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A note on submonoids of $\mathbb{N}^k$

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Abstract

In this work, we show that for every nontrivial submonoid H of $\mathbb{N}^k$ endowed with vector addition, the least positive integer r for which H is isomorphic to a submonoid of $\mathbb{N}^r$ coincides with the largest positive integer t such that H contains a free submonoid of rank t. We also provide characterizations of when this least positive integer r equals 1.

Keywords: Submonoid, isomorphism, rank, free submonoid.

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1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers, and let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ denote the set of positive integers.

The set $\mathbb{N}^k$, together with vector addition, is a commutative monoid with identity $\mathbf{0} = (0, \ldots, 0)$. This monoid is cancellative and $\mathbf{0}$ is its only invertible element. A *submonoid of* $\mathbb{N}^k$, or simply a *k-monoid*, is a subset $H \subseteq \mathbb{N}^k$ containing $\mathbf{0}$ and closed under addition. A subset $F \subseteq H$ is called a *k-submonoid of H* if it is itself a k-monoid.

If $A \subseteq \mathbb{N}^k$, then the set $\langle A \rangle$ consisting of all linear combinations of the form $\sum_{i=1}^{r} n_i \mathbf{v}_i$, where $r \in \mathbb{Z}^+$, $n_1, \ldots, n_r \in \mathbb{N}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_r \in A$, forms a k-monoid, which we call the *k-monoid generated by A*. If $H = \langle A \rangle$ for a subset $A \subseteq H$, then we say that A *generates* H, or equivalently that A is a *generating set* of H. A k-monoid H is said to be *finitely generated* if it has a finite generating set.

If H is a k-monoid, then the set $\beta(H)$ consisting of all nonzero elements of H that cannot be expressed as sum of two nonzero elements of H, is known to generate H. Moreover, $\beta(H)$ is contained in any other generating set of H. For this reason, the set $\beta(H)$ is called the *minimal generating set* of H. We define the rank of H, $\operatorname{rank}(H)$, to be the cardinality of $\beta(H)$. For instance, $\beta(\mathbb{N}^k) = \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$, where $\mathbf{e}_i$ denotes the k-tuple whose i-th coordinate is 1 and all other coordinates are 0. Hence, $\mathbb{N}^k$ is finitely generated and $\operatorname{rank}(\mathbb{N}^k) = k$.

If $k = 1$ and H is a *nontrivial* 1-monoid (that is, $H \neq \{0\}$), then the set $S = \{h/d : h \in H\}$, where $d = \gcd(H)$, is a *numerical semigroup*, that is, a submonoid of $\mathbb{N}$ with finite complement in $\mathbb{N}$. It is well known that every submonoid of $\mathbb{N}$ is finitely generated. A numerical semigroup S has several important invariants, such as the *Frobenius number*, defined as the greatest integer not belonging to S, and the *genus*, which is the cardinality of the finite set $\mathbb{N} \setminus S$ (see [11]). In recent years, many concepts and problems from the theory of numerical semigroups have been extended to submonoids of $\mathbb{N}^k$ (see [1, 2, 3, 4, 5, 6, 7, 8]).

A k-monoid H is called *free* if there exists a subset $B \subseteq H$ such that every nonzero element of H can be written uniquely in the form

$$n_1 \mathbf{v}_1 + \cdots + n_r \mathbf{v}_r,$$

where $r > 0$, $n_1, \dots, n_r \in \mathbb{Z}^+$, and $\mathbf{v}_1, \dots, \mathbf{v}_r \in B$. If H is free, then $\beta(H) = B$.

A result of Rosales [9, Theorem 1.6] shows that a finitely generated commutative monoid S is isomorphic to a k-monoid if and only if S is cancellative, torsion free, and has no invertible elements other than the identity (see also [10, Theorem 3.11], where the result is attributed to P. A. Grillet). In [9], and algorithm is also given to compute the corresponding positive integer k (i.e, the "dimension" in which S can be embedded into $\mathbb{N}^k$).

