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Growth of analytic solutions of linear differential equations with analytic coefficients near a finite singular point

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Abstract

In this paper, we investigate the growth of analytic solutions of the linear differential equation

$$f^{(k)} + A_{k-1}(z)\exp\left\{\frac{a_{k-1}}{(z_0-z)^n}\right\}f^{(k-1)} + \cdots + A_0(z)\exp\left\{\frac{a_0}{(z_0-z)^n}\right\}f = 0,$$

where $n \in \mathbb{N} - \{0\}, k \geq 2$ is an integer and $A_j(z)(j = 0, \ldots, k-1)$ are analytic functions in the closed complex plane except a singular point z_0 and $a_j(j = 0, \ldots, k-1)$ are complex numbers. Under some conditions, we prove that these solutions are of infinite order and their hyper-order is equal to n. We also consider the nonhomogeneous linear differential equations.

Keywords: Linear differential equations, Nevanlinna theory, Growth of solutions, Finite singular point.

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1 Introduction and Main Results

Nevanlinna theory is a branch of complex analysis that studies the value distribution of meromorphic functions. It provides essential tools to understand the growth and behavior of solutions of linear differential equations, depending on the nature of the coefficients and the singularities involved.

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic function in the complex plane $\mathbb{C}$ (see [2, 3, 14, 15, 16]). We denote respectively the order and the hyper-order of growth of a meromorphic function f, by $\sigma(f)$ and $\sigma_2(f)$.

Many authors as in ([4, 5, 6, 9, 12]) have studied the growth of solutions of linear differential equations near a finite singular point. They have investigated different forms of linear differential equations with analytic coefficients near a finite singular point by using adapted notions and properties of Nevanlinna theory. In this paper, we continue this investigation near a finite singular point.

First, we recall the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that f is meromorphic in $\overline{\mathbb{C}} \backslash \{z_0\}$, where $z_0 \in \mathbb{C}$. Define the counting function near z_0 by

$$N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n_{z_0}(t, f) - n_{z_0}(\infty, f)}{t} dt - n_{z_0}(\infty, f)\log r,$$

where $n_{z_0}(t,f)$ counts the number of poles of f in the region $\{z \in \mathbb{C} : t \leq |z_0 - z|\} \cup \{\infty\}$, each pole according to its multiplicity, and the proximity function by

$$m_{z_0}(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(z_0 - re^{i\phi})\right| d\phi.$$

The characteristic function of f is defined in the usual manner by

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f).$$

In addition, the order of a meromorphic function f near z_0 is defined by

$$\sigma_T(f,z_0) = \limsup_{r\to 0} \frac{\log^+ T_{z_0}(r,f)}{-\log r}.$$

For an analytic function f in $\overline{\mathbb{C}}\backslash\{z_0\}$, we have also the definition

$$\sigma_M(f,z_0) = \limsup_{r\to 0} \frac{\log^+\log^+ M_{z_0}(r,f)}{-\log r},$$

where $M_{z_0}(r,f) = \max_{|z_0 - z| = r} |f(z)|$.

When the order is infinite, we introduce the notion of hyper-order near z_0 that is defined as follows:

$$\sigma_{2,T}(f,z_0) = \limsup_{r\to 0} \frac{\log^+\log^+ T_{z_0}(r,f)}{-\log r},$$

$$\sigma_{2,M}(f,z_0) = \limsup_{r\to 0} \frac{\log^+\log^+\log^+ M_{z_0}(r,f)}{-\log r}.$$

Remark 1.1 ([4]). *It is shown in [9] that if $f(z)$ is a non-constant meromorphic function on $\overline{\mathbb{C}}\backslash\{z_0\}$ and $g(\omega) = f(z_0 - \frac{1}{\omega})$, then $g(\omega)$ is meromorphic on $\mathbb{C}$ and we have :*

$$T(R,g) = T_{z_0}(\frac{1}{R}, f),$$

where $R > 0$ and so $\sigma_T(f,z_0) = \sigma(g)$. Also, if f is analytic on $\overline{\mathbb{C}}\backslash\{z_0\}$, then $g(\omega)$ is entire, and thus $\sigma_T(f,z_0) = \sigma_M(f,z_0)$ and $\sigma_{2,T}(f,z_0) = \sigma_{2,M}(f,z_0)$. Then we can use the notations $\sigma(f,z_0)$ and $\sigma_2(f,z_0)$ without any ambiguity.

