Graduate J. Math. 10, Issue 1 (2025), 39 – 55

Approximate strong subdifferential calculus for convex set-valued mappings and applications to set optimization

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Abstract

In this paper, we are mainly concerned with a rule for approximate strong subdifferential, concerning the sum and the composition of cone-convex set-valued vector mappings, taking values in finite or infinite-dimensional preordred spaces. The obtained formulas is exact and holds under the connectedness conditions. This formula is applied to establish approximate necessary and sufficient optimality conditions for the existence of the approximate strong efficient solutions of a set-valued vector optimization problem.

Keywords. Set-valued vector optimization, Strong subdifferential of convex set-valued mappings, Optimality conditions.

1 Introduction

The approximate subdifferential is one of the most important concepts in Convex Analysis and Optimization. It is well known that strong vector subdifferential calculus plays an important role in vector optimization problems. knows these last decades a growing progress from a point of view of theoretical background (see [2, 12, 13]). In [2], several properties and calculus rules have been established for the approximate strong vector subdifferential of the sum of two vector valued mappings in the framework of ordered complete topological vector space, by using the so-called sandwich theorem, he has obtained the following formula expressing the calculus rules of the approximate strong subdifferential for the sum of two convex single vector mappings $g_1, g_2 : X \to Y \cup \{+\infty_Y\}$

$$\partial^s_\varepsilon(g_1+g_2)(\bar{x}) = \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2\in Y_+\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial^s_{\varepsilon_1} g_1(\bar{x}) + \partial^s_{\varepsilon_2} g_2(\bar{x}), \tag{1.1}$$

where $\partial^s g_i(\bar{x})$ is the approximate strong subdifferential at $\bar{x}$ and X is a locally convex space, Y is a partially ordered vector space by a cone Y_+ and $+\infty_Y$ is an abstract maximal element of the space Y.

Recently, Laghdir et al. established in [1], the approximate strong vector subdifferential calculus of the composed convex operator $f + g \circ h$ when f, g and h are vector valued convex mappings and g is nondecreasing.

$$\partial^s_\varepsilon(f+g\circ h)(\bar{x}) = \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2,\varepsilon_3\in K\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon}} \{\partial^s_{\varepsilon_1} f(\bar{x}) + \partial^s_{\varepsilon_2}(A\circ h)(\bar{x}),\, A\in \partial^s_{\varepsilon_3} g(h(\bar{x}))\}, \quad \forall\varepsilon\in Y_+. \tag{1.2}$$

In this article, our main objective is to attempt to prove that the equalities (1.1) and (1.2) hold in the setting of set-valued cone-convex mappings, which until now knew no attempt. Our main result enables us to establish the existence of approximate Lagrange multipliers for a general cone-convex set-valued optimization problem.

The paper is organized as follows. Section 2 contains preliminary material. Section 3 is devoted to stating the strong subdifferential calculus rules of the sum and the composition of cone-convex set-valued mappings. In Section 4, we derive from the obtained formulas the approximate optimality conditions for a set-valued cone-convex constrained optimization problem.

2 Preliminaries

Throughout this paper, X and Y are two Hausdorff locally convex topological vector spaces and Y is ordered through an ordering cone K, i.e.,

$$y_1, y_2 \in Y,\, y_1 \leq_K y_2 \iff y_2 - y_1 \in K,$$

where K is assumed to be nontrivial ($K \neq \emptyset$), convex, closed, pointed $K \cap -K = \{0_Y\}$ and with nonempty topological interior. We define the element $+\infty_Y$ as the supremum in Y with respect to the ordering K. In other words, it holds that $y \leq_K +\infty_Y$, for any $y \in \overline{Y}$, and if there exists $y \in \overline{Y}$ such that $+\infty_Y \leq_K y$, then $y = +\infty_Y$. The algebraic operations in Y are extended as follows

$$+\infty_Y + y = y + (+\infty_Y), \quad \alpha(+\infty_Y) = +\infty_Y, \quad \forall y \in Y, \forall \alpha > 0.$$

For every two nonempty subsets A, $B \subset Y$ and $\alpha \in \mathbb{R}$, we denote

$$A + B := \{a + b \,:\, (a, b) \in A \times B\}, \quad \alpha A := \{\alpha a \,:\, a \in A\}.$$

When $A = \emptyset$, we set $B + \emptyset = \emptyset + B = \emptyset$ and $\alpha\emptyset = \emptyset$. For the sake of simplicity, we write $y + A$ instead of $\{y\} + A$ for all $y \in Y$.

The order interval between two elements u and v of Y such that $u \leq_K v$ is the subset

$$[u, v] := \{w \in Y \,:\, u \leq_K w \leq_K v\}.$$

A subset B of Y is order bounded if there exist u, v in Y such that $B \subseteq [u, v]$. A subset B of Y is majorized (resp. minorized) if there is an element $b \in B$ such that $v \leq_K b$ (resp. $b \leq_K v$) for all $v \in B$. For the subset $B \subset Y$, if there exists $v \in Y$ such that

(i) $b \leq_K v$ for all $b \in B$,

(ii) $v \leq_K w$ whenever $b \leq_K w$ for all $b \in B$,

then v is called the supremum of B and we write $v = \sup B$. We write the infimum of B, as $\inf B$. We say that (Y, K) is order complete, if every minorized subset of Y has an infimum. This is, in fact, equivalent to saying that every majorized subset of Y has a supremum. In addition, (Y, K) is order complete lattice, if (Y, K) is order complete and for any pair of elements u, v in Y , $\sup(u, v)$ and $\inf(u, v)$ exists in Y. We will often assume that K is normal, i.e., there exists a basis of open neighbourhoods $\mathfrak{B}$ of the origin such that

$$W = (W - K) \cap (W + K), \quad \forall W \in \mathfrak{B}.$$

Given an arbitrary set-valued mapping $G : X \rightrightarrows Y$, we denote the effective domain, image, graph, and epigraph of G by $\text{dom}G$, $\text{Im}G$, $\text{gr}G$ and $\text{epi}_K G$ respectively, i.e.,

$$\begin{array}{lcl}
\text{dom}\, G & := & \{x \in X \,:\, G(x) \neq \emptyset\}, \\
\text{gr}\, G & := & \{(x, y) \in X \times Y : y \in G(x)\}, \\
\text{Im}\, G & := & \bigcup\limits_{x \in X} G(x),
\end{array}$$

and
$$\text{epi}_K G := \text{gr}(G+K) = \{(x,y) \in X \times Y : \; y \in G(x)+K\},$$
where
$$(G+K)(x) = G(x)+K, \quad \forall x \in X.$$

Definition 2.1. [8] The set-valued mapping G is said to be

1) K-convex, if its epigraph is a convex subset of $X \times Y$.
2) Proper, if its effective domain dom$G \neq \emptyset$.

Let us recall the concept of connectedness and the concept of continuity of a set-valued mapping.

Definition 2.2. [6, 14] Let $G : \; X \rightrightarrows Y$ be a set-valued mapping.

1) G is said to be connected at $x_0 \in X$, if there exists a mapping $g : \; X \to Y$ satisfaying $g(v) \in G(v)$ for all v in some neighborhood of x_0 and g is continuous at x_0.
2) G is said to be upper-semicontinuous at x_0, if for any open subset $V \supseteq G(x_0)$, there exists a neighborhood U of x_0, such that $G(U) \subseteq V$.
3) G is said to be lower-semicontinuous at x_0, if for any open subset V satisfying $V \cap G(x_0) \neq \emptyset$, there exists a neighborhood U of x_0, such that $G(U) \cap V \neq \emptyset$.

We say that G is continuous at x_0, if it is upper-semicontinuous and lower-semicontinuous at x_0 and we say that G is continuous on X, if it is continuous at each point $x \in X$.

The following definition introduces the well-known concepts of approximate strong minimal of a nonempty subset $B \subset Y$.

Definition 2.3. [4] Let $B \subset Y$ be a nonempty subset and $\varepsilon \in K$. $\bar{y} \in B$ is said to be a strongly ε-minimal point of the subset B, if $B \subseteq \bar{y} - \varepsilon + K$.

Consider now the following set-valued optimization problem
$$(\text{P}_S) \quad \begin{cases} \text{Min G(x)}, \\ x \in S, \end{cases}$$
where $G : X \rightrightarrows Y$ is a proper set-valued mapping and $S \subset X$ is the feasible set.

Definition 2.4. Let $\varepsilon \in K$, a pair $(\bar{x}, \bar{y}) \in X \times Y$ is said to be strong ε-minimum solution of (P_S), if $\bar{x} \in S$ and $\bar{y}$ is a strongly ε-minimal point of the subset $G(S)$, i.e.
$$\bar{y} \leq_K y + \varepsilon, \quad \forall x \in S, \, \forall y \in G(x).$$

The set of strong ε-minimum solutions of (P_S) will be denoted by $K_{\varepsilon,s}(G,S,K)$.