For a k-monoid H, we introduce two invariants that play a central role in this work. The *free rank* of H is the largest positive integer t such that H contains a free submonoid of rank t, and the *isomorphism rank* of H is the smallest positive integer r such that H is isomorphic to some r-monoid. Among other properties, we prove that the free rank can be computed as the dimension of the vector subspace of $\mathbb{Q}^k$ generated by the set H. Our main result is that for every nontrivial k-monoid H, the free rank and the isomorphism rank coincide.

Theorem 1.1. *If H is a nontrivial k-monoid, then* $\operatorname{iso}(H) = \operatorname{free}(H)$.

This theorem is proved in §3. In addition, we show that a k-monoid H has isomorphism rank 1 if and only if there exists $\mathbf{w} \in \mathbb{N}^k$ such that $H \subseteq \langle \mathbf{w} \rangle$, and we study the properties of such vectors $\mathbf{w}$.

2 Free k-monoids and free rank

It is convenient to consider the $\mathbb{Q}$-vector space $\mathbb{Q}^k$ endowed with the usual operations. The following result provides a characterization of free k-monoids.

Proposition 2.1. *A k-monoid H is free if and only if $\beta(H)$ is linearly independent in $\mathbb{Q}^k$. In particular, if H is free, then* $\operatorname{rank}(H) \leq k$.

Proof. By clearing denominators, we see that an equation of the form

$$\sum_{j=1}^{n} c_j \mathbf{v}_j = \mathbf{0}, \tag{2.1}$$

where $\mathbf{v}_1, \dots, \mathbf{v}_n \in \beta(H)$, with rational coefficients has a solution if and only if it also has a solution with integer coefficients.

Suppose that $\beta(H)$ is linearly dependent in $\mathbb{Q}^k$, then there exist integers $c_1, \dots, c_n$, not all zero, and vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \beta(H)$ such that (2.1) holds, which yields

$$\sum_{c_j \geq 0} c_j \mathbf{v}_j = \sum_{c_j < 0} (-c_j) \mathbf{v}_j.$$

This provides two different representations of the same element of H as a linear combination of elements of $\beta(H)$. Hence, H is not free.

Conversely, if H is not free, two representations of a nonzero element of H as a linear combination of elements of $\beta(H)$ with integers coefficients lead to an equation of the form (2.1), where not all the coefficients are zero. Hence, $\beta(H)$ is not linearly independent in $\mathbb{Q}^k$.

Finally, since an independent subset of $\mathbb{Q}^k$ has at most k elements, it follows that if H is free, then $\operatorname{rank}(H) = |\beta(H)| \leq k$. ☐

For any k-monoid H we define the *free rank* of H, denoted by $\operatorname{free}(H)$, to be the largest integer t such that H contains a free k-submonoid of rank t. By Proposition 2.1, $\operatorname{free}(H)$ is well defined and $\operatorname{free}(H) \leq k$. The free rank is invariant under isomorphisms.

Example 2.2. Consider the k-monoid

$$H = \{\mathbf{v} = (v_1, \dots, v_k) \in \mathbb{N}^k : v_1 > 0\} \cup \{\mathbf{0}\}.$$

It is easy to see that $\beta(H) = \{\mathbf{v} \in \mathbb{N}^k : v_1 = 1\}$. For $k > 1$, $\beta(H)$ is an infinite set, and hence H is not free. Observe that the k vectors

$$\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \ldots, \mathbf{e}_1 + \mathbf{e}_k$$

belong to H and generate a free k-submonoid of rank k. Thus, $\mathrm{free}(H) = k$.

For any subset $A \subseteq \mathbb{N}^k$, let $\mathrm{L}_{\mathbb{Q}}(A)$ denote the subspace of $\mathbb{Q}^k$ spanned by A. If H is the k-monoid generated by A, then clearly $\mathrm{L}_{\mathbb{Q}}(H) = \mathrm{L}_{\mathbb{Q}}(A)$. In particular, we have $\mathrm{L}_{\mathbb{Q}}(H) = \mathrm{L}_{\mathbb{Q}}(\beta(H))$.