The linear differential equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0, \tag{1.1}$$

where $A(z)$ and $B(z)$ are entire functions is investigated by many authors; see for example [1, 2, 3, 10, 15]. Kwon [15] proved that if $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ with $0 < c < 1$, then every solution $f \not\equiv 0$ of equation (1.1) is of infinite order.

To investigate the counterpart of Kwon's result near a finite singular point, Fettouch and Hamouda proved the following result :

Theorem 1.1 ([9]). *Let z_0, a, b be complex constants, such that $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Let $A(z), B(z) \not\equiv 0$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\max\{\sigma(A,z_0), \sigma(B,z_0)\} < n$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\}f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\}f = 0$$

satisfies $\sigma(f,z_0) = +\infty$ with $\sigma_2(f,z_0) = n$.

In [4], Cherief and Hamouda have generalized the above result to higher order differential equations and proved the following result :

Theorem 1.2 ([4])**.** *Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer and $A_j(z)(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) < n$ and let $a_j(j = 0, ..., k-1)$ be complex numbers, such that $a_0 = |a_0|e^{i\theta_0}$, $a_s = |a_s|e^{i\theta_s}$, $a_0 a_s \neq 0$ $(0 < s < l \leq k-1)$, $\theta_0, \theta_s \in [0, 2\pi)$, $\theta_0 \neq \theta_s$, $A_0 A_s \not\equiv 0$ and for $j \neq 0, s$, a_j satisfies either $a_j = d_j a_0$ $(d_j < 1)$ or $\arg a_j = \arg a_s$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) \exp\left\{\frac{a_{k-1}}{(z_0 - z)^n}\right\} f^{(k-1)} + ... + A_0(z) \exp\left\{\frac{a_0}{(z_0 - z)^n}\right\} f = 0 \tag{1.2}$$

that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$ satisfies $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) = n$.

Fan and Chen in [8] considered the linear differential equation

$$f^{(k)} + H_{k-1}(z) f^{(k-1)} + ... + H_1(z) f' + H_0(z) f = 0, \tag{1.3}$$

where $k \geq 2$ is an integer and $H_j(z)(j = 0, 1, .., k-1)$ are entire functions. They proved the following result :

Theorem 1.3. *Let $k \geq 2$ be an integer and $a_j(j = 0, ..., k-1)$ be complex numbers. Suppose that there exist a_s and a_l such that $s < l$, $a_s = d_s e^{i\phi}$, $a_l = -d_l e^{i\phi}$, $d_s > 0$, $d_l > 0$ and for $j \neq s, l$, $a_j = d_j e^{i\phi}$ or $a_j = -d_j e^{i\phi}(d_j \geq 0)$, and $\max\{d_j : j \neq s, l\} = d < \min\{d_s, d_l\}$. If $H_j(z) = h_j(z) e^{a_j z^n}$, where $n \in \mathbb{N}\backslash\{0\}$ and $h_j(z)$ are entire functions with $\sigma(h_j) < n$, $h_s h_l \not\equiv 0$, then every transcendental solution f of equation (1.3) is of infinite order and satisfies $\sigma_2(f) = n$.*

In 2017, K. Hamani and B. Belaidi have studied the linear differential equation

$$f^{(k)} + h_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + h_1(z) e^{P_1(z)} f' + h_0(z) e^{P_0(z)} f = 0, \tag{1.4}$$

where $h_j(z)(j = 0, ..., k-1)$ are meromorphic functions and $P_j(z)(j = 0, ..., k-1)$ are polynomials. They proved the following Theorem