Definition 2.5. [3, 6] Let $G : X \rightrightarrows Y$ be a set-valued mapping, $(\bar{x}, \bar{y}) \in$ gr$G$ and $\varepsilon \in K$. The approximate strong subdifferential of G at $(\bar{x}, \bar{y})$ is defined as the following
$$\partial^s_\varepsilon G(\bar{x},\bar{y}) := \{T \in L(X,Y) : \quad T(x-\bar{x}) \leq_K y - \bar{y} + \varepsilon, \quad \forall (x,y) \in \text{gr}G\},$$
where $L(X,Y)$ is the set of all continuous linear operators from X into Y.

This definition is justified by the importance of the following immediate property
$$(\bar{x},\bar{y}) \in K_{\varepsilon,s}(G,S,K) \quad \Longleftrightarrow \quad 0 \in \partial^s_\varepsilon G(\bar{x},\bar{y}). \tag{2.1}$$
Clearly $\partial^s_{\varepsilon_1} G(\bar{x},\bar{y}) \subseteq \partial^s_{\varepsilon_2} G(\bar{x},\bar{y})$ whenever $\varepsilon_1 \leq_K \varepsilon_2$, and
$$\partial^s G(\bar{x},\bar{y}) = \partial^s_{0_Y} G(\bar{x},\bar{y}) = \bigcap_{\varepsilon \in \text{int}K} \partial^s_\varepsilon G(\bar{x},\bar{y}).$$
By convention we take $\partial^s_\varepsilon G(\bar{x},\bar{y}) = \emptyset$ if $(\bar{x},\bar{y}) \notin$ gr$G$ and we say that G is strongly ε-subdifferentiable at $(\bar{x},\bar{y})$, if $\partial^s_\varepsilon G(\bar{x},\bar{y}) \neq \emptyset$.

The following example shows the importance of the concept of the approximate Pareto subdifferential.

Example 2.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$ and $G : X \rightrightarrows Y$ defined by

$$G(x) = \begin{cases} [-\sqrt{x}, \sqrt{x}\,], & \text{if } x \geq 0\ , \\ \emptyset, & \text{otherwise}\ . \end{cases}$$

It is easy to that $\partial^s G(0,0) = \emptyset$ and for any $\varepsilon > 0$, we have

$$\partial^s_\varepsilon G(0,0) = \{\alpha \in \mathbb{R} :\ \ \alpha \leq \frac{-1}{4\varepsilon}\}.$$

3 Strong subdifferential calculus rules

In this section, we are concerned with the subdifferential calculus of the sum and composition of convex set-valued mappings.

3.1 Addition

In [2], Théra established in the framework of ordered complete topological vector space, the following theorem, which will play an important role in proving our main results.

Theorem 3.1. *[2] Let g_1, g_2 : $X \to Y \cup \{+\infty_Y\}$ be two K-convex single vector valued mappings. Suppose that the following conditions is satisfied.*

$$(H_1) \quad \begin{cases} (Y,K) \textit{ is a normal order complete Hausdorff locally convex topological vector space,} \\ g_1 \textit{ is continuous at } \bar{x} \in \mathrm{dom} g_1 \cap \mathrm{dom} g_2. \end{cases}$$

Then,

$$\partial^s_\varepsilon (g_1 + g_2)(\bar{x}) = \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial^s_{\varepsilon_1} g_1(\bar{x}) + \partial^s_{\varepsilon_2} g_2(\bar{x}), \quad \forall \varepsilon \in K. \tag{3.1}$$

Remark 3.1. The continuity assumption in (H_1) is not merely a technical condition; it is essential to guarantee the exactness of the formula (3.1) Intuitively, continuity ensures that the epigraphs of the functions f and g behave well locally, allowing the subgradients to combine properly without loss of information.

More precisely, continuity at a point $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ acts as a *constraint qualification* that prevents pathological phenomena such as "vertical walls" or jumps in the epigraph, which would otherwise break the equality.

In the context of set-valued (multivalued) mappings, continuity is often used as a key assumption to ensure the validity of certain calculus rules, such as the subdifferential or coderivative sum rules. However, several authors have noted that continuity may be too strong or inappropriate in some nonsmooth or irregular settings. To extend classical results—such as the Moreau–Rockafellar theorem—to broader frameworks, some researchers have proposed replacing continuity with a the *connectedness* of the mapping at a given point, as introduced by [7, 11].

Let us consider the vector indicator mapping $\delta^v_C : X \to Y \cup \{+\infty_Y\}$ of a nonempty subset $C \subseteq X$, defined by

$$\delta^v_C(x) := \begin{cases} 0_Y, & \text{if } x \in C, \\ +\infty_Y, & \text{else.} \end{cases}$$

Let us note that $\mathrm{epi}\,\delta^v_C = C \times K$ and therefore the K-convexity of δ^v_C follows from the convexity of C and K.

Theorem 3.2. *Let G_1, G_2 : $X \rightrightarrows Y$ be two K-convex set-valued mappings, $\bar{x} \in \mathrm{dom} G_1 \cap \mathrm{dom} G_2$, $\bar{u} \in G_1(\bar{x})$ and $\bar{v} \in G_2(\bar{x})$. Suppose that the following condition is satisfied*

(H_2) (Y,K) *is order complete Hausdorff locally convex topological vector space and* $\text{int}(\text{epi}G_1)\cap \text{epi}G_2 \neq \emptyset$.

Then,

$$\partial^s_\varepsilon(G_1+G_2)(\bar{x},\bar{u}+\bar{v}) = \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2\in K\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial^s_{\varepsilon_1}G_1(\bar{x},\bar{u})+\partial^s_{\varepsilon_2}G_2(\bar{x},\bar{v}), \qquad \forall \varepsilon\in K.$$

Proof. Let $A\in\partial^s_{\varepsilon_1}G_1(\bar{x},\bar{u})$ and $B\in\partial^s_{\varepsilon_2}G_2(\bar{x},\bar{v})$ with $\varepsilon_1+\varepsilon_2=\varepsilon$. For any $(x,y)\in\text{gr}(G_1+G_2)$, there exist $u\in G_1(x)$ and $v\in G_2(x)$ such that $y=u+v$. Hence, we get

$$A(x-\bar{x})\leq_K u-\bar{u}+\varepsilon_1, \tag{3.2}$$

and

$$B(x-\bar{x})\leq_K v-\bar{v}+\varepsilon_2. \tag{3.3}$$

By adding (3.2) and (3.3), it follows that

$$(A+B)(x-\bar{x})\leq_K y-(\bar{u}+\bar{v})+\varepsilon, \qquad \forall (x,y)\in\text{gr}(G_1+G_2),$$

which means that $A+B\in\partial^s_\varepsilon(G_1+G_2)(\bar{x},\bar{u}+\bar{v})$. For the reverse inclusion, let $T\in\partial^s_\varepsilon(G_1+G_2)(\bar{x},\bar{u}+\bar{v})$, i.e.

$$y-(\bar{u}+\bar{v})-T(x-\bar{x})+\varepsilon\geq_K 0_Y, \quad \forall (x,y)\in\text{gr}(G_1+G_2),$$

which yields that for any $x\in\text{dom}G_1\cap\text{dom}G_2$, $u\in G_1(x)$, $v\in G_2(x)$ and $\alpha,\beta\in K$

$$u+\alpha+v+\beta-(\bar{u}+\bar{v})-T(x-\bar{x})+\varepsilon\geq_K 0_Y,$$

and thus, it follows that for any $(x,u,v)\in X\times Y\times Y$

$$\delta^v_{\text{epi}G_1}(x,u)+\delta^v_{\text{epi}G_2}(x,v)u+v-(\bar{u}+\bar{v})-T(x-\bar{x})+\varepsilon\geq_K 0_Y, \tag{3.4}$$

For any $(x,u,v)\in X\times Y\times Y$, we define the following single-vector mappings

$$\begin{aligned} g_1(x,u,v) &:= \delta^v_{\text{epi}G_1}(x,u)+u-\bar{u}-T(x-\bar{x}),\\ g_2(x,u,v) &:= \delta^v_{\text{epi}G_2}(x,v)+v-\bar{v}. \end{aligned}$$

Let us observe that $\text{dom}g_1=\text{epi}\,G_1\times Y$ and $\text{dom}g_2=\varphi^{-1}(\text{epi}\,G_2)$, where φ is a continuous function defined from $X\times Y\times Y$ into $X\times Y$ by $\varphi(x,u,v):=(x,v)$ for all $(x,u,v)\in X\times Y\times Y$. Obviously, g_1 and g_2 are proper, convex and g_1 is continuous at $\bar{x}\in\text{dom}g_1\cap\text{dom}g_2$. Moreover, it follows from (3.4) that $(0,0,0)\in\partial^s_\varepsilon(g_1+g_2)(\bar{x},\bar{u},\bar{v})$ and hence by Theorem 3.1, we assert that there exist $\varepsilon_1,\varepsilon_2\in K$ satisfying $\varepsilon_1+\varepsilon_2=\varepsilon$ and

$$(0,0,0)\in\partial^s_{\varepsilon_1}g_1(\bar{x},\bar{u},\bar{v})+\partial^s_{\varepsilon_2}g_2(\bar{x},\bar{u},\bar{v}).$$

