R) = 0 = \mathrm{Nil}(R) \cdot R$ is a commutative periodic ring, albeit not necessarily torsion.

We should emphasize that R being a two-sided annihilator of $\mathrm{Nil}(R)$ is pivotal to the argument that R is nilperiod. For instance, consider the Klein four-group $\mathbb{V} = \{0, a, b, c\}$ presented additively so that $R^+ = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and endowed with the multiplication given by $0x = cx = 0$ and $ax = bx = x$ for all $x \in \mathbb{V}$. Observe that $\mathrm{Pot}(R) = \{a, b\}$ and $\mathrm{Nil}(R) = \{0, c\}$, and yet the power maps over R are merely "quasiperiodic" over $\mathrm{Pot}(R)$ in the sense that $(x+c)^n = x^n + c$ for all $x \in R\backslash\{0, c\}$ and $n \in \mathbb{N}$. This example due to Bell [6] is notable for the one-sided orthogonality of $\mathrm{Nil}(R)$ and $\mathrm{Pot}(R)$. Various aspects of such rings are discussed in [16] and [22].

4.2 Corbas (p, k, ϕ)-rings

Let us return to Example 2.2, this time with ϕ equaling the identity, where we found that

$$(a,b)^n = \begin{cases} (a^n, na^{n-1}b) & \text{if } \phi \text{ is the identity automorphism,} \\ \left(a^n, b\phi\left(a^{n-1}\right) \frac{a^n\phi\left(a^{-n}\right)-1}{a\phi\left(a^{-1}\right)-1}\right) & \text{otherwise,} \end{cases} \tag{4.1}$$

for all $a \in \mathbb{F}_{p^k}^{\times}, b \in \mathbb{F}_{p^k}$, and $n \in \mathbb{N}$. Since every non-nilpotent element of R is potent, $\mu_0 = 2$. To ascertain μ_1, we need to calculate the smallest positive integer μ_1 such that $a^{1+\mu_1} = a$ and $(1+\mu_1)a^{\mu_1} = 1$ for every $a \in \mathbb{F}_{p^k}^{\times}$. Recall that $\mathbb{F}_{p^k}$ has characteristic p, and the multiplicative group $\mathbb{F}_{p^k}^{\times}$ is a cyclic group of order $p^k - 1$. Hence $\mu_1 = \mathrm{lcm}(p, p^k - 1) = p(p^k - 1)$. There are thus $p^{k+1} - p + 1$ distinct power maps over R.

Out of these, $p^k - 1$ are periodic when ϕ is the identity. To directly see why, note that the only way for the equation

$$((a,b) + (0,y))^n = \left(a^n, na^{n-1}(b+y)\right) = (a,b)^n = \left(a^n, na^{n-1}b\right)$$

to hold over all of R is for n to be a multiple of p, in which case $(a,b)^n = (a^n, 0)$. This amount matches the summation derived in Theorem 3.1 since every nonzero nilpotent element of R has order p and so

$$\sum_{\substack{d \mid p, \\ d \neq 1}} (-1)^{\omega(d)-1} \left\lfloor \frac{\mu_0 + \mu_1 - 1}{d} \right\rfloor = (-1)^{\omega(p)-1} \left\lfloor \frac{p^{k+1} - p + 1}{p} \right\rfloor = p^k - 1 \tag{4.2}$$

4.3 Galois Rings

Let $R = \mathrm{GR}(p^k, d)$, the unique Galois extension of $\mathbb{Z}/p^k\mathbb{Z}$ of degree d, which is a local ring of characteristic p^k. It is well-known that the unique maximal ideal of R is the principal ideal (p), which is entirely comprised of all multiples of p, and that every non-nilpotent element is a unit. Furthermore, $R^{\times} \cong G_1 \times G_2$, where G_1 is a cyclic group of order $p^d - 1$ and

$$G_2 \cong \begin{cases} C_2 \times C_{2^{k-2}} \times \left(C_{2^{k-1}}\right)^{d-1} & \text{if } p = 2 \text{ and } k \geq 3, \\ \left(C_{p^{k-1}}\right)^d & \text{otherwise.} \end{cases} \tag{4.3}$$

If $d = 1$, then $R \cong \mathbb{Z}/p^k\mathbb{Z}$. This case is discussed in the next subsection. If $k = 1$, then $R \cong \mathbb{F}_{p^d}$, which by Corollary 3.2 has $p^d - 1$ distinct power maps, none of which are periodic.

Finally, if $d, k > 1$, then $\mu_0 = k$ and $\mu_1 = \operatorname{lcm}(p^d - 1, p^{k-1}) = p^{k-1}(p^d - 1)$. There are thus $p^{k-1}(p^d - 1) + k - 1$ distinct power maps over R in this case, and since R is a finite commutative ring with unity, we can apply Theorem 3.1 to see that

$$\mu_P\Big(\mathrm{GR}\Big(p^k, d\Big)\Big) = (-1)^{\omega(p)-1} \left\lfloor \frac{p^{k-1}\left(p^d - 1\right) + k - 1}{p} \right\rfloor = p^{k-2}\left(p^d - 1\right) + \left\lfloor \frac{k-1}{p} \right\rfloor \tag{4.4}$$

4.4 The integers modulo n

The enumeration of the distinct power maps over $\mathbb{Z}/n\mathbb{Z}$ is a folklore result, but we include it below for completeness. On the other hand, a description of the periodic power maps, while elementary, is perhaps new or, at the least, little-known.

