Graduate J. Math. 10, Issue 2 (2025), 46 – 54

Roman domination in upper deg-centric graphs

Timmy Tomy Thalavayalil

Abstract

The upper deg-centric graph of a simple, connected graph G, denoted by G_{ud}, is a graph constructed from G such that $V(G_{ud}) = V(G)$ and $E(G_{ud}) = \{v_i v_j : d_G(v_i, v_j) \geq \deg_G(v_i)\}$. We investigate the properties and structural characteristics of these graphs, and compute their Roman domination number for a variety of examples.

Keywords: Distance, deg-centric graph, upper deg-centric graph, domination, Roman domination.

MSC 2020: 05C15.

1 Introduction

A dominating set in a graph G with vertex set $V(G)$ is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set.

A *Roman Dominating Function* (RDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The value $\omega(f) = \sum_{v \in V} f(v)$ is called the weight of f. The least value of $\omega(f)$ among all the Roman dominating functions f on G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. A Roman dominating function f with $\omega(f) = \gamma_R(G)$ is called a γ_R-function of G.

Roman dominating functions and their variants have been in the literature for over two decades [1–4]. Cockayne et al. [3] was the first to mathematically formulate the concept of Roman dominating functions in graphs based on the defense strategy of Roman Emperor Constantine that was mentioned in the work of Ian Stewart [5].

Motivated by the above-mentioned studies, we investigate the Roman domination and some properties of upper deg-centric graphs. The *degree centric graph* or *deg-centric graph* of a graph G is the graph G_d with $V(G_d) = V(G)$ and $E(G_d) = \{v_i v_j : d_G(v_i, v_j) \leq \deg_G(v_i)\}$, where $d_G(v_i, v_j)$ is the distance between two distinct vertices v_i and v_j of G, that is the length of the shortest path joining them [7]. The notion of an *upper degree centric graph*, on the other hand, is defined in our earlier work.

Definition 1.1. [8] The *upper degree centric graph* or *upper deg-centric graph* of a graph G, denoted by G_{ud}, is the graph with $V(G_{ud}) = V(G)$ and

$$E(G_{ud}) = \{v_i v_j : d_G(v_i, v_j) \geq \deg_G(v_i)\}$$

This graph transformation is called *upper deg-centrication* of the graph.

In this paper, we determine the Roman domination number γ_R for the upper degree centric graphs of path graphs, cyclic graphs, star graphs, bistar graphs, complete bipartite graphs and some wheel helm, web and flower graphs. Our results form a set of exploratory results that establish foundational understanding for further research in this area, and should lead to the study of various graph-theoretical parameters of upper deg-centric graphs belonging to different graph classes. One can also explore other forms of graph domination within the framework of upper deg-centric structures.

2 Preliminaries

We start by collecting key lemmas and propositions used in deriving our results.

For the basic terminology of graph theory, we refer to [11]. A graph is assumed to be a simple, connected, and undirected graph throughout this paper. The number of edges of a graph G is denoted by $\varepsilon(G)$. Recall that the distance between two distinct vertices v_i and v_j of G, denoted by $d_G(v_i, v_j)$, is the length of the shortest path joining them. The eccentricity of a vertex $v_i \in V(G)$, denoted by $e(v_i)$, is the farthest distance from v_i to some vertex of G.

Although this paper considers connected graphs, the convention is that in a disconnected graph, upper deg-centrication is applied componentwise.

We remark that any function $f : V \to \{0, 1, 2\}$ on a graph induces an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. There is a one-one correspondence between these functions and these ordered partitions, thus, we can unambiguously write such a function as $f = (V_0, V_1, V_2)$.

Lemma 2.1. *[8] The upper deg-centric graph of a graph G is an empty graph if and only if $\delta(G) > diam(G)$.*

Proposition 2.2. *[3] For any graph G of order n, $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.*

Lemma 2.3. *If for any graph G of order n with minimum degree, $\delta(G) > diam(G)$, then*

$$\gamma_R(G_{ud}) = n$$

Proof. Consider a graph G of order n with minimum degree, $\delta(G) > diam(G)$, then in view of Lemma 2.1, $G_{ud} \cong \overline{K}_n$. That is, an upper deg-centric graph is an independent set with n vertices. Hence, $\gamma_R(G_{ud}) = n$. □

Proposition 2.4. *If G is a graph of order $n \geq 3$ which contains a vertex of degree $n - 1$, then $\gamma_R(G_{ud}) = 2$.*

Proof. Let vertex v of degree $n - 1$ in a graph G. Then, in view of Definition 1.1, in an upper deg-centric graph, the vertex v is adjacent with all $n-1$ vertices in graph G. In Roman domination, we can assign $f(v) = 2$, and all other $n - 1$ adjacent vertices assign the value zero. The least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((G)_{ud}) = 2$. □