Theorem 1.4 ([11])**.** *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i (j = 0, 1, ..., k-1)$ be polynomials with degree $n \geq 1$, where $a_{0,j}, ..., a_{n,j}(j = 0, ..., k-1)$ are complex numbers. Let $h_j(z)(j = 0, ..., k-1)$ be meromorphic functions with $\sigma(h_j) < n$.*

If $h_j \not\equiv 0$, then $a_{n,j} \neq 0$. Suppose that there exists $\left\{a_{n,i_1}, a_{n,i_2}, ..., a_{n,i_m}\right\} \subset \left\{a_{n,1}, a_{n,2}, ..., a_{n,k-1}\right\}$ such that $\arg a_{n,i_j}(j = 1, 2, ..., m)$ are distinct and for every nonzero $a_{n,l} \in \left\{a_{n,1}, a_{n,2}, ..., a_{n,k-1}\right\} \backslash \left\{a_{n,i_1}, a_{n,i_2}, ..., a_{n,i_m}\right\}$ there exists some $a_{n,i_j} \in \left\{a_{n,i_1}, a_{n,i_2}, ..., a_{n,i_m}\right\}$ such that $a_{n,l} = c_l^{(i_j)} a_{n,i_j} (0 < c_l^{(i_j)} < 1)$. Then every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicity of equation (1.4) is of infinite order and satisfies $\sigma_2(f) = n$. Furthermore, if $a_{n,0} = a_{n,i_{j_0}}$ or $a_{n,0} = c_0^{i_{j_0}} a_{n,i_{j_0}} (0 < c_0^{i_{j_0}} \neq c_s^{i_{j_0}} < 1)$, where $s \in \left\{1, ..., k-1\right\}$ and $a_{n,i_{j_0}} \in \left\{a_{n,i_1}, a_{n,i_2}, ..., a_{n,i_m}\right\}$, then every meromorphic solution $f(\not\equiv 0)$ whose poles are of uniformly bounded multiplicity of equation (1.4) is of infinite order and satisfies $\sigma_2(f) = n$.

In this paper, we investigate similar results as in Theorem 1.3 and Theorem 1.4 but specifically for linear differential equations of the form (1.2), where $A_j(z)(j = 0, ..., k-1)$ are analytic functions near a singular point z_0 and $a_j(j = 0, ..., k-1)$ are complex numbers. We use adapted notions and properties of Nevanlinna theory near a singular point and introduce new lemmas we recently proved. We continue to study the growth of analytic solutions of equations of the form (1.2) by considering certain conditions on the coefficients that guarantee that every non-constant analytic solution of (1.2) is of infinite order and hyper-order equal to n. We will also consider the non-homogeneous case. We will prove the following results:

Theorem 1.5. *Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer and $A_j(z)(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) < n$. Suppose that there exist $s, l \in \{1, ..., k-1\}$ such that $A_s A_l \not\equiv 0$, $a_s = d_s e^{i\phi}$, $a_l = -d_l e^{i\phi}, \phi \in [0, 2\pi)$, $d_s > 0$, $d_l > 0$ and for $j \neq s, l$, $a_j = d_j e^{i\phi}$ or $a_j = -d_j e^{i\phi}(d_j \geq 0)$ and $\max\{d_j : j \neq s, l\} = d < min\{d_s, d_l\}$. Then every non-constant solution f of equation (1.2) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, where z_0 is an essential singular point for f.*

Theorem 1.6. *Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer and $A_j(z)(j=0,...,k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) < n$. Suppose that there exists $\{a_{i_0}, a_{i_1}, ..., a_{i_m}\} \subset \{a_1, ..., a_{k-1}\}$ such that $\arg a_{i_j}(j=1,..,m)$ are distinct and for every nonzero $a_l \in \{a_1, ..., a_{k-1}\} \setminus \{a_{i_1}, ..., a_{i_m}\}$, there exists some $a_{i_s} \in \{a_{i_1}, ..., a_{i_m}\}$ such that $a_l = c_l^{(i_s)} a_{i_s} (0 < c_l^{(i_s)} < 1)$. Then every non-constant solution f of equation (1.2) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, where z_0 is an essential singular point for f.*