Proposition 2.3. *If H is a k-monoid, then* $\mathrm{free}(H) = \dim(\mathrm{L}_{\mathbb{Q}}(\mathrm{H}))$.

Proof. Let $r = \dim(\mathrm{L}_{\mathbb{Q}}(\mathrm{H}))$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis for $\mathrm{L}_{\mathbb{Q}}(H)$ contained in H, then $\langle \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle$ is a free k-submonoid of H. Thus, $r \leq \mathrm{free}(H)$.

Now, if F is any free k-submonoid of H, then, by Proposition 2.1, $\beta(F)$ is an independent subset of $\mathbb{Q}^k$ contained in H, so $\mathrm{rank}(F) = |\beta(F)| \leq \dim(\mathrm{L}_{\mathbb{Q}}(H)) = r$. It follows that $\mathrm{free}(H) \leq r$. ☐

Thus, if H is a k-monoid generated by finitely many vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$, we may form an $r \times k$ matrix whose rows are the $\mathbf{v}_i$'s, and the rank of this matrix is precisely $\mathrm{free}(H)$.

Corollary 2.4. *If H is a nontrivial k-monoid, then* $\mathrm{free}(H) \leq \mathrm{rank}(H)$. *Moreover, equality holds if and only if H is free.*

Proof. Since $\beta(H)$ is a spanning set of the $\mathbb{Q}$-vector subspace $\mathrm{L}_{\mathbb{Q}}(H)$, Proposition 2.3 gives

$$\mathrm{free}(H) = \dim(\mathrm{L}_{\mathbb{Q}}(H)) \leq |\beta(H)| = \mathrm{rank}(H).$$

Moreover, equality $\mathrm{free}(H) = \mathrm{rank}(H)$ holds if and only if $\beta(H)$ is a basis of $\mathrm{L}_{\mathbb{Q}}(H)$, which, by Proposition 2.1, is equivalent to saying that H is free. ☐

Proposition 2.5. *Let H be a nontrivial k-monoid and let r be a positive integer. Then,* $r = \mathrm{free}(H)$ *if and only if there exists a subset $B \subseteq \beta(H)$ such that the following conditions hold:*

1. *$|B| = r$;*
2. *$\langle B \rangle$ is a free k-submonoid of H;*
3. *For every $\mathbf{w} \in \beta(H) \setminus B$ there exist $\mathbf{u}, \mathbf{v} \in \langle B \rangle$ and $c \in \mathbb{Z}^+$ such that*

$$\mathbf{w} = \frac{1}{c}(\mathbf{u} - \mathbf{v}).$$

Proof. Let $r = \mathrm{free}(H)$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ a basis for $\mathrm{L}_{\mathbb{Q}}(H)$ contained in $\beta(H)$. Set $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$. Then $|B| = r$ and $\langle B \rangle$ is a free k-submonoid. Now let $\mathbf{w} \in \beta(H) \setminus B$. Since $\langle B \cup \{\mathbf{w}\} \rangle$ is not free, there exist natural numbers $n, n_1, \cdots, n_r, m, m_1, \ldots, m_r$, with $(n, n_1, \ldots, n_r) \neq (m, m_1, \ldots, m_r)$, such that

$$n_1\mathbf{v}_1 + \cdots + n_r\mathbf{v}_r + n\mathbf{w} = m_1\mathbf{v}_1 + \cdots + m_r\mathbf{v}_r + m\mathbf{w}.$$

It follows that $n \neq m$, because otherwise we would obtain

$$n_1\mathbf{v}_1 + \cdots + n_r\mathbf{v}_r = m_1\mathbf{v}_1 + \cdots + m_r\mathbf{v}_r,$$

and since $\langle B \rangle$ is free, this would force $n_i = m_i$ for all i, hence $(n, n_1, \ldots, n_r) = (m, m_1, \ldots, m_r)$, a contradiction. Suppose now that $m < n$, and set $c = n - m$. Define

$$\mathbf{v} = n_1\mathbf{v}_1 + \cdots + n_r\mathbf{v}_r, \mathbf{u} = m_1\mathbf{v}_1 + \cdots + m_r\mathbf{v}_r.$$

Then we obtain the equality $\mathbf{v} + c\mathbf{w} = \mathbf{u}$, so that $\mathbf{w} = \frac{1}{c}(\mathbf{u} - \mathbf{v})$.