Proposition 2.5. *For any upper deg-centric graph G_{ud} of order n, $\gamma(G_{ud}) = \gamma_R(G_{ud})$ if and only if $G = \overline{K_n}$.*

Proof. The result is a direct consequence of Proposition 2.2 and Definition 1.1. □

Proposition 2.6. *For a complete graph K_n, $n \geq 3$, $\gamma_R((K_n)_{ud}) = n$.*

Proof. For a complete graph K_n, $n \geq 3$, $\delta(K_n) > \underline{diam}(K_n)$, the upper deg-centric graph of a complete graph K_n of order $n \neq 2$ is an empty graph $\overline{K}_n$. In views of Lemma 2.3, $\gamma_R((K_n)_{ud}) = n$. If $n = 2$, the result is true directly by Definition 1.1. □

3 Roman Domination Number of Upper Deg-centric Graphs

For convenience, a path P_n is depicted on a horizontal line, and the vertices are labelled from left to right as v_1, v_2, v_3,…, v_n.

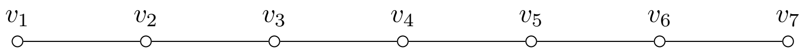
Proposition 3.1. *For any path P_n,*

$$\gamma_R((P_n)_{ud}) = \begin{cases} 1 & \text{if } n = 1; \\ 2 & \text{if } n \geq 2. \end{cases}$$

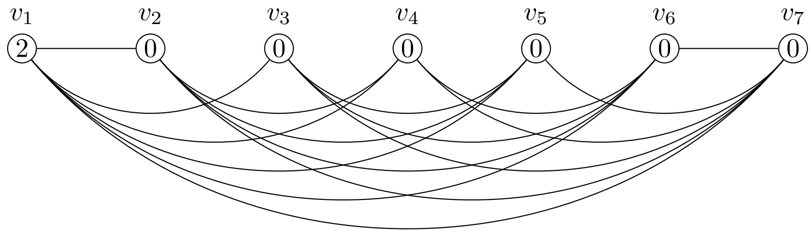
Proof. Let the functions $f : V \rightarrow \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. Consider the upper deg-centric graph P_1, clearly $\gamma_R((P_1)_{ud}) = 1$.

Consider the upper deg-centric graph $P_n, n \geq 2$, with consecutively labeled vertices $\{v_1, v_2, v_3, \dots, v_n\}$. In views of Definition 1.1 and Proposition 2.4, vertices v_1 and v_n are adjacent with all other $n-1$ vertices in $(P_n)_{ud}$. That is, the vertex v_1 is adjacent to vertices $\{v_2, v_3, v_4 \dots, v_n\}$ in $(P_n)_{ud}$. Since $f(v_1) = 2$ is assigned, then in the Roman domination, we have to assign the values $f(v_2) = 0, f(v_3) = 0, \dots, f(v_n) = 0$. The least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((P_n)_{ud}) = 2$. □

An illustration of Proposition 3.1 is given in Figure 1.



(a) P_7



(b) $(P_7)_{ud}$

Fig. 1 $\gamma_R((P_7)_{ud}) = 2$.

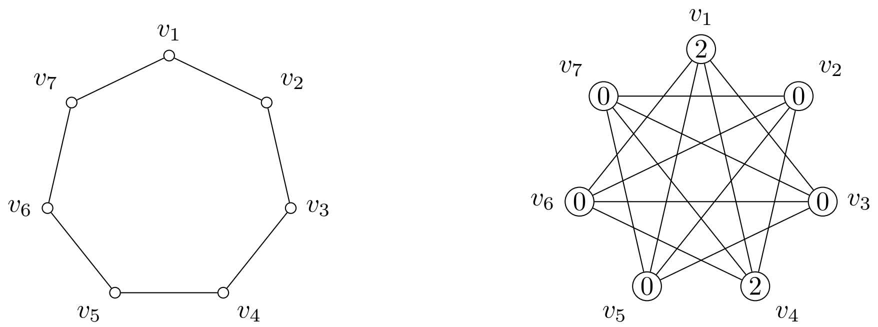
Proposition 3.2. *For any cycle graph C_n, $n \geq 3$,*

$$\gamma_R((C_n)_{ud}) = \begin{cases} 3 & \text{if } n = 3; \\ 4 & \text{if } n \geq 4. \end{cases}$$

Proof. Consider the upper deg-centric graph of C_n, $n = 3$, clearly $(C_3)_{ud}$ is disconnected and is the empty graph. That is ,$\gamma_R((C_3)_{ud}) = 3$. If $n = 4$, clearly $(C_4)_{ud}$ is a disjoint union of two K_2. That is, $\gamma_R((C_4)_{ud}) = 4$.