Thus there exist $(T_1,A_1,B_1)\in\partial^s_{\varepsilon_1}g_1(\bar{x},\bar{u},\bar{v})$ such that $(-T_1,-A_1,-B_1)\in\partial^s_{\varepsilon_1}g_2(\bar{x},\bar{u},\bar{v})$ satisfying

$$\begin{cases} T_1(x-\bar{x})+A_1(u-\bar{u})+B_1(v-\bar{v})\leq_K u-\bar{u}-T(x-\bar{x})+\varepsilon_1, & \forall (x,u,v)\in\text{dom}g_1, \quad (3.5)\\ -T_1(x-\bar{x})-A_1(u-\bar{u})-B_1(v-\bar{v})\leq_K v-\bar{v}+\varepsilon_2, & \forall (x,u,v)\in\text{dom}g_2. \quad (3.6)\end{cases}$$

By taking $x=\bar{x}$ and $u=\bar{u}$ in (3.5), we get $B_1(v-\bar{v})\leq_K\varepsilon_1$, for any $v\in Y$, since K is closed and pointed ($K\cap -K=\{0_Y\}$), it follows that $B_1=0$. Similarly, by taking $x=\bar{x}$ and $v=\bar{v}$ in (3.6), we get $-A_1(u-\bar{u})\leq_K\varepsilon_2$, for all $u\in Y$ and thus $A_1=0$. Consequently,

$$\begin{cases} (T_1+T)(x-\bar{x})\leq_K u-\bar{u}+\varepsilon_1, & \forall (x,u)\in\text{epi}\,G_1,\\ -T_1(x-\bar{x})\leq_K v-\bar{v}+\varepsilon_2, & \forall (x,v)\in\text{epi}\,G_2,\end{cases}$$

which yields that $T+T_1\in\partial^s_{\varepsilon_1}G_1(\bar{x},\bar{u})$ and $-T_1\in\partial^s_{\varepsilon_2}G_2(\bar{x},\bar{v})$, hence we obtain

$$\partial_\varepsilon(G_1+G_2)(\bar{x},\bar{u}+\bar{v}) = \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2\in K\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial_{\varepsilon_1}G_1(\bar{x},\bar{u})+\partial_{\varepsilon_2}G_2(\bar{x},\bar{v}).$$

So, we obtain the desired result. The proof is complete. □

The following theorem gives us the sum rule for two set-valued mappings under the connectedness assumption.

Theorem 3.3. *Let G_1, $G_2 : X \rightrightarrows Y$ be two set-valued mappings. Assume that the following condition holds.*

$$(MR)\quad \begin{cases} (Y,K) \text{ is a normal order complete lattice Hausdorff locally convex topological} \\ \text{vector space, } G_1 \text{ and } G_2 \text{ are } K\text{-convex,} \\ G_1 \text{ is connected at some point } x_0 \in \mathrm{dom} G_1 \cap \mathrm{dom} G_2. \end{cases}$$

Then, for any $(\bar{x},\bar{u}) \in \mathrm{gr}G_1$, $(\bar{x},\bar{v}) \in \mathrm{gr}G_2$

$$\partial^s_\varepsilon(G_1+G_2)(\bar{x},\bar{u}+\bar{v}) = \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2\in K\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial^s_{\varepsilon_1}G_1(\bar{x},\bar{u}) + \partial^s_{\varepsilon_2}G_2(\bar{x},\bar{v}), \qquad \forall \varepsilon \in K.$$

Proof. At first, we prove that $\mathrm{int}(\mathrm{epi}G_1) \neq \emptyset$. Since G_1 is connected at x_0, there exists some neighborhood U_0 of x_0 and a mapping $h : X \to Y$ continuous at x_0 such that $h(x) \in G_1(x)$ for all $x \in U_0$. Let $y_0 \in h(x_0) + \mathrm{int}\,K$, i.e. $h(x_0) \in y_0 - \mathrm{int}\,K \subset y_0 - K$ which yields that $y_0 - K$ is a neighborhood of $h(x_0)$ and hence it follows from the continuity of the mapping h at x_0 that $h^{-1}(y_0 - K)$ is a neighborhood of x_0. By putting $U = U_0 \cap h^{-1}(y_0 - K)$ which is a neighborhood of x_0, we get that $y_0 \in G_1(x) + K$ for any $x \in U$. On other hand, as $y_0 - h(x_0) \in \mathrm{int}\,K$, there exists a neighbourhood V of 0_Y such that $y_0 - h(x_0) + V \subset K$. By using the fact that $K + K = K$, we obtain for any $x \in U$ and $y \in y_0 + y_0 - h(x_0) + V$ that

$$y \in G_1(x) + y_0 - h(x_0) + V + K \subset G_1(x) + K.$$

which yields that $(x_0, y_0 + y_0 - h(x_0)) \in \mathrm{int}\,(\mathrm{epi}\,G_1)$. Secondly, we prove that $\mathrm{int}(\mathrm{epi}\,G_1) \cap \mathrm{epi}G_2 \neq \emptyset$. We proceed by contradiction: Suppose that $\mathrm{int}(\mathrm{epi}G_1) \cap \mathrm{epi}G_2 = \emptyset$, then by separation theorem [5, Theorem 1.1.3], there exist a nonzero $(x^*, y^*, \beta) \in X^* \times Y^* \times \mathbb{R}$ such that

$$\langle x^*, x\rangle + \langle y^*, y\rangle \le \beta \le \langle x^*, x'\rangle + \langle y^*, y'\rangle, \qquad \forall (x,y) \in \mathrm{epi}G_1,\ \forall (x',y') \in \mathrm{epi}G_2 \tag{3.7}$$

As $x_0 \in \mathrm{dom}\,G_1 \cap \mathrm{dom}\,G_2$, we claim that there exists $z_0 \in Y$ such that $(x_0, z_0) \in \mathrm{epi}\,G_1 \cap \mathrm{epi}\,G_2$. Indeed, let $y_1 \in G_1(x_0)$ and $y_2 \in G_2(x_0)$ and by taking $z_0 := \sup(y_1, y_2)$, we obtain $z_0 \in G_1(x_0) + K$ and $z_0 \in G_2(x_0) + K$ i.e. $(x_0, z_0) \in \mathrm{epi}\,G_1 \cap \mathrm{epi}\,G_2$. Since $(x_0, z_0 + u) \in \mathrm{epi}\,G_1 \cap \mathrm{epi}\,G_2$ for any $u \in K$, hence by taking in relation (3.7) $x = x' = x_0,\ y = z_0$ and $y' = z_0 + u$ (resp. $x = x' = x_0,\ y = z_0 + u$ and $y' = z_0$), we obtain $\langle y^*, u\rangle \ge 0$, for all $u \in K$ (resp. $\langle y^*, u\rangle \le 0$, for all $u \in K$), which yields that $y^* = 0$ since $\mathrm{int}K \neq \emptyset$. It follows from (3.7) that $\langle x^*, u\rangle \le 0$ for all $u \in (\mathrm{dom}\,G_1 - \mathrm{dom}\,G_2)$ and as G_1 is connected at $x_0 \in \mathrm{dom}\,G_1 \cap \mathrm{dom}\,G_2$, one can see easily that $0_X \in \mathrm{int}(\mathrm{dom}\,G_1 - \mathrm{dom}\,G_2)$ which yields that $x^* = 0_{X^*}$ and this leads to a contradiction. □

Now, we illustrate our man results with the help of the following example.