Conversely, suppose that there is a subset $B \subseteq \beta(H)$ satisfying conditions $(1)-(3)$. The inequality $r \leq \mathrm{free}(H)$ follows by (1) and (2). It remains to show that $\mathrm{free}(H) \leq r$. Indeed, let F be a free k-submonoid of H. Then $\beta(F)$ is an independent set in $\mathbb{Q}^k$, with $\beta(F) \subseteq H \subseteq \mathrm{L}_{\mathbb{Q}}(H)$. Since B is a basis of $\mathrm{L}_{\mathbb{Q}}(H)$, it follows that $\mathrm{rank}(F) = |\beta(F)| \leq |B| = r$. This completes the proof. ☐

Proposition 2.5 shows that if H is a nontrivial k-monoid such that $\operatorname{free}(H) = r$, then there exists a subset $B \subseteq \beta(H)$ of cardinality r such that $\langle B \rangle$ is a free k-monoid and

$$\langle B \rangle \subseteq H \subseteq \mathrm{L}_{\mathbb{Q}}(B).$$

Conversely, suppose that we are given a nonempty subset $B \subseteq H$ such that $\langle B \rangle$ is free and $\langle B \rangle \subseteq H \subseteq \mathrm{L}_{\mathbb{Q}}(B)$. Clearly, $B \subseteq \beta(H)$. If we can prove condition (3), then Proposition 2.5 implies $\operatorname{free}(H) = |B|$. Indeed, let $\mathbf{w} \in \beta(H) \setminus B$. Since $\mathbf{w} \in \mathrm{L}_{\mathbb{Q}}(B)$, it can be expressed as a linear combination of elements of B with rational coefficients. By clearing denominators, we obtain an equality of the form

$$c\mathbf{w} = \sum_i c_i \mathbf{v}_i,$$

where $c \in \mathbb{Z}^+$, $\mathbf{v}_i \in B$, and the c_i's are integers. This equality can be rewritten as

$$c\mathbf{w} = \sum_{c_i \geq 0} c_i \mathbf{v}_i - \sum_{c_i < 0} (-c_i)\mathbf{v}_i.$$

Defining $\mathbf{u} = \sum_{c_i \geq 0} c_i \mathbf{v}_i$ and $\mathbf{v} = \sum_{c_i < 0} (-c_i)\mathbf{v}_i$, we have $\mathbf{u}, \mathbf{v} \in \langle B \rangle$ and therefore $\mathbf{w} = (1/c)(\mathbf{u} - \mathbf{v})$. We have thus established the following result.

Corollary 2.6. *A nontrivial k-monoid H has free rank r if and only if there exists a subset $B \subseteq H$ with $|B| = r$ such that $\langle B \rangle$ is free and $\langle B \rangle \subseteq H \subseteq \mathrm{L}_{\mathbb{Q}}(B)$.*

3 The main result

For convenience, let us define $\mathbb{N}^0 := \{0\}$. Note that the only submonoid of $\mathbb{N}^0$, which we refer to as a 0-monoid, is $\mathbb{N}^0$ itself. For a k-monoid H, we define *the isomorphism rank* of H, denoted by $\operatorname{iso}(H)$, to be the least nonnegative integer r such that H is isomorphic to some r-monoid. It is clear that the isomorphism rank is invariant under isomorphism. Moreover, if F is a k-submonoid of H, then $\operatorname{iso}(F) \leq \operatorname{iso}(H)$.

If H is a nontrivial free k-monoid with rank r, then H is isomorphic to $\mathbb{N}^r$, and therefore $\operatorname{iso}(H) = r$. In addition, note that $\operatorname{iso}(\{\mathbf{0}\}) = 0$.