Consider the upper deg-centric graph $C_n, n \geq 5$, with consecutively labeled vertices $\{v_1, \dots, v_n\}$. In view of Definition 1.1, $\deg_{C_n}(v_i) = 2$, for all $v_i \in V(C_n)$, any vertex v_i in $(C_n)_{ud}$ is adjacent to all vertices in $V(C_n) \backslash N_{C_n}[v_i]$. It immediately follows that $(C_n)_{ud}$ is always a $(n-3)$-regular graph. Consider the Roman domination of $(C_n)_{ud}$, the vertices $\{v_1, v_2, v_3, \dots, v_n\}$, assigned the values $f(v_1) = 2$, then the vertex set $\{v_3, v_4, v_5, \dots, v_{n-1}\}$ are adjacent to v_1, and we can assign the values $f(v_3) = 0, f(v_4) = 0, \dots, f(v_{n-1}) = 0$. Then, consider the vertex $v_{\lceil \frac{n}{2} \rceil}$, is adjacent to the $n-3$ vertices, we can assign the values $f(v_{\lceil \frac{n}{2} \rceil}) = 2$ and all adjacent vertices assign value as zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2 + 2 = 4$. Hence, $\gamma_R((C_n)_{ud}) = 4$. □

An illustration of Proposition 3.2 is given in Figure 2.



(a) C_7 (b) $(C_7)_{ud}$

Fig. 2 $\gamma_R((C_7)_{ud}) = 4$.

A *star graph*, denoted by $K_{1,n}$, $n \geq 0$, is obtained by attaching n pendant vertices (also called leaves) to a central vertex v_0.

Proposition 3.3. *For a star graph $K_{1,n}$, $n \geq 0$, we hae*

$$\gamma_R((K_{1,n})_{ud}) = \begin{cases} 1 & \text{if } n = 0; \\ 2 & \text{if } n \geq 1. \end{cases}$$

Proof. The star graph $K_{1,n,}$, $n \geq 0$, is of the order $n+1$. Let $V(K_{1,n,}) = \{v_0, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. Let the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. If $n = 0$, then isolated vertex, $f(v) = 1$, $\gamma_R((K_1)_{ud}) = 1$.

Consider the upper deg-centric graph $K_{1,n}, n \geq 1$, with consecutively labeled vertices $\{v_0, \ldots, v_n\}$. Then, by Definition 1.1, the upper deg-centric graph is the complete graph K_{n+1}. Now, in view of Proposition 2.4, the Roman domination of $(K_{1,n})_{ud}$, the vertices $\{v_0, v_1, v_2, \ldots, v_n\}$, assigned the values $f(v_i) = 2$, the all other n vertices are adjacent with v_i, assign value as zero all these vertices. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((K_{1,n})_{ud}) = 2$. □

A non-trivial *bistar graph*, denoted by $S_{a,b}$, is a graph obtained by joining the centers of two non-trivial star graphs $K_{1,a}$, $a \geq 1$ and $K_{1,b}$, $b \geq 1$ with the edge v_0u_0.

Proposition 3.4. *For a bistar graph $S_{a,b}$, $a, b \geq 1$, $\gamma_R((S_{a,b})_{ud}) = 2$.*

Proof. The bistar graph $S_{a,b}$, $a, b \geq 1$. Let the pendant vertices of $K_{1,a}$ be the set $X = \{v_1, v_2, \ldots, v_a\}$ and let the pendant vertices of $K_{1,b}$ be the set $Y = \{u_1, u_2, \ldots, u_b\}$. Finally let $W = \{v_0, u_0\}$. From Definition 1.1, all pendant vertices of $S_{a,b}$ will be adjacent to all other vertices in the upper deg-centric graph,$(S_{a,b})_{ud}$. Also, the central vertices of $S_{a,b}$ cannot be adjacent to each other in the upper deg-centric graph, since they are at a distance of one and their degree is greater than one. Therefore, the upper deg-centric graph of $S_{a,b}$ is isomorphic to $K_{a+b+2} - \{u_0, v_0\}$ [8]. Then, in view of Proposition 2.4, assign $f(v_i) = 2$, any of the vertices, and all other adjacent vertices of values zero, Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((S_{a,b})_{ud}) = 2$. □

Recall that a complete bipartite graph $K_{n,m}$, $n, m \geq 1$ is a graph whose vertex set can be partitioned into two independent sets X, $|X|= n$ and Y, $|Y|= m$ and each vertex in X is adjacent to all vertices in Y.