Example 3.1. By taking $X = \mathbb{R}$, $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, let us consider the following set-valued mappings F, $G : X \rightrightarrows Y$ defined respectively, by

$$G_1(x) = \begin{cases} [-\sqrt{x}, \sqrt{x}\,], & \text{if } x \ge 0\ , \\ \emptyset, & \text{otherwise}\ . \end{cases}$$

$$G_2(x) = \{y \in \mathbb{R} :\ |x| \le y \le |x| + 1\}.$$

It is easy to check that G_1 and G_2 are $\mathbb{R}_+$-convex and connected at 0, hence the condition (MR) is satisfied. so f

$$\partial^s_\varepsilon(G_1+G_2)(0,0) = \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2\in K\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial^s_{\varepsilon_1}G_1(0,0) + \partial^s_{\varepsilon_2}G_2(0,0), \qquad \forall \varepsilon \in K.$$

Indeed, let ε, ε_1, $\varepsilon_2 \in \mathbb{R}_+$, we get easily that

$$\begin{aligned}
\partial^s_{\varepsilon_1} G_1(0,0) &= \{\alpha \in \mathbb{R} : \alpha x \leq y + \varepsilon_1, \ \forall (x,y) \in \mathrm{gr} G_1\}, \\
&= \{\alpha \in]-\infty; 0[: \ \varepsilon_1 \geq \frac{-1}{4\alpha}\}
\end{aligned}$$

$$\begin{aligned}
\partial^s_{\varepsilon_2} G_2(0,0) &= \{\alpha \in \mathbb{R} : \alpha x \leq y + \varepsilon_2, \ \forall (x,y) \in \mathrm{gr} G_2\}, \\
&= [-1; 1].
\end{aligned}$$

$$\begin{aligned}
\partial^s_{\varepsilon} (G_1 + G_2)(0,0) &= \{\alpha \in \mathbb{R} : \alpha x \leq y + \varepsilon, \ \forall (x,y) \in \mathrm{gr}(G_1 + G_2)\}, \\
&= \{\alpha \in]-\infty; 1[: \ \varepsilon \geq \frac{1}{4(1-\alpha)}\}.
\end{aligned}$$

On other hand, by taking $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 0$, we see easily that $\partial^s_{\varepsilon}(G_1 + G_2)(0,0) \subseteq \partial^s_{\varepsilon_1} G_1(0,0) + \partial^s_{\varepsilon_2} G_2(0,0)$. Thus

$$\partial^s_{\varepsilon}(G_1 + G_2)(0,0) \subseteq \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial^s_{\varepsilon_1} G_1(0,0) + \partial^s_{\varepsilon_2} G_2(0,0), \qquad \forall \varepsilon \in K.$$

For the reverse inclusion, let $\alpha_1 \in \partial^s_{\varepsilon_1} G_1(0,0)$ and $\alpha_2 \in \partial^s_{\varepsilon_2} G_2(0,0)$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$.

$$\begin{aligned}
\alpha_1 \in \partial^s_{\varepsilon_1} G_1(0,0) \quad \text{and} \quad \alpha_2 \in \partial^s_{\varepsilon_2} G_2(0,0) \ &\Rightarrow \ \alpha_1 < 0, \quad \varepsilon_1 \geq \frac{-1}{4\alpha_1}, \quad \alpha_2 \in [-1; 1], \\
&\Rightarrow \ \alpha_1 + \alpha_2 < 1, \quad \varepsilon_1 \geq \frac{-1}{4\alpha_1}, \\
&\Rightarrow \ \alpha_1 + \alpha_2 < 1, \quad \alpha_1 + \alpha_2 - 1 \leq \frac{-1}{4\varepsilon_1}, \\
&\Rightarrow \ \alpha_1 + \alpha_2 < 1, \quad \varepsilon \geq \frac{1}{4(1-(\alpha_1+\alpha_2))}, \qquad \text{since } \varepsilon \geq \varepsilon_1.
\end{aligned}$$

Therfore, we obtain the equality

$$\partial^s_{\varepsilon}(G_1 + G_2)(0,0) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial^s_{\varepsilon_1} G_1(0,0) + \partial^s_{\varepsilon_2} G_2(0,0), \qquad \forall \varepsilon \in K.$$

3.2 Composition

In this subsection, we provide the approximate strong subdifferential calculus of the composition of two set-valued mappings. In what follows, Z is a real locally convex topological vector space equipped with a nonempty pointed convex cone Q. We shall work also with the following definitions: for $(x,z) \in X \times Z$, $(A,B) \in L(X,Y) \times L(Z,Y)$, we set $(A,B)(x,z) := A(x) + B(z)$. Let $F : X \rightrightarrows Y$, $H : X \rightrightarrows Z$ and $G : Z \rightrightarrows Y$ be three set-valued mappings.

The composed set-valued mapping $G \circ H : X \rightrightarrows Y$ is defined by

$$(G \circ H)(x) = G(H(x)) := \begin{cases} \bigsqcup\limits_{z \in H(x)} G(z), & \text{if } x \in \mathrm{dom} H, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We have $\mathrm{dom}(G \circ H) = H^{-1}(\mathrm{dom} G) \cap \mathrm{dom} H$ where $H^{-1}(\mathrm{dom} G) := \{x \in X : H(x) \cap \mathrm{dom} G \neq \emptyset\}$.

For a nonempty subset $S \subseteq X$, the set-valued indicator mapping $R^v_S : X \rightrightarrows Y$ is defined by

$$R^v_S(x) := \begin{cases} \{0_Y\}, & \text{if } x \in S, \\ \emptyset, & \text{else.} \end{cases}$$

Definition 3.1. [8] Let $G : Z \rightrightarrows Y$ be a set-valued mapping and $S \subseteq Z$. G is said to be (Z_+, K)-nondecreasing over S, if for any $(z_1, z_2) \in S \times S$ satisfying $z_1 \leq_{Z_+} z_2$ we have $G(z_2) \subseteq G(z_1) + K$.

In what follows, we will need the following definition

$$L_+(Z, Y) := \{B \in L(Z, Y) : \ B(Q) \subseteq K\}.$$

Lemma 3.1. *Let $G : Z \rightrightarrows Y$ be (Q, K)-nondecreasing set-valued mapping, $(\bar{z}, \bar{y}) \in \mathrm{gr}G$ and $\varepsilon \in K$. Then*

$$\partial^s_\varepsilon G(\bar{z}, \bar{y}) \subseteq L_+(Z, Y).$$

Proof. Let $A \in \partial^s_\varepsilon G(\bar{z}, \bar{y})$, then

$$G(z) \subseteq \bar{y} + A(z - \bar{z}) - \varepsilon + K, \ \ \forall z \in Z. \tag{3.8}$$

Let $w \in Z_+$ and by taking $z = \bar{z} - w$ in (3.8), we obtain

$$G(\bar{z} - w) \subseteq \bar{y} - A(w) - \varepsilon + K. \tag{3.9}$$

As G is (Z_+, K)-nondecreasing, we have $G(\bar{z}) \subseteq G(\bar{z} - w) + K$. Since $\bar{y} \in G(\bar{z})$ and by using the fact $K + K = K$, it follows from (3.9) that

$$A(w) + \varepsilon \in K, \ \ \forall w \in Z_+.$$

Since Q is a cone, it follows that

$$A(w) + \frac{\varepsilon}{n} \in K, \ \ \forall w \in Z_+, \forall n \in \mathbb{N}^*,$$

which yields by letting $n \nearrow +\infty$ that $A(Z_+) \subseteq K$. □

Remark 3.2. It follows immediately from above lemma that if H is Q-convex, G is (Z_+, K)-nondecreasing and K-convex, then for any $A \in \partial^s_\varepsilon G(\bar{z}, \bar{y})$, the mapping $A \circ H$ is K-convex.

Let us consider the following auxiliary set-valued mappings

$$\begin{array}{rcl} \tilde{F} : X \times Z & \rightrightarrows & Y \\ (x, z) & \mapsto & F(x) + R^v_{\mathrm{epi}H}(x, z) \\ \tilde{G} : X \times Z & \rightrightarrows & Y \\ (x, z) & \mapsto & G(z) \end{array}$$

Note that $\mathrm{dom}\tilde{F} = (\mathrm{dom}F \times Z) \cap \mathrm{epi}H$, $\mathrm{dom}\tilde{G} = X \times \mathrm{dom}G$ and $\mathrm{gr}\tilde{G} = X \times \mathrm{gr}G$. In addition $\mathrm{epi}\tilde{G} = X \times \mathrm{epi}G$ and $\mathrm{epi}\tilde{F} = (\mathrm{epi}H \times Y) \cap \varphi^{-1}(\mathrm{epi}F)$, where φ is a continuous function defined from $X \times Z \times Y$ into $X \times Y$ by $\varphi(x, z, y) := (x, y)$ for all $(x, z, y) \in X \times Z \times Y$. It is easy to see that if F and G are K-convex and H is Z_+-convex, then $\tilde{F}$ and $\tilde{G}$ are K-convex. Now, we are going to show that the study of the formula $\partial^s(F + G \circ H)$ can be reduced to that for $\partial^s(\tilde{F} + \tilde{G})$. For this, we need the following lemma, which studies the relationship between the subdifferentials of $\tilde{F}$, $\tilde{G}$ and the subdifferentials of F, H and G.

Lemma 3.2. *Let $\bar{x} \in \mathrm{dom}F \cap \mathrm{dom}(G \circ H)$, $\bar{u} \in F(\bar{x})$, $\bar{z} \in H(\bar{x})$, $\bar{v} \in G(\bar{z})$ and $\varepsilon \in K$, we have*

(i) If G is (Z_+, K)-nondecreasing, then

$$A \in \partial^s_\varepsilon(F + G \circ H)(\bar{x}, \bar{u} + \bar{v}) \quad \Longleftrightarrow \quad (A, 0) \in \partial^s_\varepsilon(\tilde{F} + \tilde{G})((\bar{x}, \bar{z}), \bar{u} + \bar{v}). \tag{3.10}$$

(ii) $\partial^s_\varepsilon \tilde{G}((\bar{x}, \bar{z}), \bar{v}) = \{0\} \times \partial^s_\varepsilon G(\bar{z}, \bar{v})$.