Theorem 1.7. *Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer, $A_j(z)$ and $a_j(j=0,...,k-1)$ satisfy hypotheses of Theorem 1.5 or those of Theorem 1.6 . Let $F \not\equiv 0$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ of order $\sigma = \sigma(F, z_0) < n$. Then every solution f of equation*

$$f^{(k)} + A_{k-1}(z) \exp\left\{\frac{a_{k-1}}{(z_0-z)^n}\right\} f^{(k-1)} + ... + A_0(z) \exp\left\{\frac{a_0}{(z_0-z)^n}\right\} f = F \tag{1.5}$$

that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, where z_0 is an essential singular point for f, with at most one exceptional analytic solution f_0 of finite order in $\overline{\mathbb{C}}\backslash\{z_0\}$, where z_0 is an essential singular point for f_0.

2 Preliminary Lemmas

Lemma 2.1 ([9])**.** *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. Let $\alpha > 0$ be a given real constant and $j \in \mathbb{N}$. Then there exists a set $E_1 \subset (0,1)$ of finite logarithmic measure, that is $\int_0^1 \chi_{E_1}(t)\frac{dt}{t} < \infty$ and a constant $A > 0$ that depends on α and j, such that for all $r = |z - z_0|$ satisfying $r \notin E_1$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq A \left[\frac{1}{r^2} T_{z_0}(\alpha r, f) \log T_{z_0}(\alpha r, f)\right]^j,$$

where χ_{E_1} is the characteristic function of the set E_1.

Lemma 2.2 ([7])**.** *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. For $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = max_{|z_0-z|=r}|f(z)|$. Then there exist a constant $\delta_r > 0$ and a set $E_2 \subset (0,1)$ of finite logarithmic measure, such that for all z satisfying $| z_0 - z |= r \notin E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \leq 2r^j \ (j \in \mathbb{N}),$$

where z_0 is an essential singular point for f.

Lemma 2.3 ([9])**.** *Let $A(z)$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\sigma(A, z_0) < n$ $(n \in \mathbb{N}\backslash\{0\})$. Set $g(z) = A(z)\exp\left\{\frac{a}{(z_0-z)^n}\right\}$, where $a = \alpha + i\beta \neq 0$ is a complex number, $z_0 - z = re^{i\phi}$, $\delta_a(\phi) = \alpha\cos(n\phi) + \beta\sin(n\phi)$, and $H = \{\phi \in [0, 2\pi) : \delta_a(\phi) = 0\}$ (obviously , H is a finite set). Then for any given $\varepsilon > 0$ and for any $\phi \in [0, 2\pi)\backslash H$, there exists $r_0 > 0$, such that for $0 < r < r_0$, we have*
(i) *if $\delta_a(\phi) > 0$, then*

$$\exp\left\{(1-\varepsilon)\delta_a(\phi)\frac{1}{r^n}\right\} \leqslant |g(z)| \leqslant \exp\left\{(1+\varepsilon)\delta_a(\phi)\frac{1}{r^n}\right\}, \tag{2.1}$$

(ii) *if $\delta_a(\phi) < 0$, then*

$$\exp\left\{(1+\varepsilon)\delta_a(\phi)\frac{1}{r^n}\right\} \leqslant |g(z)| \leqslant \exp\left\{(1-\varepsilon)\delta_a(\phi)\frac{1}{r^n}\right\}. \tag{2.2}$$

Lemma 2.4 ([4])**.** *Let $k \geq 2$ be an integer and $A_j(z)(j=0,...,k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) \leqslant \alpha < \infty$. If f is a solution of equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f + A_0(z) f = 0 \tag{2.3}$$

that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, then $\sigma_2(f, z_0) \leqslant \alpha$.

Lemma 2.5 ([7]). *Let $k \geq 2$ be an integer, $A_j(z)(j = 0, ..., k-1)$ and $F(\not\equiv 0)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that* $\max\left\{\sigma(A_j, z_0), \sigma(F, z_0)\right\} \leq \alpha < \infty$. *If f is an infinite order solution of equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + ... + A_1(z)f' + A_0(z)f = F \tag{2.4}$$

that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, then $\sigma_2(f, z_0) \leq \alpha$.