Proposition 3.5. *For a complete bipartite graph $K_{n,m}$, $n, m \geq 3$, we have*

$$\gamma_R((K_{n,m})_{ud}) = n + m$$

Proof. In view of Lemma 2.1, the upper deg-centric graph of a complete bipartite graph $K_{n,m}$, $n, m \geq 3$ is the empty graph $\overline{K}_{n+m}$. Then direct consequence of Lemma 2.3, $\gamma_R((K_{n,m})_{ud}) = n + m$. □

Proposition 3.6. *For a complete bipartite graph $K_{2,m}$, $m \geq 3$, $\gamma_R((K_{2,m})_{ud}) = 4$.*

Proof. The complete bipartite graph $K_{2,m}$, $m \geq 3$, in view of Definition 1.1, the upper deg-centric graph is the disjoint union of the empty graph $\overline{K}_2$ and the complete graph K_m. Let the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. Then, in Roman domination, in the case of an empty graph $\overline{K_2}$, we can assign each vertex as a value of one. Similarly, in K_m, assign the value of any vertex v_i as two and the remaining $m - 1$ adjacent vertices as the value of zero. Then, after summation, the least value of $\omega(f) = \sum_{v\in V} f(v) = 2 + 1 + 1 = 4$. Hence, $\gamma_R((K_{2,m})_{ud}) = 4$. □

A *wheel graph* denoted by $W_{1,n}$, $n \geq 3$ is obtained by taking a cycle C_n, $n \geq 3$ (the rim with rim-vertices) and adding the central vertex v_0 with *spokes* namely, edges v_0v_i, $1 \leq i \leq n$.

Proposition 3.7. *For a wheel graph $W_{1,n}$, $n \geq 3$, then, $\gamma_R((W_{1,n})_{ud}) = n + 1$.*

Proof. In view of Definition 1.1, the upper deg-centric graph of a wheel graph $W_{1,n}$, $n \geq 3$ is the empty graph $\overline{K}_{n+1}$. Then direct consequence of Lemma 2.3, $\gamma_R((W_{1,n})_{ud}) = n + 1$. □

A *helm graph*, denoted by $H_{1,n,}$, $n \geq 3$ is a graph obtained from a wheel graph $W_{1,n}$ by attaching a pendant vertex u_i to each corresponding *rim vertex* v_i.

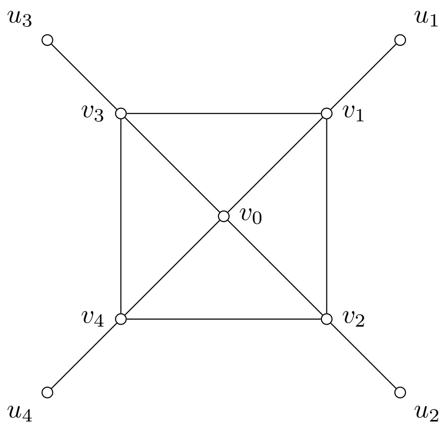
Proposition 3.8. *For a helm graph $H_{1,n}$, $n \geq 3$, $\gamma_R((H_{1,n})_{ud}) = 2$.*

Proof. The helm graph $H_{1,n,}$, $n \geq 3$, is of the order $2n + 1$. Let

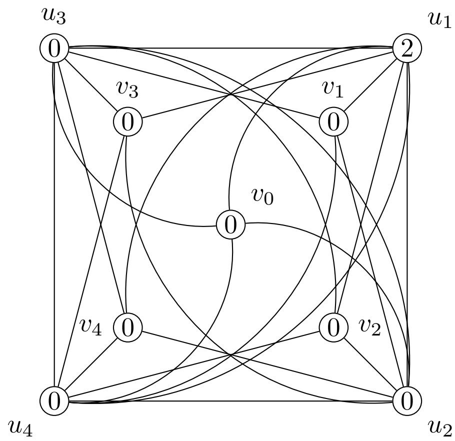
$$V(H_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}} \}$$

Then, by Definition 1.1, $\deg_{G_{cd}}(u_n) = 2n$, $\deg_{G_{cd}}(v_0) = n$, $\deg_{G_{cd}}(v_n) = n$. Let the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. Then, in view of Proposition 2.4, assign $f(u_i) = 2$, any of the vertices, and all other adjacent vertices $2n$ of values zero. Then, the least value of $\omega(f) = \sum_{v\in V} f(v) = 2$. Hence, $\gamma_R((H_{1,n})_{ud}) = 2$. □

An illustration to Proposition 3.8 is given in Figure 3.



(a) $H_{1,4}$



(b) $(CH_{1,4})_{ud}$

Fig. 3 $\gamma_R((H_{1,4})_{ud}) = 2$.

A *closed helm graph* denoted by $CH_{1,n}$, $n \geq 3$ is the graph obtained from a helm graph $H_{1,n}$ by cyclically joining the pendant vertices to form an outer rim.