The following is Theorem 1.1 of the introduction, and constitutes our main result: for a nontrivial k-monoid, the free rank and the isomorphism rank coincide.

Theorem 3.1. *If H is a nontrivial k-monoid, then* $\operatorname{iso}(H) = \operatorname{free}(H)$.

Proof. Let $\ell = \operatorname{free}(H)$ and $r = \operatorname{iso}(H)$. First, we prove that $\ell \leq r$. In fact, H is isomorphic to an r-monoid F, let K be a free k-submonoid of H with $\operatorname{rank}(K) = \ell$. Then K is isomorphic to some r-submonoid F_0 of F, so that

$$\ell = \operatorname{rank}(K) = \operatorname{rank}(F_0) \leq r,$$

where the last inequality follows by Proposition 2.1.

To prove that $r \leq \ell$, it suffices to show that H is isomorphic to an ℓ-monoid. Since $\ell = \operatorname{free}(H) = \operatorname{rank}(\mathrm{L}_{\mathbb{Q}}(H))$, there exist vectors $\mathbf{v}_1, \ldots, \mathbf{v}_\ell \in \beta(H)$ forming a basis of $\mathrm{L}_{\mathbb{Q}}(H)$. Consider the $\ell \times k$ matrix whose rows are the vectors $\mathbf{v}_j$:

$$A = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{\ell 1} & v_{\ell 2} & \cdots & v_{\ell k} \end{bmatrix}$$

This matrix has rank ℓ, so there are ℓ independent columns over $\mathbb{Q}$, say those indexed by $j_1, \ldots, j_\ell$, and the remaining $k - \ell$ columns can be expressed as linear combinations of these independent columns. This implies that $k - \ell$ coordinates of the vectors $\mathbf{v}_j$ can be written as linear combinations of the remaining ℓ coordinates (those indexed by $j_1, \ldots, j_\ell$). Since $\mathbf{v}_1, \ldots, \mathbf{v}_\ell$ form a basis, the same property holds for all vectors in H. We define a function $\varphi : H \to \mathbb{N}^\ell$, which is in fact a projection, by $\varphi(\mathbf{v}) = (v_{j_1}, \dots, v_{j_\ell})$, for all $\mathbf{v} = (v_1, v_2, \dots, v_k) \in H$. It is clear that φ is additive and $\varphi(\mathbf{0}) = \mathbf{0}$. Note that if $\varphi(\mathbf{v}) = (v_{j_1}, \dots, v_{j_\ell})$, then all coordinates of $\mathbf{v}$ can be uniquely determined as linear combinations of $v_{j_1}, \dots, v_{j_\ell}$; from this fact it is easily seen that φ is injective. Therefore, H is isomorphic to the ℓ-monoid $F = \varphi(H)$, and hence $r \leq \ell$. $\square$

Theorem 3.1 shows that the *obstruction* for a nontrivial k-monoid H to embed into $\mathbb{N}^r$ is the existence of a free k-submonoid of rank $r + 1$.

The remainder of this section is devoted to characterizations of k-monoids having isomorphism rank 1. We begin with the following Lemma.

Lemma 3.2. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{N}^k$. *If* $c\mathbf{u} = d\mathbf{v}$ *for some positive integers* c, d, *then there exist* $\mathbf{w} \in \mathbb{N}^k$ *and positive integers* r, s *such that* $\mathbf{u} = r\mathbf{w}$ *and* $\mathbf{v} = s\mathbf{w}$. *In particular,* $\mathbf{u}, \mathbf{v} \in \langle \mathbf{w} \rangle$.