3 Proof of Theorems

Proof of theorem 1.5. Assume that f is a non-constant analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of equation (1.2), where z_0 is an essential singular point for f.

By Lemma 2.1, there exist a set $E_1 \subset (0, 1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all $r = |z_0 - z|$ satisfying $r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \lambda \left[\frac{1}{r}T_{z_0}(\alpha r, f)\right]^{2j} \ (j = 1, ..., k). \tag{3.1}$$

For each sufficiently small $|z_0-z| = r$, let $z_r = z_0-re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0-z|=r} |f(z)|$.

By Lemma 2.2, there exist a constant $\delta_r > 0$ and a set $E_2 \subset (0, 1)$ of finite logarithmic measure, such that for all z satisfying $|z_0 - z| = r \notin E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \leq 2r^j \ (j = 1, ..., k). \tag{3.2}$$

Set

$$H = \{\theta \ \in [0, 2\pi) \ : \cos(\phi + n\theta) = 0 \ \}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$, we have

$$\cos(\phi + n\theta) > 0 \quad \text{or} \quad \cos(\phi + n\theta) < 0.$$

Case 1. $\cos(\phi + n\theta) > 0$. Thus by Lemma 2.3, for any given $\varepsilon(0 < 2\varepsilon < \frac{d_s-d}{d_s+d})$, for all z satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$, we have

$$\left|A_s(z) \exp\left\{\frac{a_s}{(z_0 - z)^n}\right\}\right| \geq \exp\left\{(1 - \varepsilon)\frac{d_s \cos(\phi + n\theta)}{r^n}\right\} \tag{3.3}$$

and

$$\left|A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \leq \exp\left\{(1 + \varepsilon)\frac{d \cos(\phi + n\theta)}{r^n}\right\} \ \ (j \neq s). \tag{3.4}$$

By (1.2), it follows that

$$-A_s(z) \exp\left\{\frac{a_s}{(z_0 - z)^n}\right\} = \frac{f^{(k)}}{f^{(s)}} + \sum_{j=s+1}^{k-1} A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\frac{f^{(j)}}{f^{(s)}} + \sum_{j=0}^{s-1} A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\frac{f^{(j)}}{f}\frac{f}{f^{(s)}}. \tag{3.5}$$

Substituting (3.1), (3.2), (3.3), (3.4) into (3.5), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_2$, $r \to 0$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$, we obtain

$$\exp\left\{(1 - \varepsilon)\frac{d_s \cos(\phi + n\theta)}{r^n}\right\} \leq M_1 r^s \exp\left\{(1 + \varepsilon)\frac{d \cos(\phi + n\theta)}{r^n}\right\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \tag{3.6}$$

where $M_1(> 0)$ is a constant. Hence by (3.6), we obtain $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 2.4, we have $\sigma_2(f, z_0) \leq n$. Hence $\sigma_2(f, z_0) = n$.

Case 2. $\cos(\phi+n\theta) < 0$. We use the same reasoning as in the case 1 by replacing $A_s(z) \exp\left\{\frac{a_s}{(z_0-z)^n}\right\}$ by $A_l(z) \exp\left\{\frac{a_l}{(z_0-z)^n}\right\}$ to prove that $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) \geq n$. From this and Lemma 2.4, we have $\sigma_2(f, z_0) = n$. □

Proof of theorem 1.6. Assume that f is a non-constant analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of equation (1.2), where z_0 is an essential singular point for f.

By Lemma 2.1, there exist a set $E_1 \subset (0,1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all $r = |z_0 - z|$ satisfying $r \notin E_1$, we have (3.1).

For each sufficiently small $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0-z|=r} |f(z)|$.

By Lemma 2.2, there exist a constant $\delta_r > 0$ and a set $E_2 \subset (0,1)$ of finite logarithmic measure, such that for all z satisfying $|z_0 - z| = r \notin E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have (3.2).