Proposition 3.9. *For a closed helm graph $CH_{1,n}$, $n \geq 3$,*

$$\gamma_R((CH_{1,n})_{ud}) = \begin{cases} 2n+1 & \text{if } n \leq 4; \\ 9 & \text{if } n = 5; \\ 7 & \text{if } 6 \leq n \leq 8; \\ 6 & \text{if } n = 9; \\ 5 & \text{if } n \geq 10. \end{cases}$$

Proof. The closed helm graph $CH_{1,n}$, $n \geq 3$, is clearly of the order $2n+1$. Let $V(CH_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$, $\deg(v_0) = n$, $\deg(v_i) = 4$ and $\deg(u_i) = 3$. Let the functions $f : V \to \{0,1,2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0,1,2$. Consider the upper deg-centric graph $CH_{1,n}$, $n = 3$, clearly $(CH_{1,3})_{ud}$ is disconnected and is the empty graph $\overline{K}_7$. That is ,$\gamma_R((CH_{1,3})_{ud}) = 7$. If $n = 4$, clearly $(CH_{1,4})_{ud}$ is a disjoint union of four K_2 and one isolated vertex. That is, $\gamma_R((CH_{1,4})_{ud}) = 9$. Hence, $\gamma_R((CH_{1,n})_{ud}) = 2n+1$. If $n = 5$, clearly $(CH_{1,5})_{ud}$ is a disjoint union of the cycle C_5 and one isolated vertex. We know that $\gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$. For the isolated vertex, we can assign value one in Roman domination. That is, $\gamma_R((CH_{1,5})_{ud}) = 2\lceil \frac{2n}{3} \rceil + 1 = 8 + 1 = 9$.

If $6 \leq n \leq 8$, $deg_{G_{ud}}(v_0) = 0$, then the Roman domination assign $f(v_0) = 1$ for the vertex v_0. Also, $deg_{G_{ud}}(v_n) = n-3$,and $\deg_{G_{ud}(u_n)} = 2n-8$ for all n vertices, consecutively labeled outer rim n vertices $u_1, u_2, u_3, \ldots, u_n$. Then, assign value two to the vertices u_1, u_n, and $u_{\lceil \frac{n}{2} \rceil}$. That is, $f(u_1) = f(u_n) = f(u_{\lceil \frac{n}{2} \rceil}) = 2$, then, adjacent to all the vertices, assign the values zero. Then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2+2+2+1 = 7$. Hence, $\gamma_R((CH_{1,n})_{ud}) = 7$. If $n = 9$, $deg_{G_{ud}}(v_0) = 0$, then the Roman domination assign $f(v_0) = 1$ for the vertex v_0. Roman domination, assign that value two, $f(u_1) = f(u_{\lceil \frac{n}{2} \rceil}) = 2$, and all adjacent vertices assign value zero. Then, one vertex v_3 is the only non-assigned vertex; assign value one to that vertex. Then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2+2+1+1 = 6$. Hence, $\gamma_R((CH_{1,9})_{ud}) = 6$.

If $n \geq 10$, $deg_{G_{ud}}(v_0) = 0$, then the Roman domination assign $f(v_0) = 1$ for the vertex v_0. Also, $\deg_{G_{ud}}(v_n) = n-3$,and $\deg_{G_{ud}}(u_n) = 2n-8$ for all n vertices, consecutively labeled outer rim n vertices $u_1, u_2, u_3, \ldots, u_n$. Then, assign $f(u_1) = 2$, then vertex u_1 adjacent to $2n-8$ vertices of u_1, that assign the values zero. Then assign the value two to the $\lceil \frac{n}{2} \rceil$th vertex u_i $f(u_{\lceil \frac{n}{2} \rceil}) = 2$, all other u_i adjacent vertices $f(u_i) = 0$. That is, we assign $f(v_1) = 2$ and $f(u_{\lceil \frac{n}{2} \rceil}) = 2$, all other vertices can assign value zero. Then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2+2+1 = 5$. Hence, $\gamma_R((CH_{1,n})_{ud}) = 5$. □

A *double wheel* DW_n is obtained by taking two copies of a wheel W_n $n \geq 3$ and *merging* the two central vertices.

Proposition 3.10. *For a double wheel graph DW_n, $n \geq 3$, $\gamma_R((DW_n)_{ud}) = 2n+1$.*

Proof. The double wheel graph DW_n, $n \geq 3$, is of the order $2n+1$. Let $V(DW_n) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. In view of Lemma 2.1, the upper deg-centric graph of a double wheel DW_n, $n \geq 3$ is the empty graph $\overline{K}_{2n+1}$. Then direct consequence of Lemma 2.3, $\gamma_R((DW_n)_{ud}) = 2n+1$. □

A *gear graph*, denoted by G_n,$n \geq 3$, is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph W_n.