(iii) If G is connected at $\bar{z} \in H(\bar{x})$, then $\tilde{G}$ is connected at $(\bar{x}, \bar{z})$.

Proof. (i) Let $A \in \partial^s_\varepsilon(F + G \circ H)(\bar{x}, \bar{u} + \bar{v})$, then we have

$$F(x) + (G \circ H)(x) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K, \ \forall x \in X,$$

and so, we obtain

$$F(x) + R^v_{\mathrm{epi}H}(x, z) + (G \circ H)(x) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K, \ \forall (x, z) \in X \times Z,$$

which imply that

$$\tilde{F}(x, z) + (G \circ H)(x) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K, \ \forall (x, z) \in \mathrm{epi}H. \tag{3.11}$$

As G is (Z_+, K)-nondecreasing, then for any $(x, z) \in \mathrm{epi}H$, we have $G(z) \subset (G \circ H)(x) + K$ and so by relation (3.11), it follows that

$$\tilde{F}(x, z) + G(z) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K + K \subseteq K,$$

hence we get

$$\tilde{F}(x, z) + \tilde{G}(x, z) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K, \ \forall (x, z) \in X \times Z,$$

which yields $(A, 0) \in \partial^s_\varepsilon(\tilde{F} + \tilde{G})((\bar{x}, \bar{z}), \bar{u} + \bar{v})$. Conversely, let us take any $(A, 0) \in \partial^s_\varepsilon(\tilde{F} + \tilde{G})((\bar{x}, \bar{z}), \bar{u} + \bar{v})$, then

$$\tilde{F}(x, z) + \tilde{G}(x, z) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K, \ \forall (x, z) \in X \times Z,$$

i.e.

$$F(x) + R_{\mathrm{epi}H}(x, z) + G(z) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K, \ \forall (x, z) \in X \times Z.$$

Therefore for all $(x, z) \in \mathrm{epi}H$, we have

$$F(x) + G(z) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K,$$

which implies that for all $x \in X$

$$F(x) + \bigcup_{z \in H(x)} G(z) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K,$$

i.e.

$$F(x) + (G \circ H)(x) - \bar{u} - \bar{v} - A(x - \bar{x}) + \varepsilon \subseteq K, \ \forall x \in X.$$

Finally, $A \in \partial^s_\varepsilon(F + G \circ H)(\bar{x}, \bar{u} + \bar{v})$.
(ii) Let $(A, B) \in \partial^s_\varepsilon \tilde{G}((\bar{x}, \bar{z}), \bar{y})$, then for all $((x, z), y) \in \mathrm{gr}\tilde{G} = X \times \mathrm{gr}G$.

$$A(x - \bar{x}) + B(z - \bar{z}) \leq_K y - \bar{y} + \varepsilon. \tag{3.12}$$

By taking $z = \bar{z}$ and $y = \bar{y}$ in (3.12), it follows that for all $x \in X$ and $n \in \mathbb{N}^*$

$$A(x) \leq_K \frac{1}{n}(A(\bar{x}) + \varepsilon).$$

Since K is closed and pointed, we get $A = 0$, consequently $\partial^s_\varepsilon \tilde{G}((\bar{x}, \bar{z}), \bar{y}) \subseteq \{0\} \times \partial^s_\varepsilon G(\bar{z}, \bar{y})$. For the reverse inclusion, let $B \in \partial^s_\varepsilon G(\bar{z}, \bar{y})$, i.e.

$$B(z - \bar{z}) \leq_K y - \bar{y} + \varepsilon, \ \forall (z, y) \in \mathrm{gr}G.$$

As $\mathrm{gr}\tilde{G} = X \times \mathrm{gr}G$, we deduce that $\{0\} \times \partial^s_\varepsilon G(\bar{z}, \bar{y}) \subseteq \partial^s_\varepsilon \tilde{G}((\bar{x}, \bar{z}), \bar{y})$.
(iii) As G is connected at $\bar{z} \in H(\bar{x})$, there exists a neighborhood V of $\bar{z}$ and a mapping $g : Z \to Y$ such that $g(z) \in G(z)$ for all $z \in V$ and g is continuous at $\bar{z}$. Define the following function

$$\begin{array}{rcl} \tilde{g} : X \times Z & \to & Y \\ (x, z) & \mapsto & g(z) \end{array}$$

It is clear that $\tilde{g}$ is continuous at $(\bar{x}, \bar{z})$ and $\tilde{g}(x, z) \in \tilde{G}(x, v)$ for all $(x, z) \in X \times V$. Hence $\tilde{G}$ is connected at $(\bar{x}, \bar{z})$. □

Now, we are ready to state our main results in this subsection.

Theorem 3.4. *Let $F : X \rightrightarrows Y$, $H : X \rightrightarrows Z$ and $G : Z \rightrightarrows Y$ be three set-valued mappings, $(\bar{x}, \bar{u}) \in \mathrm{gr}F$, $(\bar{x}, \bar{z}) \in \mathrm{gr}H$ and $(\bar{z}, \bar{v}) \in \mathrm{gr}G$. Suppose also that the following condition holds.*

$$(MR)_1 \quad \begin{cases} (Y, K) \text{ is a normal order complete lattice Hausdorff locally convex topological} \\ \text{vector space, } (Z, Q) \text{ is a Hausdorff locally convex space,} \\ F,\, G \text{ are } K\text{-convex and } H \text{ is } Z_+\text{-convex ,} \\ G \text{ is } (Q, K)\text{-nondecreasing,} \\ \exists a \in \mathrm{dom}F \cap \mathrm{dom}H \text{ such that } G \text{ is connected at some point } b \in H(a). \end{cases}$$

Then,

$$\partial^s_\varepsilon (F + G \circ H)(\bar{x}, \bar{u} + \bar{v}) = \bigcup_{\substack{\varepsilon_1,\, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial^s_{\varepsilon_1} (F + A \circ H)(\bar{x}, \bar{u} + A(\bar{z})),\, A \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{v})\}, \quad \forall \varepsilon \in K.$$

Proof. Let $\varepsilon \in K$ and prove at first that

$$\partial^s_\varepsilon (F + G \circ H)(\bar{x}, \bar{u} + \bar{v}) \supseteq \bigcup_{\substack{\varepsilon_1,\, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial^s_{\varepsilon_1} (F + A \circ H)(\bar{x}, \bar{u} + A(\bar{z})),\, A \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{v})\}.$$

Let $A \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{v})$ and $B \in \partial^s_{\varepsilon_1} (F + A \circ H)(\bar{x}, \bar{u} + A(\bar{z}))$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$, then

$$(F + A \circ H)(x) - \bar{u} - A(\bar{z}) - B(x - \bar{x}) + \varepsilon_2 \subseteq K, \quad \forall x \in X,$$

which means that for any $(x, u) \in \mathrm{gr}F$ and $(x, z) \in \mathrm{gr}H$

$$B(x - \bar{x}) \leq_K u - \bar{u} + A(z - \bar{z}) + \varepsilon_1. \tag{3.13}$$

As $A \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{v})$, we have

$$G(z) - \bar{v} - A(z - \bar{z}) + \varepsilon_2 \subseteq K, \quad \forall z \in Z,$$

which yields that

$$\bigcup_{z \in H(x)} (G - A)(z) - \bar{v} + A(\bar{z}) + \varepsilon_2 \subseteq K, \quad \forall x \in X,$$

i.e.

$$(G \circ H)(x) - (A \circ H)(x) - \bar{v} + A(\bar{z}) + \varepsilon_2 \subseteq K, \quad \forall x \in X.$$

Therefore,

$$A(z - \bar{z}) \leq_K v - \bar{v} + \varepsilon_2, \quad \forall (x, z) \in \mathrm{gr}H, (z, v) \in \mathrm{gr}G. \tag{3.14}$$

From the inequalities (3.13) and (3.14), we get

$$B(x - \bar{x}) \leq_K u + v - \bar{u} - \bar{v} + \varepsilon, \quad \forall (x, u + v) \in \mathrm{gr}(F + G \circ H),$$

i.e. $B \in \partial^s_\varepsilon (F + G \circ H)(\bar{x}, \bar{u} + \bar{v})$. For the reverse inclusion, let us take any $B \in \partial^s_\varepsilon (F + G \circ H)(\bar{x}, \bar{u} + \bar{v})$. According to Lemma 3.2 (i), we have

$$(B, 0) \in \partial^s_\varepsilon (\tilde{F} + \tilde{G})((\bar{x}, \bar{z}), \bar{u} + \bar{v}). \tag{3.15}$$