Proof. Let $\mathbf{u} = (u_1, \dots, u_k)$ and $\mathbf{v} = (v_1, \dots, v_k)$. The equality $c\mathbf{u} = d\mathbf{v}$ means that $cu_i = dv_i$, for all $i \in \{1, \dots, k\}$. We may assume that c and d are coprime. Then, for each $i \in \{1, \dots, k\}$, there exist $u_i', v_i' \in \mathbb{N}$ such that $u_i = du_i'$ and $v_i = cv_i'$. Substituting into the relation above gives $cdu_i' = cu_i = dv_i = dcv_i'$, so $u_i' = v_i'$, for all i. Define $\mathbf{w} = (u_1', \dots, u_k') \in \mathbb{N}^k$. Then we have $\mathbf{u} = d\mathbf{w}$ and $\mathbf{v} = c\mathbf{w}$, which completes the proof. $\square$

It is immediate that if $\mathbf{w} \in \mathbb{N}^k$, then every nontrivial k-submonoid of $\langle \mathbf{w} \rangle$ has isomorphism rank 1. Moreover, we have the following result.

Proposition 3.3. *A nontrivial* k*-monoid* H *has isomorphism rank 1 is and only if there exists* $\mathbf{w} \in \mathbb{N}^k$ *such that* $H \subseteq \langle \mathbf{w} \rangle$.

Proof. We only need to prove that if $\operatorname{iso}(H) = 1$, then there exists $\mathbf{w} \in \mathbb{N}^k$ such that $H \subseteq \langle \mathbf{w} \rangle$.

Assume H is nontrivial. Since H is isomorphic to a 1-monoid, it is finitely generated. Let $\beta(H) = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$, where $r = \operatorname{rank}(H)$. The vector subspace $\mathrm{L}_{\mathbb{Q}}(H)$ has dimension 1, so we may assume that $\{\mathbf{v}_1\}$ generates $\mathrm{L}_{\mathbb{Q}}(H)$. In particular, there exists a fraction c_1/d_1 with $\mathbf{v}_2 = (c_1/d_1)\mathbf{v}_1$, that is $c_1\mathbf{v}_1 = d_1\mathbf{v}_2$. By Lemma 3.2, it follows that there exists $\mathbf{w}_1 \in \mathbb{N}^k$ such that $\mathbf{v}_1, \mathbf{v}_2 \in \langle \mathbf{w}_1 \rangle$.

Also, there exist $c_2, d_2 \in \mathbb{Z}^+$ such that $c_2\mathbf{v}_1 = d_2\mathbf{v}_3$. Since $\mathbf{v}_1 \in \langle \mathbf{w}_1 \rangle$, there exists $e \in \mathbb{Z}^+$ such that $\mathbf{v}_1 = e\mathbf{w}_1$, and hence $(c_2 e)\mathbf{w}_1 = d_2\mathbf{v}_3$. Again, by Lemma 3.2, there exists $\mathbf{w}_2 \in \mathbb{N}^k$ such that $\mathbf{w}_1, \mathbf{v}_3 \in \langle \mathbf{w}_2 \rangle$ and therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \langle \mathbf{w}_2 \rangle$. Proceeding in this way, we obtain $\mathbf{w} = \mathbf{w}_{r-1} \in \mathbb{N}^k$ such that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r \in \langle \mathbf{w} \rangle$. Finally, we have $H \subseteq \langle \mathbf{w} \rangle$. $\square$

The element $\mathbf{w}$ given in Proposition 3.3 is not unique. We now describe how to determine all possible $\mathbf{w}$ such that $H \subseteq \langle \mathbf{w} \rangle$.

Given a nonzero $\mathbf{v} \in \mathbb{N}^k$, we can uniquely write $\mathbf{v} = d\mathbf{v}'$, where $d = \gcd(v_1, \dots, v_k)$ and $\mathbf{v}' = (v_1', \dots, v_k')$ satisfies $\gcd(v_1', \dots, v_k') = 1$. We say that d is the *content* of $\mathbf{v}$ and that $\mathbf{v}'$ is the *primitive part* of $\mathbf{v}$. We denote the content of $\mathbf{v}$ by $\operatorname{cont}(\mathbf{v})$ and its primitive part by $\pi(\mathbf{v})$. We also say that $\mathbf{v}$ is *primitive* if $\operatorname{cont}(\mathbf{v}) = 1$. If H is a nontrivial k-monoid, we define the *content* of H, denoted by $\operatorname{cont}(H)$, as the greatest common divisor of the set $\{\operatorname{cont}(\mathbf{v}) : \mathbf{v} \in H \setminus \{\mathbf{0}\}\}$.