Proposition 3.11. *For a gear graph G_n, $n \geq 3$, $\gamma_R((G_n)_{ud}) = 4$.*

Proof. For a gear graph G_n, $n \geq 3$ is of the order $2n+1$. Let $V(G_n) = \{v_0, \ldots, v_n, u_1, \ldots, u_n\}$. Since $\deg_{G_n}(v_0) = n \geq 3 > e(v_0) = 2$, no edges formed from v_0 in $(G_n)_{ud}$. However, since $\deg(v_i) = 3$, there are $n-2$ edges incident on any vertex v_i and since $\deg(u_i) = 2$, there are $2n-2$ edges incident on any vertex u_i in $(G_n)_{ud}$. Then, $\deg_(v_i) = n-2$, $\deg(u_i) = 2n-2$ and $\deg(v_0) = n$ in $(G_n)_{ud}$. Then, in Roman domination, assign the value two to any u_i vertex, that is $f(u_i) = 2$, there are $2n-2$ edges incident on any vertex u_i in $(G_n)_{ud}$. We can assign a zero value to all these vertices. Note that two v_i vertices are not adjacent to u_i, which are adjacent in the gear graph. Assign value one to these two vertices. Then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2 + 1 + 1 = 4$. Hence, $\gamma_R((G_n)_{ud}) = 4$. □

A *web graph*, denoted by $Wb_{1,n}$, $n \geq 3$ is the graph obtained by attaching a pendant edge to each vertex of the outer cycle (or rim) of the closed helm graph $CH_{1,n}$.

Proposition 3.12. *For a web graph $Wb_{1,n}$, $n \geq 3$, $\gamma_R((Wb_{1,n})_{ud}) = 2$.*

Proof. The web graph $Wb_{1,n}$, $n \geq 3$, is of the order $3n + 1$. Let

$$V(Wb_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_{n-1}, v_n, u_1, u_2, u_3, \ldots, u_n, \underbrace{w_1, w_2, w_3, \ldots, w_n}_{\text{pendant vertices}}\}$$

. Then, by Definition 1.1, $\deg_{G_{ud}(w_n)} = 3n$, $\deg_{G_{ud}(u_n)} = n$, $\deg_{G_{ud}(v_0)} = n$, $\deg_{G_{ud}(v_n)} = n$. Let the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. Then, in view of Proposition 2.4, assign $f(w_i) = 2$, any of the vertices, and all other $3n$ adjacent vertices of values zero. Then, the least value of $\omega(f) = \sum_{v,u,w \in V} f(v) + f(u) + f(w) = 2$. Hence, $\gamma_R((Wb_{1,n})_{ud}) = 2$. □

A *flower graph*, $F_{1,n}$, $n \geq 3$ is a graph obtained from a helm graph $H_{1,n,}$ by joining each of its pendant vertices u_i's to its central vertex v_0.

Proposition 3.13. *For a flower graph $F_{1,n}$, $n \geq 3$, $\gamma_R((F_{1,n})_{ud}) = 4$.*

Proof. Consider a flower graph $F_{1,n,}$, $n \geq 3$, is of the order $2n+1$. Let $V(F_{1,n,}) = \{v_0, \ldots, v_n, u_1, \ldots, u_n\}$. In view of Definition 1.1, $\deg_{F_{1,n}}(v_0) = n > e(v_0) = 2$, no edges incident at v_0. That is, $\deg(v_0) = 0$ in $(F_{1,n})_{ud}$, in Roman domination, assign $f(v_0) = 1$. However, since $deg(u_i) = 2$ in $F_{1,n}$, each u_i forms the edge with the distance of two or more vertices from u_i. Then, by Definition 1.1, the n vertices $u_1, u_2, \ldots, u_n$ are adjacent with $2n-2$ vertices that are, $\deg(u_n) = 2n-2$ in $(F_{1,n})_{ud}$, assign vertex u_i as value two, that is, $f(u_i) = 2$, and $2n - 2$ adjacent vertices assign the value zero. Now in $(F_{1,n})_{ud}$, only one vertex v_i is non-assigned the value, that v_i is adjacent in u_i in the flower graph. Then, vertex v_i assign the value one, $f(v_i) = 1$. Then, the least value of $\omega(f) = \sum_{v,u \in V} f(v) + f(u) = 2 + 1 + 1$. Hence, $\gamma_R((F_{1,n})_{ud}) = 4$. □

The *sunflower graph*, denoted by $SF_{1,n}$, $n \geq 3$ is obtained from the wheel $W_{1,n}$ by attaching n vertices u_i, $1 \leq i \leq n$ such that each u_i is adjacent to v_i and v_{i+1} and count the suffix is taken modulo n.

Proposition 3.14. *For a sunflower graph $SF_{1,n}$, $n \geq 3$, $\gamma_R((SF_{1,n})_{ud}) = 4$.*

