

Entropy formula for C^1 expanding maps



ELEONORA CATSIGERAS AND FERNANDO VALENZUELA GÓMEZ

Abstract

We establish that the pseudo-physical or SRB-like measures—the existence of which is guaranteed—of C^1 expanding dynamical systems on a compact Riemannian manifold satisfy Pesin’s entropy formula. Furthermore, we provide explicit examples of $C^1 \setminus C^{1+\alpha}$ expanding maps on the circle and on the 2-torus, and investigate their pseudo-physical measures.

Keywords: Equilibrium states, pseudo-physical/SRB-like measures, Pesin’s entropy formula, expanding maps.

MSC 2020. Primary: 37D20; Secondary: 37D35, 37A35, 37A60.

1 Introduction

We consider the dynamical system generated by the iteration of a C^1 map $f : M \rightarrow M$ on a finite-dimensional compact Riemannian manifold M . A central objective in ergodic theory is the search for “natural” invariant measures that describe the statistical behavior of an observable set of orbits. Typically, the observability of a set of orbits is associated with its volume; specifically, a set of orbits is considered observable if it has a positive Lebesgue measure. This requirement is particularly restrictive when the map is not Lebesgue-preserving. If the system does not preserve volume, the ergodicity of an invariant measure alone does not guarantee that the measure is “natural”. Indeed, while an ergodic measure μ describes the statistical behavior of μ -almost all orbits, the set of such orbits may have zero volume if μ is mutually singular with respect to the Lebesgue measure.

1.1 Physical and pseudo-physical measures.

We denote by $\mathcal{M}$ the set of Borel probability measures on M , endowed with the weak*-topology (see, for example, [36]). The weak* topology on $\mathcal{M}$ is characterized by the following condition:

$$\lim_{n \rightarrow +\infty} \int \phi d\mu_n = \int \phi d\mu \text{ for every continuous function } \phi : M \rightarrow \mathbb{R}.$$

We denote by $\mathcal{M}_f \subset \mathcal{M}$ the subset of f -invariant measures, i.e., $f^*\mu = \mu$, where $f^* : \mathcal{M} \rightarrow \mathcal{M}$ is the pull-back operator acting on the measures, defined by $f^*\mu(B) = \mu(f^{-1}(B))$ for any Borel-measurable set B .

Following [22], a measure $\mu \in \mathcal{M}_f$ is called *natural* if it satisfies

$$\mu = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} (f^*)^j \nu,$$

for some (not necessarily invariant) Borel probability measure ν that is absolutely continuous with respect to the Lebesgue measure. The primary limitation of this definition of natural measures is that such measures do not always exist.

From a statistical point of view on a positive-volume set of orbits, one of the most relevant and useful characterizations of an invariant measure is the property of being “physical”. This property is defined below:

Definition 1.1 (Empirical probabilities and basin of statistical attraction). For any initial point $x \in M$, the *empirical probability measure* $\sigma_n(x) \in \mathcal{M}$ up to time n of the orbit of x is defined as

$$\sigma_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)},$$

where δ_y denotes the Dirac delta probability measure supported at y . That is, the empirical probability is uniformly supported on the points of the finite segment of the orbit from x to $f^{n-1}(x)$.

For any point $x \in M$, the sequence $\{\sigma_n(x)\}_{n \geq 1} \subset \mathcal{M}$ of empirical probabilities possesses convergent subsequences, as $\mathcal{M}$ is a compact space under the weak*-topology. The set of probability measures that arise as weak* limits of the convergent subsequences of $\{\sigma_n(x)\}_{n \geq 1}$ is called the *p-omega limit* of the orbit of x (i.e., the omega-limit set in the space of probabilities) and is denoted by $p\omega(x)$. More precisely,

$$p\omega(x) = \{\nu \in \mathcal{M} : \nu = \lim_{j \rightarrow +\infty}^* \sigma_{n_j}(x) \text{ for some convergent subsequence } \{\sigma_{n_j}(x)\}_{j \geq 1}\}.$$

For any invariant measure μ , the *basin* $B(\mu)$ of *statistical attraction* of μ is the set

$$B(\mu) := \{x \in M : \mu = \lim_{n \rightarrow +\infty}^* \sigma_n(x)\} = \{x \in M : p\omega(x) = \{\mu\}\}.$$

Definition 1.2 (Physical measures). We call an invariant probability measure μ *physical* if its basin of statistical attraction has positive Lebesgue measure, i.e.

$$m(B(\mu)) > 0,$$

where m denotes the Lebesgue measure.