Lemma 3.4. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{N}^k$ *be nonzero. Then* $\mathbf{u}$ *and* $\mathbf{v}$ *have the same primitive part if and only if* $c\mathbf{u} = d\mathbf{v}$ *for some* $c, d \in \mathbb{Z}^+$.

Proof. Suppose $\mathbf{u}$ and $\mathbf{v}$ have the same primitive part and let $\mathbf{w} = \pi(\mathbf{u}) = \pi(\mathbf{v})$. Then $\mathbf{u} = d\mathbf{w}$ and $\mathbf{v} = c\mathbf{w}$ for some positive integers c and d, hence $c\mathbf{u} = d\mathbf{v}$.

Now, if $c\mathbf{u} = d\mathbf{v}$ for positive integers c and d, then, by Lemma 3.2, there exist $r, s \in \mathbb{Z}^+$ and $\mathbf{w} \in \mathbb{N}^k$ such that $\mathbf{u} = r\mathbf{w}$ and $\mathbf{v} = s\mathbf{w}$. Then $\mathbf{u} = (r \cdot \operatorname{cont}(\mathbf{w}))\pi(\mathbf{w})$, hence $\pi(\mathbf{u}) = \pi(\mathbf{w})$. Similarly, $\pi(\mathbf{v}) = \pi(\mathbf{w})$, which shows that $\mathbf{u}$ and $\mathbf{v}$ have the same primitive part. $\square$

Proposition 3.5. *Let* H *be the* k*-monoid generated by the nonzero vectors* $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$. *Then,* $\operatorname{iso}(H) = 1$ *if and only if* $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ *all have the same primitive part.*

Proof. If $\operatorname{iso}(H) = 1$, then by Proposition 3.3, $H \subseteq \langle \mathbf{w} \rangle$ for some $\mathbf{w} \in \mathbb{N}^k$. Clearly, $\mathbf{w} \neq \mathbf{0}$. For each $i \in \{1, \dots, r\}$ we have $\mathbf{v}_i = d_i \mathbf{w}$ for some $d_i \in \mathbb{Z}^+$. By Lemma 3.4, $\mathbf{v}_i$ and $\mathbf{w}$ have the same primitive part, so all the $\mathbf{v}_i$'s have the same primitive part, namely $\pi(\mathbf{w})$.

Conversely, if all the $\mathbf{v}_i$'s have the same primitive part, say $\mathbf{w}$, then $H \subseteq \langle \mathbf{w} \rangle$. By Proposition 3.3, it follows that $\operatorname{iso}(H) = 1$. ☐

By Proposition 3.3, if H is a k-monoid and $\operatorname{iso}(H) = 1$, then there exists a nonzero $\mathbf{w} \in \mathbb{N}^k$ such that $H \subseteq \langle \mathbf{w} \rangle$. Therefore, $H \subseteq \langle \mathbf{w}' \rangle$, where $\mathbf{w}'$ is the primitive part of $\mathbf{w}$. We now show that this primitive vector $\mathbf{w}'$ is unique.

Proposition 3.6. *Let H be a nontrivial k-monoid such that* $\operatorname{iso}(H) = 1$. *Then there exists a unique primitive $\mathbf{w}' \in \mathbb{N}^k$ such that $H \subseteq \langle \mathbf{w}' \rangle$. Moreover, if $\mathbf{w} \in \mathbb{N}^k$ satisfies $H \subseteq \langle \mathbf{w} \rangle$, then $\mathbf{w} = c\mathbf{w}'$, for some $c \in \mathbb{Z}^+$.*