Set

$$H_1 = \cup_{j=0}^{k-1}\{\theta \in [0, 2\pi) : \delta_{a_j}(\theta) = 0\}$$

and

$$H_2 = \cup_{1\leq s<d\leq m}\{\theta \in [0, 2\pi) : \delta_{a_s}(\theta) = \delta_{a_d}(\theta)\}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have $\delta_{a_j}(\theta) \neq 0$ $(j = 0, ..., k-1)$, $\delta_{a_s}(\theta) \neq \delta_{a_d}(\theta)$ $(1 \leq s < d \leq m)$.

Since a_{i_j} $(j = 1, ..., m)$ are distinct complex numbers, then there exists only one $t \in \{1, ..., m\}$, such that

$$\delta_t = \delta_{a_{i_t}}(\theta) = \max\{\delta_{a_{i_j}}(\theta) : j = 1, ..., m\}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have $\delta_{a_{i_t}}(\theta) > 0$ or $\delta_{a_{i_t}}(\theta) < 0$.

Case 1. $\delta_t > 0$. For $l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$, we have $a_l = c_l^{(i_t)} a_{i_t}$ or $a_l = c_l^{(i_s)} a_{i_s}$ $s \neq t$.

Hence for $l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$, we have $\delta_l(\theta) < \delta_t$.

Set $\delta = \max\{\delta_{a_j}(\theta) : j \neq i_t\}$, thus $\delta < \delta_t$.

Subcase 1.1. $\delta > 0$. Thus, by Lemma 2.3, for any given $\varepsilon(0 < 2\varepsilon < \frac{\delta_t-\delta}{\delta_t+\delta})$, for all z satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\left|A_{i_t}(z) \exp\left\{\frac{a_{i_t}}{(z_0 - z)^n}\right\}\right| \geq \exp\left\{(1-\varepsilon)\frac{\delta_t}{r^n}\right\} \tag{3.7}$$

and

$$\left|A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \leq \exp\left\{(1+\varepsilon)\frac{\delta}{r^n}\right\} \quad (j \neq i_t). \tag{3.8}$$

We can rewrite (1.2) as

$$-A_{i_t}(z) \exp\left\{\frac{a_{i_t}}{(z_0 - z)^n}\right\} = \frac{f^{(k)}}{f^{(i_t)}} + \sum_{j=i_t+1}^{k-1} A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\frac{f^{(j)}}{f^{(i_t)}} + \sum_{j=0}^{i_t-1} A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\frac{f^{(j)}}{f}\frac{f}{f^{(i_t)}}. \tag{3.9}$$

Substituting (3.1), (3.2), (3.7), (3.8) into (3.9), for all z satisfying $|z_0 - z| = r$, $r \notin E_1 \cup E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1 \cup H_2$, we obtain

$$\exp\left\{(1-\varepsilon)\frac{\delta_t}{r^n}\right\} \leq M_2 r^{i_t} \exp\left\{(1+\varepsilon)\frac{\delta}{r^n}\right\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \tag{3.10}$$

where $M_2(> 0)$ is a constant. Hence by (3.10), we obtain $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 2.4, we have $\sigma_2(f, z_0) \leq n$. Hence $\sigma_2(f, z_0) = n$.

Subcase 1.2. $\delta < 0$. By Lemma 2.3, for any given $\varepsilon(0 < 2\varepsilon < 1)$, for all z satisfying $|z_0-z| = r, r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have (3.7) and

$$\left|A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \leq \exp\left\{(1-\varepsilon)\frac{\delta_{a_j}(\theta)}{r^n}\right\} < 1 \quad (j \neq i_t). \tag{3.11}$$

Substituting (3.1), (3.2), (3.7), (3.11) into (3.9), for all z satisfying $|z_0 - z| = r$, $r \notin E_1 \cup E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash \notin H_1 \cup H_2$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_t}{r^n}\Big\} \leq M_3 r^{i_t}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.12}$$

where $M_3(> 0)$ is a constant. Hence by (3.12), we obtain $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 2.4, we have $\sigma_2(f, z_0) \leq n$. Hence $\sigma_2(f, z_0) = n$.

Case 2. $\delta_t < 0$. Set $c = \min\{c_l^{(i_j)} : l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$ and $j = 1, ...m\}$.