Physical measures are also known as Sinai–Ruelle–Bowen (SRB) measures, in recognition of the pioneering work of Ya. Sinai, D. Ruelle, and R. Bowen in the 1970s, who introduced physical measures for smooth dynamical systems with uniform hyperbolicity [31, 12, 11, 29]. When working with the C^1 topology, we prefer the term physical measures over SRB measures to avoid confusion. Indeed, there is an abundant literature studying physical or SRB measures for smooth systems, or even $C^{1+\alpha}$ systems with $\alpha > 0$ (see Definition 5.1). Under such regularity, crucial properties emerge: the conditional measures of the physical probabilities along the unstable submanifolds are absolutely continuous with respect to the Lebesgue measures on these submanifolds [26] [20]. In the literature, these properties are typically required or established before an invariant measure is called an SRB measure [37]. However, in our context, where the regularity is only C^1 (and not necessarily $C^{1+\alpha}$), the existence of unstable submanifolds may fail [27]. Moreover, even in specific cases where unstable manifolds do exist, the absolute continuity properties generally do not hold [7].

One of the most fundamental problems in ergodic theory is to prove the existence of physical measures, since a priori the sequence of empirical probabilities may fail to converge for a set of orbits with positive volume. The existence of physical measures is primarily established under some form of uniform or non-uniform hyperbolicity or, at least, domination of the expanding directions. Specifically, the existence of physical measures was proved for C^1 -generic expanding maps of the circle in [13], for C^1 -generic diffeomorphisms possessing a hyperbolic attractor in [28], and for $C^{1+\alpha}$

diffeomorphisms with a dominated splitting in [6]. More recently, a characterization of the existence of SRB measures on surfaces within the $C^{1+\alpha}$ setting was provided in [17].

Besides existence, the problem of uniqueness or, at least, finitude of physical measures is an active object of research, particularly for partially hyperbolic systems. For instance, the existence and uniqueness of SRB measures for a certain class of C^2 partially hyperbolic systems were established in [23]. Furthermore, in [2], the authors proved the existence of at most a finite number of SRB measures for a class of $C^{1+\alpha}$ partially hyperbolic dynamical systems.

As mentioned above, the existence of physical or SRB measures has been primarily established for systems with $C^{1+\alpha}$ regularity (and under some form of hyperbolicity or expanding properties), with only a few articles exploring their existence within the C^1 topology. Addressing an intermediate setting, the author in [9] established the existence of SRB measures for hyperbolic systems that possess higher regularity than C^1 but strictly weaker regularity than $C^{1+\alpha}$.

To overcome the potential nonexistence of physical measures, a generalization of these measures was introduced in [15], namely the concept of pseudo-physical or SRB-like measures.

Definition 1.3 (Epsilon-weak basin of statistical attraction). Choose a distance dist^* in $\mathcal{M}$ that endows the weak* topology. For any f -invariant probability measure μ , and for $\epsilon > 0$, we call the following set the ϵ -weak basin of statistical attraction of μ :

$$B_\epsilon(\mu) := \{x \in M : \text{dist}^*(p\omega(x), \mu) < \epsilon\}$$

We note that the basin of statistical attraction defined in 1.1 may not coincide with the zero-weak basin of statistical attraction. In fact, the weak*-distance between $p\omega(x)$ and μ may be zero, even though the sequence of empirical probabilities does not converge, instead possessing convergent subsequences whose limits differ from μ .

Definition 1.4 (Pseudo-physical measures). We say that an invariant probability measure μ is *pseudo-physical* or *SRB-like* if its ϵ -weak basin of statistical attraction has positive Lebesgue measure for all $\epsilon > 0$. In brief,

$$m(B_\epsilon(\mu)) > 0 \quad \text{for all } \epsilon > 0.$$

The following paragraph summarizes the main properties established in [15]:

Pseudo-physical measures always exist for any continuous map f and they do not depend on the choice of the metric dist^* on the space of probability measures, provided it induces the weak*-topology. Furthermore, for Lebesgue-almost every orbit, any convergent subsequence of empirical probabilities converges to a pseudo-physical measure. In other words, the set of all pseudo-physical measures completely describes the observable statistical behavior of the system, provided that the criterion for observability is based on orbits of positive volume. Since every physical measure is pseudo-physical, pseudo-physicality serves as a genuine generalization of physicality. If the set of all pseudo-physical measures is finite, then all such measures are physical.

In Section 5, we present examples of C^1 -expanding maps (see Definition 1.9) on the circle and on the torus that fail to be $C^{1+\alpha}$ for any $\alpha > 0$, and we study their pseudo-physical measures.

1.2 Equilibrium states and Pesin's entropy formula.

Other important definitions in the study of the statistical properties of a dynamical system, originating from statistical mechanics, include the concept of equilibrium states of a variational principle (see, for instance, [11, 19]), and, in particular, the set of measures that satisfy Pesin's entropy formula. We recall the definitions of these concepts in the following paragraphs.