Proof. Let us suppose that $H \subseteq \langle \mathbf{w}' \rangle$ and $H \subseteq \langle \mathbf{w}'' \rangle$, where $\mathbf{w}', \mathbf{w}'' \in \mathbb{N}^k$ are primitive. Let $\mathbf{v} \in H \setminus \{\mathbf{0}\}$. Then there exist $c', c'' \in \mathbb{Z}^+$ such that $\mathbf{v} = c'\mathbf{w}'$ and $\mathbf{v} = c''\mathbf{w}''$. It follows that $c''\mathbf{w}' = c'\mathbf{w}''$, so by Lemma 3.4, $\mathbf{w}'$ and $\mathbf{w}''$ have the same primitive part. Since $\mathbf{w}'$ and $\mathbf{w}''$ are primitive, we obtain $\mathbf{w}' = \mathbf{w}''$.

Now, suppose $H \subseteq \langle \mathbf{w} \rangle$ and $H \subseteq \langle \mathbf{w}' \rangle$, where $\mathbf{w}'$ is primitive. Let $\mathbf{v} \in H \setminus \{\mathbf{0}\}$. Then $\mathbf{v} = c_1\mathbf{w}$ and $\mathbf{v} = c_2\mathbf{w}'$, where $c_1, c_2 \in \mathbb{Z}^+$. It follows that $c_1\mathbf{w} = c_2\mathbf{w}'$ and, again by Lemma 3.4, we obtain that $\mathbf{w}$ and $\mathbf{w}'$ have the same primitive part. Since $\mathbf{w}'$ is primitive, it follows that $\pi(\mathbf{w}) = \mathbf{w}'$. Therefore, $\mathbf{w} = c\mathbf{w}'$, where $c = \operatorname{cont}(\mathbf{w})$. ☐

Let H be a k-monoid generated by nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ that have the same primitive part $\mathbf{w}'$. Let us define a function $\varphi : H \to \mathbb{N}$ by $\varphi(\mathbf{0}) = 0$ and $\varphi(\mathbf{v}) = \operatorname{cont}(\mathbf{v})$ for all $\mathbf{v} \in H \setminus \{\mathbf{0}\}$. Then φ is additive and injective, so H is isomorphic to the 1-monoid $\langle d_1, d_2, \dots, d_r \rangle$, where $d_i = \operatorname{cont}(\mathbf{v}_i)$, $i = 1, 2, \dots, r$.

Let $d = \gcd(d_1, d_2, \dots, d_r)$. It is easy to see that $d = \operatorname{cont}(H)$. If $H \subseteq \langle \mathbf{w} \rangle$ for some $\mathbf{w} \in \mathbb{N}^k$, then $\mathbf{w} = c\mathbf{w}'$ for some factor c of d. In fact, by Proposition 3.6, we can write $\mathbf{w} = c\mathbf{w}'$ for some positive integer c. Now, if $\mathbf{v} \in H$ is nonzero, then $\mathbf{v} = e\mathbf{w} = (ce)\mathbf{w}'$ for some positive integer e, so $\operatorname{cont}(\mathbf{v}) = ce$, which shows that c divides $\operatorname{cont}(\mathbf{v})$ for all nonzero $\mathbf{v} \in H$. Thus, c divides $\operatorname{cont}(H)$. We summarize these observations in the following result.

Proposition 3.7. *Let H be a nontrivial k-monoid such that* $\operatorname{iso}(H) = 1$. *Then H is isomorphic to the 1-monoid* $\{\operatorname{cont}(\mathbf{v}) : \mathbf{v} \in H \setminus \{\mathbf{0}\}\} \cup \{0\}$. *Moreover, if $\mathbf{w}'$ is primitive and $H \subseteq \langle \mathbf{w}' \rangle$, then $H \subseteq \langle \mathbf{w} \rangle$ for $\mathbf{w} \in \mathbb{N}^k$ if and only if $\mathbf{w} = c\mathbf{w}'$ for some $c \mid \operatorname{cont}(H)$. In particular, there are finitely many such $\mathbf{w}$, and more precisely, their number equals the number of positive divisors of* $\operatorname{cont}(H)$.

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