By Lemma 2.3, for any given $\varepsilon(0 < 2\varepsilon < 1)$, for all z satisfying $|z_0 - z| = r, r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1-\varepsilon)\frac{c\delta_t}{r^n}\Big\} \quad (j = 0, .., k-1). \tag{3.13}$$

By (1.2), we get

$$-1 = A_{k-1}(z)\exp\Big\{\frac{a_{k-1}}{(z_0 - z)^n}\Big\}\frac{f^{(k-1)}}{f}\frac{f}{f^{(k)}} + ... + A_0(z)\exp\Big\{\frac{a_0}{(z_0 - z)^n}\Big\}\frac{f}{f^{(k)}}. \tag{3.14}$$

Substituting (3.1), (3.2), (3.13) into (3.14), for all z satisfying $|z_0 - z| = r$, $r \notin E_1 \cup E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$1 \leq M_4 r^k \exp\Big\{(1+\varepsilon)\frac{c\delta_t}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.15}$$

where $M_4(> 0)$ is a constant. Hence by (3.15), we obtain $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 2.4, we have $\sigma_2(f, z_0) = n$. □

Proof of theorem 1.7. First we show that (1.5) can possess at most one exceptional analytic solution f_0 of finite order in $\overline{\mathbb{C}}\backslash\{z_0\}$.

In fact, if f^* is another analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite order of equation (1.5) where z_0 is an essential singular point for f^*, then $f_0 - f^*$ is a non-constant analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite order of the corresponding homogeneous equation of (1.5). This contradicts Theorem 1.5 and Theorem 1.6.

We assume that f is an infinite order analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of equation (1.5), where z_0 is an essential singular point for f. By Lemma 2.1, there exist a set $E_1 \subset (0, 1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all z satisfying $|z_0 - z| = r \notin E_1$, we have (3.1).

For each sufficiently small $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0 - z| = r}|f(z)|$. By Lemma 2.2, there exist a constant $\delta_r > 0$ and a set $E_2 \subset (0, 1)$ of finite logarithmic measure such that for all z satisfying $|z_0 - z| = r \notin E_2$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have (3.2). Since $|f(z)|$ is continous in $|z_0 - z| = r$, then there exists a constant $\lambda_r > 0$ such that for all z satisfying $|z_0 - z| = r$ sufficiently small and $\arg(z_0 - z) = \theta \in [\theta_r - \lambda_r, \theta_r + \lambda_r]$, we have

$$\frac{1}{2}|f(z_r)| < |f(z)| < \frac{3}{2}|f(z_r)|. \tag{3.16}$$

On the other hand, for any given $\varepsilon(0 < 2\varepsilon < n - \sigma)$, there exists $r_0 > 0$, such that for all $0 < r = |z_0 - z| < r_0$, we have

$$|F(z)| \leq \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\}. \tag{3.17}$$

Since $M_{z_0}(r, f) \geq 1$ as $r \to 0$, it follows from (3.16) and (3.17) that

$$\Big|\frac{F(z)}{f(z)}\Big| \leq 2\exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \quad \text{as} \quad r \to 0. \tag{3.18}$$

Set $\gamma = \min\{\delta_r, \lambda_r\}$.

(i) Suppose that $a_j(j = 0, .., k-1)$ satisfy hypotheses of Theorem 1.5.

Case 1. $\cos(\phi + n\theta) > 0$. From (3.1), (3.2), (3.3), (3.4), (3.18) and (1.5), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H$($H$ is defined above), we obtain

$$\exp\left\{(1-\varepsilon)\frac{d_s\cos(\phi+n\theta)}{r^n}\right\} \le B_1 r^s \exp\left\{\frac{1}{r^{\sigma+\varepsilon}}\right\}\exp\left\{(1+\varepsilon)\frac{d\cos(\phi+n\theta)}{r^n}\right\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \quad (3.19)$$

where $B_1(>0)$ is a constant. From (3.19), we get $\sigma_2(f, z_0) \ge n$. This and the fact that $\sigma_2(f, z_0) \le n$ by Lemma 2.5, we have $\sigma_2(f, z_0) = n$.