Let n be a positive integer and let $p_1^{\beta_1} p_2^{\beta_2} \cdots p_v^{\beta_v}$ be its prime factorization. By the Chinese Remainder Theorem,

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{\beta_1}\mathbb{Z} \times \mathbb{Z}/p_2^{\beta_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_v^{\beta_v}\mathbb{Z} \tag{4.5}$$

Based on (4.3) and the discussion surrounding it, $\mu_1(\mathbb{Z}/p_i^{\beta_i}\mathbb{Z}) = \lambda(p_i^{\beta_i})$, where λ is the Carmichael function defined on prime powers p^β by

$$\lambda(p^\beta) = \begin{cases} 2^{\beta-2} & \text{if } p = 2 \text{ and } \beta \geq 3, \\ p^{\beta-1}(p-1) & \text{otherwise.} \end{cases}$$

It follows that the exponential period of $\mathbb{Z}/n\mathbb{Z}$ is

$$\mu_1(\mathbb{Z}/n\mathbb{Z}) = \operatorname{lcm}\Big(\lambda(p_1^{\beta_1}), \lambda(p_2^{\beta_2}), \ldots, \lambda(p_v^{\beta_v})\Big) = \lambda(n) \tag{4.6}$$

whereas the onset of exponential periodicity is $E(n) := \max \beta_i$, that is, the index of

$$\mathrm{Nil}(\mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Nil}\Big(\mathbb{Z}/p_1^{\beta_1}\mathbb{Z}\Big) \times \mathrm{Nil}\Big(\mathbb{Z}/p_2^{\beta_2}\mathbb{Z}\Big) \times \cdots \times \mathrm{Nil}\Big(\mathbb{Z}/p_v^{\beta_v}\mathbb{Z}\Big) \tag{4.7}$$

There are thus $\lambda(n) + E(n) - 1$ distinct power maps over $\mathbb{Z}/n\mathbb{Z}$.

Table 2 below summarizes this data for some simple values of n. This settles Blomberg and Whitmore's conjectures for sequence A109746 of the OEIS [27].

Tab. 2: Some power map enumerations in $\mathbb{Z}/n\mathbb{Z}$

Value of n	$\lambda(n)$	$E(n)$	*# of power maps*	*# that are periodic*
prime p	$p-1$	1	$p-1$	0
$2p$ (for odd prime p)	$p-1$	1	$p-1$	0
$\prod_{\text{prime factors } p_i} p_i$	$\operatorname{lcm}\{p_i - 1\}$	1	$\operatorname{lcm}\{p_i - 1\}$	0
p^2 (for any prime p)	$p^2 - p$	2	$p^2 - p + 1$	$p-1$
$2^k, k > 2$	2^{k-2}	k	$2^{k-2} + k - 1$	$2^{k-3} + \lfloor \frac{k-1}{2} \rfloor$
$p^k, k > 2$ (for odd prime p)	$p^{k-1}(p-1)$	k	$p^{k-1}(p-1) + k - 1$	$p^{k-2}(p-1) + \left\lfloor \frac{k-1}{p} \right\rfloor$

To help us see which power maps over $\mathbb{Z}/n\mathbb{Z}$ are periodic, we recall that $\mathbb{Z}/p_i^{\beta_i}\mathbb{Z}$ is a field precisely when $\beta_i = 1$. For this reason, p_i divides the order of some nonzero nilpotent element of $\mathbb{Z}/n\mathbb{Z}$ if and only if $\beta_i > 1$ (which is equivalent to the condition that $p_i \mid (n/\mathrm{rad}(n))$). Hence, by Theorem 3.1,

$$\mu_P(\mathbb{Z}/n\mathbb{Z}) = \sum_{\substack{d \,|\, \mathrm{rad}(n/\mathrm{rad}(n)), \\ d \neq 1}} (-1)^{\omega(d)-1} \left\lfloor \frac{\lambda(n) + E(n) - 1}{d} \right\rfloor. \tag{4.8}$$

4.5 Endomorphism rings of finite abelian groups

Let G be a finite abelian group, and let $\mathrm{End}(G)$ denote the endomorphism ring of G. By the fundamental theorem of finite abelian groups,

$$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \tag{4.9}$$

for some integers $n_1, \ldots, n_k \geq 2$. So

$$\mathrm{End}(G) \cong \prod_{i=1}^{k} \prod_{j=1}^{k} \mathbb{Z}_{\gcd(n_i, n_j)}. \tag{4.10}$$