Under the condition (MR_1) and by virtue of Lemma 3.2 (iii), the mappings $\tilde{F}$ and $\tilde{G}$ satisfy together all assumptions of Theorem 3.3, hence we obtain

$$\partial^s_\varepsilon (\tilde{F} + \tilde{G})((\bar{x}, \bar{z}), \bar{u} + \bar{v}) = \bigcup_{\substack{\varepsilon_1,\, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial^s_{\varepsilon_1} \tilde{F}((\bar{x}, \bar{z}), \bar{u}) + \partial^s_{\varepsilon_2} \tilde{G}((\bar{x}, \bar{z}), \bar{v}).$$

Then, there exists ε_1, $\varepsilon_2 \in K$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$, and $(T, A) \in \partial^s_{\varepsilon_2} \tilde{G}((\bar{x}, \bar{z}), \bar{v})$ such that

$$(B - T, -A) \in \partial^s_{\varepsilon_2} \tilde{F}((\bar{x}, \bar{z}), \bar{u}).$$

By virtue of Lemma 3.2 (ii), we obtain that $T = 0$ and $A \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{v})$. Now, let us show that $B \in \partial^s_{\varepsilon_1}(F + A \circ H)(\bar{x}, \bar{u} + A(\bar{z}))$. As $(B, -A) \in \partial^s_{\varepsilon_1} \tilde{F}((\bar{x}, \bar{z}), \bar{u})$, we have for all $(x, z) \in X \times Z$

$$F(x) + R^v_{\text{epi}H}(x, z) - \bar{u} - B(x - \bar{x}) + A(z - \bar{z}) + \varepsilon_1 \subseteq K,$$

which implies that

$$F(x) - \bar{u} - B(x - \bar{x}) + A(z - \bar{z}) + \varepsilon_1 \subseteq K, \ \forall (x, z) \in \text{epi}H.$$

Hence for all $x \in X$, we have

$$F(x) + \bigcup_{z \in H(x)} A(z) - (\bar{u} + A(\bar{z})) - B(x - \bar{x}) + \varepsilon_1 \subseteq K,$$

i.e.

$$(F + A \circ H)(x) - (\bar{u} + A(\bar{z})) - B(x - \bar{x}) + \varepsilon_1 \subseteq K.$$

Therefore, $B \in \partial^s_{\varepsilon_1}(F + A \circ H)(\bar{x}, \bar{u} + A(\bar{z}))$. □

Corollary 3.1. *Let $H : X \rightrightarrows Z$ and $G : Z \rightrightarrows Y$ be two set-valued mappings, $\bar{z} \in H(\bar{x})$ and $\bar{y} \in G(\bar{z})$. Suppose that the following condition holds*

$$(MR)_2 \quad \begin{cases} (Y, K) \text{ is a normal order complete lattice Hausdorff locally convex topological} \\ \text{vector space, } (Z, Q) \text{ is a Hausdorff locally convex space,} \\ G \text{ is } K\text{-convex and } H \text{ is } Z_+\text{-convex ,} \\ G \text{ is } (Q, K)\text{-nondecreasing,} \\ G \text{ is connected at some point of } \text{Im}H. \end{cases}$$

Then,

$$\partial^s_\varepsilon(G \circ H)(\bar{x}, \bar{y}) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial^s_{\varepsilon_1}(A \circ H)(\bar{x}, A(\bar{z})), \, A \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{y})\}, \quad \forall \varepsilon \in K.$$

Consider now the case of composition with a linear operator. Let $A : X \to Z$ be a linear operator and $G : Z \rightrightarrows Y$ be a K-convex set-valued mapping. By putting $Q = \{0_Z\}$, the function G is obviously (Q, K)-nondecreasing and A is Q-convex. So applying Corollary 3.1, one gets the following result.

Corollary 3.2. *Let $\bar{x} \in X$ and $\bar{y} \in G(A(\bar{x}))$. Assume that the following condition holds*

$$(MR)_3 \quad \begin{cases} (Y, K) \text{ is a normal order complete lattice Hausdorff locally convex topological} \\ \text{vector space, } (Z, Q) \text{ is a Hausdorff locally convex topological vector space,} \\ G \text{ is } K\text{-convex,} \\ G \text{ is connected at some point of } \text{Im}A. \end{cases}$$

Then,

$$\partial^s_\varepsilon(G \circ A)(\bar{x}, \bar{y}) \quad = \quad \bigcup_{\eta \in K} \partial^s_{\varepsilon - \eta} G(A(\bar{x}), \bar{y}) \circ A, \quad \forall \varepsilon \in K.$$

Proof. It follows from Corollary 3.1 that

$$\partial^s_\varepsilon(G \circ A)(\bar{x}, \bar{y}) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial^s_{\varepsilon_1}(B \circ A)(\bar{x}, A(\bar{x})), \, B \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{y})\}. \tag{3.16}$$

Moveover, it follows from the definition of the approximate strong subdifferential and the fact K is closed and pointed that $\partial^s_{\varepsilon_1}(B \circ A)(\bar{x}, A(\bar{x})) = \{B \circ A\}$, hence by (3.17), we get

$$\begin{aligned} \partial^s_\varepsilon(G \circ A)(\bar{x}, \bar{y}) \quad &= \quad \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{B \circ A : \quad B \in \partial^s_{\varepsilon_2} G(A(\bar{x}), \bar{y})\}, \\ &= \quad \bigcup_{\eta \in K} \partial^s_{\varepsilon - \eta} G(A(\bar{x}), \bar{y}) \circ A. \end{aligned}$$

□

Corollary 3.3. *Under the assumptions of Theorem 3.4, if we assume that F or H is connected at some point of* $\mathrm{dom}F \cap \mathrm{dom}H$*, then*

$$\partial^s_\varepsilon(F + G \circ H)(\bar{x}, \bar{u} + \bar{v}) = \bigcup_{\substack{\varepsilon_1,\, \varepsilon_2, \varepsilon_3 \in K \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \{\partial^s_{\varepsilon_1} F(\bar{x}, \bar{u}) + \partial^s_{\varepsilon_2}(A \circ H)(\bar{x}, A(\bar{z})),\ A \in \partial^s_{\varepsilon_3} G(\bar{z}, \bar{v})\}., \quad \forall \varepsilon \in K.$$

Proof. According to Theorem 3.4, we have

$$\partial^s_\varepsilon(F + G \circ H)(\bar{x}, \bar{u} + \bar{v}) = \bigcup_{\substack{\varepsilon_1,\, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial^s_{\varepsilon_1}(F + A \circ H)(\bar{x}, \bar{u} + A(\bar{z})),\ A \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{v})\}.$$

Let us note that $A \circ H$ is K-convex since $A \in L_+(Z, Y)$. As F or H is connected at some point of $\mathrm{dom}F \cap \mathrm{dom}H$, we can see easily that F or $A \circ H$ is connected at some point of $\mathrm{dom}F \cap \mathrm{dom}(A \circ H) = \mathrm{dom}F \cap \mathrm{dom}H$. Hence from Theorem 3.3 we have

$$\begin{aligned} \partial^s_\varepsilon(F + G \circ H)(\bar{x}, \bar{u} + \bar{v}) &= \bigcup_{\substack{\varepsilon_1,\, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{ \bigcup_{\substack{\varepsilon_3,\, \varepsilon_4 \in K \\ \varepsilon_3 + \varepsilon_4 = \varepsilon_1}} \{\partial^s_{\varepsilon_3} F(\bar{x}, \bar{u}) + \partial^s_{\varepsilon_4}(A \circ H)(\bar{x}, A(\bar{z}))\},\ A \in \partial^s_{\varepsilon_2} G(\bar{z}, \bar{v})\}. \\ &= \bigcup_{\substack{\varepsilon_1,\, \varepsilon_2, \varepsilon_3 \in K \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \{\partial^s_{\varepsilon_1} F(\bar{x}, \bar{u}) + \partial^s_{\varepsilon_2}(A \circ H)(\bar{x}, A(\bar{z})),\ A \in \partial^s_{\varepsilon_3} G(\bar{z}, \bar{v})\}. \end{aligned}$$

□

The following corollary is a result obtained in [1].