Case 2. $\cos(\phi+n\theta) < 0$. We use the same reasoning as in the case 1 by replacing $A_s(z)\exp\left\{\frac{a_s}{(z_0-z)^n}\right\}$ by $A_l(z)\exp\left\{\frac{a_l}{(z_0-z)^n}\right\}$ to prove that $\sigma_2(f, z_0) \ge n$. This and the fact that $\sigma_2(f, z_0) \le n$ yield $\sigma_2(f, z_0) = n$.

(ii) Suppose that $a_j(j = 0, .., k-1)$ satisfy hypotheses of Theorem 1.6.

Since a_{i_j} $(j = 1, ..., m)$ are distinct complex numbers, then there exists only one $t \in \{1, ..., m\}$ such that

$$\delta_t = \delta_{a_{i_t}}(\theta) = \max\{\delta_{a_{i_j}}(\theta) : j = 1, ..., m\}.$$

For any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, where H_1 and H_2 are defined above, we have $\delta_{a_{i_t}}(\theta) > 0$ or $\delta_{a_{i_t}}(\theta) < 0$.

Case 1. $\delta_t > 0$. For $l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$, we have $a_l = c_l^{(i_t)} a_{i_t}$ or $a_l = c_l^{(i_s)} a_{i_s}$ $s \ne t$. Hence for $l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$, we have $\delta_l < \delta_t$.

Set $\delta = \max\{\delta_{a_j}(\theta) : j \ne i_t\}$ thus $\delta < \delta_t$.

Subcase 1.1. $\delta > 0$. From (3.1), (3.2), (3.7), (3.8), (3.18) and (1.5) for all z satisfying $|z_0 - z| = r$, $r \notin E_1 \cup E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H_1 \cup H_2$, we obtain

$$\exp\left\{(1-\varepsilon)\frac{\delta_t}{r^n}\right\} \le B_2 r^{i_t}\exp\left\{\frac{1}{r^{\sigma+\varepsilon}}\right\}\exp\left\{(1+\varepsilon)\frac{\delta}{r^n}\right\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \quad (3.20)$$

where $B_2(>0)$ is a constant. Hence by (3.20), we obtain that $\sigma_2(f, z_0) \ge n$. This and the fact that $\sigma_2(f, z_0) \le n$ by Lemma 2.5, we have $\sigma_2(f, z_0) = n$.

Subcase 1.2. $\delta < 0$. From (3.1), (3.2), (3.7), (3.13), (3.18) and (1.5) for all z satisfying $|z_0 - z| = r$, $r \notin E_1 \cup E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H_1 \cup H_2$, we obtain

$$\exp\left\{(1-\varepsilon)\frac{\delta_t}{r^n}\right\} \le B_3 r^{i_t}\exp\left\{\frac{1}{r^{\sigma+\varepsilon}}\right\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \quad (3.21)$$

where $B_3(>0)$ is a constant. Hence by (3.21), we obtain that $\sigma_2(f, z_0) \ge n$. This and the fact that $\sigma_2(f, z_0) \le n$ by Lemma 2.5, we have $\sigma_2(f, z_0) = n$.

Case 2. $\delta_t < 0$. Set $c = \min\left\{c_l^{(i_j)} : l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\} \text{ and } j = 1, .., m\right\}$.

From (3.1), (3.2), (3.13), (3.18) and (1.5) for all z satisfying $|z_0 - z| = r$, $r \notin E_1 \cup E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, we obtain

$$1 \le B_4 r^k \exp\left\{\frac{1}{r^{\sigma+\varepsilon}}\right\}\exp\left\{(1+\varepsilon)\frac{c\delta_t}{r^n}\right\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \quad (3.22)$$

where $B_4(>0)$ is a constant. Hence by (3.22), we obtain that $\sigma_2(f, z_0) \ge n$. This and the fact that $\sigma_2(f, z_0) \le n$ by Lemma 2.5, we have $\sigma_2(f, z_0) = n$. □

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