It thus follows from the work of the previous subsection that

$$\mu_0(\mathrm{End}(G)) = \max_{1 \leq i,j \leq k} E(\gcd(n_i, n_j)), \tag{4.11}$$

$$\mu_1(\mathrm{End}(G)) = \mathrm{lcm}\{\lambda(\gcd(n_i, n_j)) : 1 \leq i, j \leq k\}, \tag{4.12}$$

$$\mu_P(\mathrm{End}(G)) = \prod_{i=1}^{k} \prod_{j=1}^{k} \mu_P(\mathbb{Z}_{\gcd(n_i, n_j)}). \tag{4.13}$$

Abyzov, Barati, and Danchev established in [2] that $\mathrm{End}(G)$ is periodic exactly when G is finite. For this reason, the combinatorics of periodic power maps over endomorphism rings is now completely resolved.

4.6 Matrix rings over fields.

Consider $R = \mathcal{M}_n(\mathbb{F}_q)$, where q is a prime power p^k. Almkvist [3] proved that $\mu_1(\mathcal{M}_n(\mathbb{F}_2))$ is the exponent of $\mathrm{GL}(n, \mathbb{F}_2)$. Here we modify Almkvist's methods to show that $\mu_1(\mathcal{M}_n(\mathbb{F}_q))$ is the exponent of $\mathrm{GL}(n, \mathbb{F}_q)$ and $\mu_0(\mathcal{M}_n(\mathbb{F}_q)) = n$ in general.

Let $\tau \in R$, let $\psi(t) \in \mathbb{F}_q[t]$ be the minimal polynomial of τ, and let $\psi_1(t)^{\beta_1} \cdots \psi_v(t)^{\beta_v}$ be the prime factorization of $\psi(t)$. Treat τ as a linear operator of the vector space $V := \mathbb{F}_q^n$ and set $V_i = \ker \psi_i(\tau)^{\beta_i}$ for each i. Then $V = \bigoplus_{i=1}^{v} V_i$ and each V_i is invariant under τ. For each i, let $\tau_i : V_i \to V_i$ be the restriction of τ to V_i so that $\psi_i(\tau_i)^{\beta_i}$ vanishes. It suffices to flesh out the dynamics of the sequence $\{\tau_i^j\}_{j=1}^{\infty}$.

If $\psi_i(t) = t$, then τ_i is nilpotent and there is nothing more to prove, so assume instead that $\psi_i(t) \neq t$. Suppose that $\deg \psi_i(t) = m$. Then $\psi_i(t) \mid (t^{q^m} - t)$ since the product of all monic irreducible polynomials in $\mathbb{F}_q[t]$ with degree dividing m is $t^{q^m} - t$. Hence $(\tau_i^{q^m} - \tau_i)^{\beta_i} = 0$, from which it follows that τ_i is potent. Moreover, because $\beta_i \leq n \leq p^{\lceil \log_p n \rceil}$, we have that

$$(t^{q^m} - t)^{\beta_i} \mid (t^{q^m} - t)^{p^{\lceil \log_p n \rceil}} = t^{p^{\lceil \log_p n \rceil} q^m} - t^{p^{\lceil \log_p n \rceil}}. \tag{4.14}$$

It follows that

$$\tau_i^{p^{\lceil \log_p n \rceil} q^m} = \tau_i^{p^{\lceil \log_p n \rceil}}, \tag{4.15}$$

and so the exponential periodicity of R is a divisor of

$$\begin{aligned} &\mathrm{lcm}\left\{p^{\lceil \log_p n \rceil}(q-1), p^{\lceil \log_p n \rceil}\left(q^2 - 1\right), \ldots, p^{\lceil \log_p n \rceil}(q^n - 1)\right\} \qquad (4.16)\\ = \;& p^{\lceil \log_p n \rceil} \,\mathrm{lcm}\left\{q - 1, q^2 - 1, \ldots, q^n - 1\right\}. \end{aligned}$$

Yet (4.16) is precisely the exponent of $\mathrm{GL}(n, \mathbb{F}_q)$ [15, Corollary 1], and so this yields $\mu_1(R)$. Finally, $\mu_0(R)$ is the largest possible index of a nilpotent operator on a subspace of V, which is n.

5 Concluding Remarks

While we have explicitly specified the periodic power maps over finite commutative rings with unity, things quickly turn more exotic in broader classes of rings. For instance, Example 2.5 shows that μ_P can potentially reach the upper bound of $\mu_1 + \mu_0 - 2$ where every non-identity power map is periodic. In any event, we pose the following question.

Question 5.1. *What is the distribution of μ_P over random finite commutative rings?*

Investigations into the oscillatory behavior of generic power maps over rings can proceed in at least two directions. One is taxonomic: can we achieve at least a partial characterization of the nilperiod rings? Another is combinatorial: is it possible to detect and enumerate the periodic power maps over any periodic ring? To the author's surprise, these questions appear mostly unexplored. We hope that this article generates some interest in what may potentially be a rich and attractive topic that could present some low-hanging fruit, even for students.

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