Corollary 3.4. *Let $f : X \to Y \cup \{+\infty_Y\}$, $h : X \to Z \cup \{+\infty_Z\}$ and $g : Z \to Y \cup \{+\infty_Y\}$ be three set-valued mappings, $\bar{x} \in \mathrm{dom}F \cap \mathrm{dom}h$ such that $h(\bar{x}) \in \mathrm{dom}g$. Suppose also that the following condition holds.*

$$(MR)_4 \quad \begin{cases} (Y, K) \textit{ is a normal order complete lattice Hausdorff locally convex topological} \\ \textit{vector space, } (Z, Q) \textit{ is a Hausdorff locally convex topological vector space,} \\ f,\, g \textit{ are } K\textit{-convex and } h \textit{ is } Q\textit{-convex ,} \\ g \textit{ is } (Q, K)\textit{-nondecreasing,} \\ g \textit{ is continuous at some point of } h(\mathrm{dom}f \cap \mathrm{dom}h). \end{cases}$$

Then,

$$\partial^s(f + g \circ h)(\bar{x}) = \bigcup_{A \in \partial^s g(\bar{z})} \partial^s(f + A \circ h)(\bar{x}).$$

Proof. Let us consider the following set-valued mappings

$$F_f(x) = \begin{cases} \{f(x)\}, & \text{if } x \in \mathrm{dom}\, f, \\ \emptyset, & \text{otherwise,} \end{cases} \quad , \quad H_h(x) = \begin{cases} \{h(x)\}, & \text{if } x \in \mathrm{dom}\, h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

$$G_g(x) = \begin{cases} \{g(x)\}, & \text{if } x \in \mathrm{dom}\, g, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easy to check that if g is continuous at some point of $h(\mathrm{dom}f \cap \mathrm{dom}h)$ then the set-valued mapping g is connected at some point of $H_h(\mathrm{dom}F_f \cap \mathrm{dom}H_h)$ and also we check easily that the set-valued mappings F_f, H_h and G_g satisfy all the assumptions of Theorem 3.4. Therefore, we get for all $\varepsilon \in K$

$$\partial^s_\varepsilon(F_f + G_g \circ H_h)(\bar{x}, f(\bar{x}) + g(h(\bar{x}))) = \bigcup_{\substack{\varepsilon_1,\, \varepsilon_2 \in K \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial^s_{\varepsilon_1}(F_f + A \circ H_h)(\bar{x}, f(\bar{x}) + A(h(\bar{x}))),\ A \in \partial^s_{\varepsilon_2} G_g(h(\bar{x}), g(h(\bar{x})))\}.$$

□

4 Application to vector set optimization problem

This section is devoted to establish approximate optimality conditions for the following constrained vector set-valued optimization problem using approximate strong subdifferential and vector ε-normal set

$$(\mathrm{P}_S) \quad \begin{cases} \text{Min F(x)}, \\ x \in S, \end{cases}$$

where $F : X \rightrightarrows Y$ is a set-valued mapping and S is a nonempty convex closed subset of X. The indicator vector set-valued mapping $R^v_S : X \rightrightarrows Y$ is defined for the nonempty subset $S \subseteq X$ by

$$R^v_S(x) := \begin{cases} \{0_Y\}, & \text{if } x \in S, \\ \emptyset, & \text{elsewhere.} \end{cases}$$

It is obvious that R^v_S is proper. The problem (P_S) becomes equivalent to the unconstrained vector set-valued minimization problem

$$(\mathrm{P}) \quad \begin{cases} \text{Min } (F + R^v_S)(x), \\ x \in X, \end{cases}$$

in the following sense.

$$\mathrm{K}_{\varepsilon,\ s}(F, S, K) = \mathrm{K}_{\varepsilon,s}(F + R^v_S, X, K).$$

Lemma 4.1. *(i) If S is convex and closed then R^v_S is K-convex and for all $\bar{x} \in S$,*

$$\partial^s_\varepsilon R^v_S(\bar{x}, 0_Y) = N^v_\varepsilon(\bar{x}, S), \quad \forall \varepsilon \in K,$$

where

$$N^v_\varepsilon(\bar{x}, S) := \{A \in L(X, Y) : A(x - \bar{x}) \leq_K \varepsilon,\ \forall x \in S\}$$

is the set of ε-normal vectors at $\bar{x} \in S$.
(ii) If $\mathrm{int}(S) \neq \emptyset$ then, R^v_S is connected on $\mathrm{int}(S)$.

Proof. (i) The epigraph of R^v_S is given by

$$\mathrm{epi} R^v_S = \{(x, y) \in X \times Y : \quad y \in R^v_S(x) + K\} = S \times K,$$

and its K-convexity follows easily from the convexity of S and K.
(ii) Let us consider the single following mapping $h : X \longrightarrow Y$ defined by $h(x) := 0_Y$ for all $x \in X$. Since $0_Y \in R^v_S(x)$, for any $x \in S$, it follows that $h(x) \in R^v_S(x)$, for any $x \in \mathrm{int}(S)$, which ensures that R^v_S is connected on $\mathrm{int}(S)$. □

We are now ready to establish optimality conditions for problem (P_S).

Theorem 4.1. *Let $F : X \rightrightarrows Y$ be a set-valued mapping, S be a nonempty convex closed subset of X, $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ with $\bar{x} \in S$ and $\varepsilon \in K$. If the following qualification condition holds*

$(\mathrm{MR})_5$ $\begin{cases}$ *(Y, K) is a normal order complete lattice Hausdorff locally convex topological vector space, F is K-convex,*
 $\mathrm{dom}F \cap \mathrm{int}(S) \neq \emptyset$ *or F is connected at some point of* $\mathrm{dom}F \cap S$.

Then $(\bar{x}, \bar{y}) \in \mathrm{K}_{\varepsilon,\ s}(F, S, K)$ if and only if, there exist $\varepsilon_1, \varepsilon_2 \in K$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $A \in \partial^s_{\varepsilon_1} F(\bar{x}, \bar{y})$ such that $-A \in N^v_{\varepsilon_2}(\bar{x}, S)$.

Proof. Since the problem (P_S) is equivalent to the unconstrained set-valued minimization problem (P), we have $(\bar{x}, \bar{y}) \in \mathrm{K}_{\varepsilon,\ s}(F, S, K)$ for (P_S) if and only if

$$0 \in \partial^s_\varepsilon (F + R^v_S)(\bar{x}, \bar{y} + 0_Y). \tag{4.1}$$

The conditions $(\mathrm{MR})_5$ and Lemma 4.1, show together that the mappings F and R^v_S satisfy all the hypotheses of Theorem 3.3. Hence, we get

$$0 \in \bigcup_{\substack{\varepsilon_1,\,\varepsilon_2\in K\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial^s_{\varepsilon_1} F(\bar{x},\bar{y}) + \partial^s_{\varepsilon_2} R^v_S(\bar{x}, 0_Y),$$

i.e. there exist $\varepsilon_1, \varepsilon_2 \in K$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $A \in \partial^s_{\varepsilon_1} F(\bar{x},\bar{y})$ such that $-A \in \partial^s_{\varepsilon_2} R^v_S(\bar{x}, 0_Y) = N^v_{\varepsilon_2}(\bar{x}, S)$. □

Example 4.1. Let us consider the following constrained vector set-valued optimization problem

$$(\mathrm{P}) \quad \begin{cases} \text{Min F(x)}, \\ x \in [0,1], \end{cases}$$

where $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ is defined by

$$F(x) := \{(a,b) \in \mathbb{R}^2 :\ a \geq |x|,\ \ b \geq x^2\}.$$

It is easy to see that F satisfies the condition $(\mathrm{MR})_5$ of Theorem 4.1, and for any $(\eta_1,\eta_2) \in \mathbb{R}^2_+$

$$\partial^s_{(\eta_1,\eta_2)} F(0,(0,0)) = \{(\mu,\nu) \in \mathbb{R}^2 :\ -1 \leq \mu \leq 1, \quad -2\sqrt{\eta_2} \leq \nu \leq 2\sqrt{\eta_2}\}.$$

By taking $\varepsilon = (1,1)$, $\varepsilon_1 = (\frac{1}{2}, \frac{1}{2})$ and $\varepsilon_2 = (\frac{1}{2}, \frac{1}{2})$, we have

$$N^v_{\varepsilon_2}(0,[0,1]) = \{(\alpha,\beta) \in \mathbb{R}^2 :\ \alpha \leq \frac{1}{2}, \quad \beta \leq \frac{1}{2}\}.$$

Obviously, $(0,0) \in \partial^s_{\varepsilon_1} F(0,(0,0))$, $(0,0) \in -N^v_{\varepsilon_2}(0,[0,1])$ and $\varepsilon_1 + \varepsilon_2 = \varepsilon$. Hence, $(0,0)$ is a strong $(1,1)$-minimiser solution for $(\mathrm{P}_{[0,1]})$.

Let us consider the following general convex set-valued mathematical programming problem

$$(\mathrm{R}) \quad \begin{cases} \text{Minimize F(x)}, \\ H(x) \cap -Q \neq \emptyset, \\ x \in C. \end{cases}$$

where $F : X \rightrightarrows Y$ and $H : X \rightrightarrows Z$ are two set-valued mappings, Z is a real locally convex topological vector space, Q is a closed convex pointed cone with nonempty topological interior and C be a nonempty closed convex set of X. For establishing the optimality conditions of this problem, we will need the following lemma

Lemma 4.2. *(i) If Z is a real locally convex topological vector space and $Q \subseteq Z$ be a closed convex cone, then the strong subdifferential of the indicator set-valued mapping $R^v_{-Q} : Z \rightrightarrows Y$ is given by*

$$\partial^s_\varepsilon R^v_{-Q}(\bar{z}, 0_Y) = \{A \in L_+(Z,Y) : \quad A(\bar{z}) \in [-\varepsilon, 0_Y]\}, \quad \forall \varepsilon \in K.$$

(ii) The indicator set-valued mapping R^v_{-Q} is (Q,K)-nondecreasing on Z.