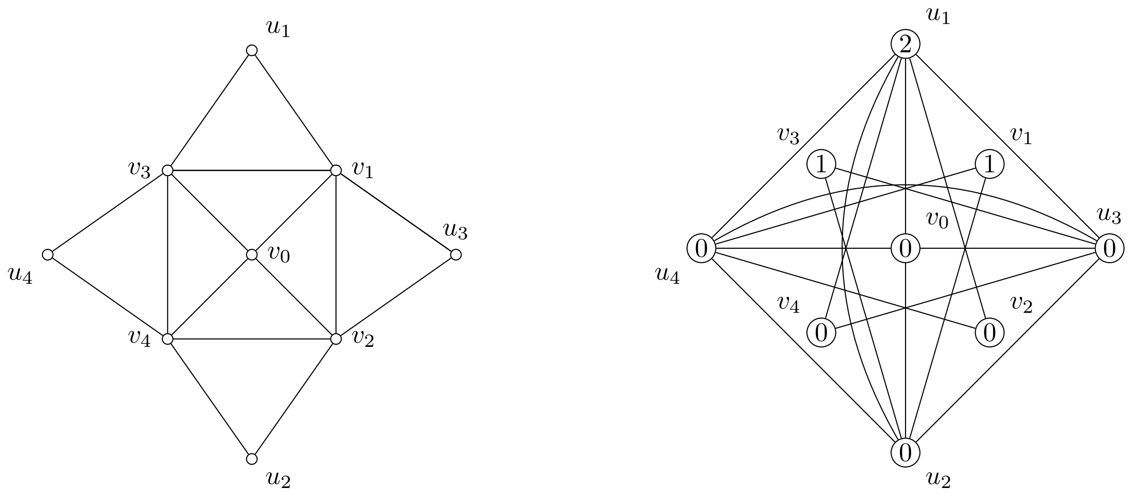
Proof. The sunflower graph $SF_{1,n,}$, $n \geq 3$, is of the order $2n+1$. Let $V(SF_{1,n,}) = \{v_0, \ldots, v_n, u_1, \ldots, u_n\}$ as mentioned in the definition. By Definition 1.1, in the center vertex v_0 is adjacent with all u_i that is $\deg(v_0) = n$ in $(SF_{1,n})_{ud}$ and $v_1, v_2, v_3 \ldots v_n$ are adjacent with $n-2$ vertices hence, $\deg(v_n) = n-2$ in $(SF_{1,n})_{ud}$. Also, the n vertices $u_1, u_2, \ldots, u_n$ are adjacent with all other $2n - 2$ vertices that are $\deg(u_n) = 2n - 2$ in $(SF_{1,n})_{ud}$, all u_n vertices are adjacent each other. In Roman domination, assign any vertex u_i as value two, that is, $f(u_i) = 2$, and $2n - 2$ adjacent vertices assign the value zero. Now, in $(SF_{1,n})_{ud}$, two vertices v_i is non-assigned the value, that vertices v_i is adjacent in u_i in the sunflower graph. Then, these vertices v_i assign the value one, $f(v_i) = 1$. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2 + 1 + 1$. Hence, $\gamma_R((SF_{1,n})_{ud}) = 4$. □

An illustration to Proposition 3.14 is given in Figure 4.

A *closed sunflower graph* $CSF_{1,n}$ is obtained by adding the edge u_iu_{i+1} of the sunflower graph.

Proposition 3.15. *For a closed sunflower graph $CSF_{1,n}$, $n \geq 3$,*

$$\gamma_R((CSF_{1,n})_{ud}) = \begin{cases} 2n + 1 & \text{if } n \leq 8; \\ n + 7 & \text{if } 9 \leq n \leq 12; \\ n + 6 & \text{if } n = 13; \\ n + 5 & \text{if } n \geq 14. \end{cases}$$



(a) $SF_{1,4}$ (b) $(SF_{1,4})_{ud}$

Fig. 4 $\gamma_R((SF_{1,4})_{ud}) = 4$.

Proof. The closed sunflower graph $CSF_{1,n}$,$n \geq 3$, is clearly of the order $2n+1$. Let $V(CSF_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$, $\deg(v_0) = n$,$\deg(v_i) = 5$ and $\deg(u_i) = 4$. Let the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. In $CH_{1,n}$, $n \leq 7$, clearly $(CSF_{1,n})_{ud}$ is disconnected and is the empty graph $\overline{K}_{2n+1}$. That is, $\gamma_R((CSF_{1,n})_{ud}) = 2n+1$. If $n = 8$, clearly $(CSF_{1,8})_{ud}$ is a disjoint union of four k_2 and eight isolated vertices. That is, $\gamma_R((CSF_{1,8})_{ud}) = 2n+1$.

If $9 \leq n \leq 12$, $\deg_{G_{ud}}(v_0) = 0$, $\deg_{G_{ud}}(v_n) = 0$, then the Roman domination assign $f(v_i) = 1$ for these $n+1$ vertices. Also, $deg_{G_{ud}}(u_n) = n-7$ for all n vertices, consecutively labeled these n vertices $u_1, u_2, u_3, \ldots, u_n$. Then, assign $f(u_1) = 2$, then vertex u_1 adjacent to $n-7$ vertices of u_1, that assign the values zero that is,$f(u_i) = 0$. Then assign the values $f(u_4) = 2, f(u_2) = 1$, $f(u_3) = 1$ and, all other u_i vertices $f(u_i) = 0$. Then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = n+1+2+2+1+1 = n+7$. Hence, $\gamma_R((CSF_{1,n})_{ud}) = n+7$.