Theorem 4.2. *Let $F : X \rightrightarrows Y$ and $H : X \rightrightarrows Z$ be two set-valued mappings, $(\bar{x},\bar{y}) \in \mathrm{gr}F$ with $\bar{x} \in C$ and $\bar{z} \in H(\bar{x}) \cap (-Q) \neq \emptyset$. If the following condition holds*

$$(\mathrm{MR})_6 \quad \begin{cases} (Y,K) \textit{ is a normal order complete lattice Hausdorff locally convex topological} \\ \textit{vector space, } (Z,Q) \textit{ is a Hausdorff locally convex topological vector space,} \\ F \textit{ is } K\textit{-convex and } H \textit{ is } Q\textit{-convex,} \\ \mathrm{int}(-Q) \cap H(C \cap \mathrm{dom}F \cap \mathrm{dom}H) \neq \emptyset. \end{cases}$$

Then $(\bar{x},\bar{y})$ is a strong ε-minimiser solution of problem (R)*, if and only if, there exist $\varepsilon_1, \varepsilon_2 \in K$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $A \in L_+(Z,Y)$ such that*

(a) $A(\bar{z}) \in [-\varepsilon_2, 0_Y]$.

(b) $0 \in \partial^s_{\varepsilon_1}(F + A \circ H + R^v_C)(\bar{x}, \bar{y} + A(\bar{z}))$.

Proof. The feasible set associated to problem (R) is given by $S = \{x \in X : H(x) \cap -Q \neq \emptyset\} \cap C$, and it is easy to check that $R^v_S = R^v_C + R^v_{-Q} \circ H$. Hence the problem (R) becomes equivalent to the unconstrained set-valued minimization problem

$$\begin{cases} \text{Minimize}(F + R^v_C + R^v_{-Q} \circ H)(x), \\ x \in X. \end{cases}$$

Thus, $(\bar{x}, \bar{y})$ is a strong ε-minimiser solution of problem (Q), if and only if,

$$0 \in \partial^s_{\varepsilon}(F + R^v_C + R^v_{-Q} \circ H)(\bar{x}, \bar{y}).$$

Let us observe that

$$\text{epi}(F + R^v_C) = \text{epi}F \cap (C \times Y),$$

which assert that the convexity of the set-valued mapping $F + R^v_C$ follows from the convexity of the epigraph of F and the convexity of the subset C. Also, let us note that the conditions $(\bar{x}, \bar{y}) \in \text{gr}F$ with $\bar{x} \in C$ and $\bar{z} \in H(\bar{x}) \cap (-Q) \neq \emptyset$ may be written equivalently as $(\bar{x}, \bar{y}) \in \text{gr}(F + R^v_C)$, $(\bar{x}, \bar{z}) \in \text{gr}H$ and $(\bar{z}, 0_Y) \in \text{gr}R^v_{-Q}$. According to Lemma 4.1 and Lemma 4.2, the set-valued mappings $F + R^v_C$, H, and R^v_{-Q} satisfy together all the assumptions of Theorem 3.4 and thus we obtain that $(\bar{x}, \bar{y})$ is a strong ε-minimiser, if and only if, there exist $\varepsilon_1, \varepsilon_2 \in K$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $A \in \partial^s_{\varepsilon_2} R^v_{-Q}(\bar{z}, 0_Y) = \{A \in L_+(Z, Y) : \; A(\bar{z}) \in [-\varepsilon_2, 0_Y]\}$ such that

$$0 \in \partial^s_{\varepsilon_1}(F + R^v_C + A \circ H)(\bar{x}, \bar{y} + A(\bar{z})).$$

The proof of theorem is complete. □

Corollary 4.1. *Under the assumptions of Theorem 4.2, we assume, in addition, that F is connected at some point of C and H is connected at some point of C. Then $(\bar{x}, \bar{y})$ is a strong minimiser of problem* (R) *if and only if, there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in K$ with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = \varepsilon$, $A \in L_+(Z, Y)$, $B \in \partial^s_{\varepsilon_1} F(\bar{x}, \bar{y})$ and $T \in \partial^s_{\varepsilon_3}(A \circ H)(\bar{x}, 0_Y)$ such that*

(a) $A(\bar{z}) \in [-\varepsilon_2, 0_Y]$.

(b) $-T - B \in N^v_{\varepsilon_4}(\bar{x}, C)$.

Proof. According to Theorem 4.2, we have $(\bar{x}, \bar{y})$ is a strong ε-minimiser of problem (R) if and only if, there exist $\eta_1, \eta_2 \in K$ with $\eta_1 + \eta_2 = \eta$ and $A \in L_+(Z, Y)$ such that $A(\bar{z}) \in [-\eta_2, 0_Y]$ and

$$0 \in \partial^s_{\eta_1}(F + A \circ H + R^v_C)(\bar{x}, \bar{y} + A(\bar{z})).$$

The fact that $A \in L_+(Z, Y)$ yields that $A \circ H$ is K-convex. As H is connected at some point of C, it is easy to check that $A \circ H$ is connected at some point of C. The convexity of the set-valued mapping $A \circ H + R^v_C$ follows from the convexity of the epigraph of $A \circ H$ and the convexity of the subset C. The set-valued mappings F and $A \circ H + R^v_C$ satisfy together all the assumptions of Theorem 3.3 and hence we obtain

$$0 \in \partial^s_{\eta_1}(F + A \circ H + R^v_C)(\bar{x}, \bar{y} + A(\bar{z})) = \bigcup_{\substack{\eta_3, \eta_4 \in K \\ \eta_3 + \eta_4 = \eta_1}} \partial^s_{\eta_3} F(\bar{x}, \bar{y}) + \partial^s_{\eta_4}(A \circ H + R^v_C)(\bar{x}, A(\bar{z})).$$

On other hand, it is clearthat the set-valued mappings $A \circ H$ and R^v_C satisfy together all the hypothesis of Theorem 3.3 and hence we get

$$0 \in \partial^s_{\eta_1}(F + A \circ H + R^v_C)(\bar{x}, \bar{y} + A(\bar{z})) = \bigcup_{\substack{\eta_3, \eta_4 \in K \\ \eta_3 + \eta_4 = \eta_1}} \left\{ \partial^s_{\eta_3} F(\bar{x}, \bar{y}) + \bigcup_{\substack{\eta_5, \eta_6 \in K \\ \eta_5 + \eta_6 = \eta_4}} \{\partial^s_{\eta_5}(A \circ H)(\bar{x}, A(\bar{z})) + \partial^s_{\eta_6} R^v_C(\bar{x}, 0_Y)\} \right\}.$$

i.e., $\eta_3, \eta_4, \eta_5, \eta_6 \in K$ with $\eta_3 + \eta_4 = \eta_1$, $\eta_5 + \eta_6 = \eta_4$ and $B \in \partial^s_{\eta_3} F(\bar{x}, \bar{y})$ and $T \in \partial^s_{\eta_5}(A \circ H)(\bar{x}, 0_Y)$ such that $-T - B \in \partial^s_{\eta_6} R^v_C(\bar{x}, 0_Y) = N^v_{\eta_6}(\bar{x}, C)$. By putting, $\varepsilon_1 = \eta_3$, $\varepsilon_2 = \eta_2$, $\varepsilon_3 = \eta_5$ and $\varepsilon_4 = \eta_6$, we get $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = \varepsilon$, which completes the proof of theorem. □

Tab. 1: Paper Notation Summary

Symbol	Meaning
X, Y	Topological vector spaces
$K \subset X$	Closed, convex cone (used to define an order)
$x \leq_K y$	Ordering relation: $y - x \in K$
$\mathrm{int}(A)$	Interior of the set A
$\mathrm{dom}(G)$	Domain of the set-valued mapping F, i.e., $\{x \in X : G(x) \neq \emptyset\}$
$\mathrm{gph}(G)$	Graph of the set-valued mapping F, i.e., $\{(x, y) \mid y \in G(x)\}$
$\mathrm{epi}_K G$ $\partial^s F(\bar{x}, \bar{y})$	The epigraph of the set-valued mapping Strong subdifferential of the set valued mapping F at $(\bar{x}, \bar{y})$
$\partial^s_\varepsilon F(\bar{x}, \bar{y})$	Approximate Strong subdifferential of the set valued mapping F at $(\bar{x}, \bar{y})$
$N_C(x)$	Normal cone to the set C at point x
$\partial^s f(x)$	Strong Subdifferential of a vector convex function f at point x
$\mathrm{dom}(F)$	Domain of the set-valued mapping F
$\mathrm{cl}(A)$	Closure of the set A
$\mathrm{co}(A)$	Convex hull of the set A

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