Similarly, if $n = 13$, then the Roman domination assign $f(v_i) = 1$ for these $n+1$ vertices. $\deg_{G_{ud}}(u_n) = 6$ for all thirteen vertices, consecutively labeled these thirteen vertices $u_1, u_2, u_3, \ldots, u_{13}$. Then, assign $f(u_1) = 2$, then vertex u_1 adjacent to six vertices $\{u_5, u_6, u_7, u_8, u_9, u_{10}\}$ of u_1, that assign the values zero. Then assign the value $f(u_7) = 2$, that is $\lceil \frac{n}{2} \rceil$ vertex and, all six adjacent vertices $\{u_3, u_2, u_1, u_{13}, u_{12}, u_{11}\}$ of u_7 assign value zero. Then, the remaining vertex u_4, assigns the value of one; then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = 13+1+2+2+1 = 19$. Hence, $\gamma_R((CSF_{1,13})_{ud}) = n+6$.

If $n \geq 14$, $\deg_{G_{ud}}(v_0) = 0$, $\deg_{G_{ud}}(v_n) = 0$, then the Roman domination assign $f(v_i) = 1$ for these $n+1$ vertices. Also, $\deg_{G_{ud}(u_n)} = n-7$ for all n vertices, consecutively labeled these n vertices $u_1, u_2, u_3, \ldots, u_n$. Then, assign $f(u_1) = 2$, then vertex u_1 adjacent to $n-7$ vertices of u_1, that assign the values zero that is,$f(u_i) = 0$. Then assign the value two to the $\lceil \frac{n}{2} \rceil$th vertex u_i $f(u_{\lceil \frac{n}{2} \rceil}) = 2$, all other u_i adjacent vertices $f(u_i) = 0$. That is, we assign $f(v_1) = 2$ and $f(u_{\lceil \frac{n}{2} \rceil}) = 2$, all other vertices can assign value zero. Then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = n+1+2+2 = n+5$. Hence, $\gamma_R((CSF_{1,n})_{ud}) = n+5$. □

A *blossom graph*, denoted by $Bl_{1,n}$, is obtained by making each u_i adjacent to the central vertex of the closed sunflower graph.

Proposition 3.16. *For a blossom graph $Bl_{1,n}$, $n \geq 3$, $\gamma_R((Bl_{1,n})_{ud}) = 2n+1$.*

Proof. In view of Lemma 2.1, the upper deg-centric graph of a Blossom graph $Bl_{1,n}$, $n \geq 3$ is the empty graph $\overline{K}_{2n+1}$. Then direct consequence of Lemma 2.3, $\gamma_R((Bl_{1,n})_{ud}) = 2n+1$. □

A *sunlet graph* , denoted by Sl_n, $n \geq 3$, is a graph obtained by attaching a pendant vertex to every vertex of a cycle graph c_n, $n \geq 3$. In other words, a sunlet graph on $2n$ vertices is obtained by taking the corona product $C_n \circ K_1$.

Proposition 3.17. *For a sunlet graph Sl_n, $n \geq 3$, $\gamma_R((Sl_n)_{ud}) = 2$.*

Proof. For a sunlet graph Sl_n, $n \geq 3$. The sunlet graph is of the order $2n$. Let $V(Sl_n) = \{v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. Since all u_i are pendant vertices, each u_i forms the edge u_iv_i. In view of Definition 1.1, the n pendant vertices $u_1, u_2, \ldots, u_n$ are adjacent to all other $2n-1$ vertices that is $\deg(u_n) = 2n-1$ in $(Sl_n)_{ud}$. Let the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. Then, in view of Proposition 2.4, assign $f(u_i) = 2$, any of the vertices, and all other $2n-1$ adjacent vertices of values zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((Sl_n)_{ud}) = 2$. □

A *djembe graph*, denoted by $D_{1,n}$, is obtained by joining the vertices $u_i's$; $1 \leq i \leq n$ of a closed helm graph $CH_{1,n}$ to its central vertex v_0.

Proposition 3.18. *For a upper deg-centric graph of a djembe graph $(D_{1,n})_{ud}$, $n \geq 3$,*

$$\gamma_R((D_{1,n})_{ud}) = 2n + 1.$$

Proof. In view of Definition 1.1, the upper deg-centric graph of a graph $(D_{1,n})_{ud}$, $n \geq 3$, is the empty graph $\overline{K}_{2n+1}$. Then direct consequence of Lemma 2.3, $\gamma_R((D_{1,n})_{ud}) = 2n + 1$. □

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Timmy Tomy Thalavayalil

Department of Mathematics
Kristu Jayanti University, Bengaluru-560077, INDIA

E-mail address: timmy@kristujayanti.com