To define equilibrium states, we employ the *metric entropy* of the map f with respect to an f -invariant probability measure μ . For a precise definition of metric entropy, see Section 3 of this article. For a more detailed exposition of the properties of entropy, we refer the reader to, for instance, the book [36].

Definition 1.5 (Equilibrium States). Let $\psi : M \mapsto \mathbb{R}$ be a continuous function, called a *potential*.

The supremum $P(f, \psi)$, defined below, is called the *pressure* of f with respect to the potential ψ :

$$P(f, \psi) := \sup_{\nu \in \mathcal{M}_f} \left\{ h_\nu(f) + \int \psi d\nu \right\},$$

where $h_\nu(f)$ denotes the metric entropy of f with respect to the f -invariant measure ν .

An f -invariant measure μ is called an *equilibrium state* of f with respect to the potential ψ if

$$h_\mu(f) + \int \psi d\mu = P(f, \psi).$$

In the 1970s, Sinai, Ruelle, and Bowen established fundamental connections between equilibrium states and physical measures for smooth hyperbolic dynamical systems [31, 12, 11, 29]. More recently, [1] proved the existence of equilibrium states for a certain class of C^1 partially hyperbolic endomorphisms and for specific types of potentials. In [18], the existence and uniqueness of equilibrium states were established for non-uniformly expanding skew products and Hölder continuous potentials. Furthermore, the uniqueness of equilibrium states, in addition to their existence, was proved in [24] for a class of flows satisfying a version of the specification property among other conditions. Finally, [4] established the existence of finitely many ergodic equilibrium states for a type of non-uniformly expanding maps with respect to Hölder continuous potentials.

Equilibrium states are used mainly when the potential is related to the positive Lyapunov exponents, which characterize the chaotic behavior of the dynamics along the tangent space under iterations of f . Oseledets' Theorem (see, for instance, [8]) states that for any f -invariant measure μ , at μ -almost every point x , there exists a splitting of the tangent space $T_x M = \bigoplus_{i=1}^{k(x)} E_x^i$ into

Df -invariant measurable subspaces E_x^i along which the Lyapunov exponents exist according to the following definition:

Definition 1.6 (Lyapunov exponents). The *Lyapunov exponent* $\chi_i(x)$ of f at the point $x \in M$ along the measurable Df -invariant tangent subspace E_x^i is defined as

$$\chi_i(x) := \lim_{n \rightarrow \pm\infty} \frac{\|Df_x^n v\|}{n} \quad \text{for all } 0 \neq v \in E_x^i.$$

Geometrically, the Lyapunov exponents quantify the exponential rate of growth (if positive) or decay (if negative) of tangent vectors under iterations of $Df : TM \mapsto TM$.

We denote by $\sum_{i=1}^{m(x)} \chi_i^+(x)$ the sum of the positive Lyapunov exponents at the point x , counting each exponent with its multiplicity. If all Lyapunov exponents are nonpositive, this sum is understood to be zero. The following theorem, due to Margulis [21] and Ruelle [30], provides an upper bound for the metric entropy in terms of the positive Lyapunov exponents.

Theorem 1.7 (Margulis-Ruelle inequality). *For any f -invariant measure μ ,*

$$h_\mu(f) \leq \int \sum_{i=1}^{m(x)} \chi_i^+(x) d\mu$$

For a proof see for instance [36].

Definition 1.8 (Pesin's entropy formula). An f -invariant measure μ satisfies *Pesin's entropy formula* if

$$h_\mu(f) = \int \sum_{i=1}^{m(x)} \chi_i^+(x) d\mu$$

Measures satisfying Pesin's entropy formula may fail to exist. However, if such a measure exists, its metric entropy achieves the maximum possible value with respect to the chaotic behavior of f encoded by the positive Lyapunov exponents.

When the system possesses a continuous Df -invariant unstable sub-bundle $U \subset TM$, the integral of the sum of the positive Lyapunov exponents equals the integral of $\phi := \log |\det Df|_U|$. If this function is continuous, its opposite $\psi = -\phi$ can be employed as the potential to study the corresponding equilibrium states. In this case, the Margulis–Ruelle inequality implies that the pressure satisfies $P(f, \psi) \leq 0$. Consequently, the measures satisfying Pesin's entropy formula, if any exist, are precisely the equilibrium states of f with respect to the potential ψ , and the associated pressure is zero.

Ya. B. Pesin [26] initiated what is now known as Pesin theory, establishing fundamental relations between Lyapunov exponents and the existence of measures satisfying Pesin's entropy formula for certain smooth systems. This was achieved in a setting where there exist invariant measures that are absolutely continuous with respect to the Lebesgue measure along the unstable submanifolds. Later, Ledrappier and Young [20] proved that this absolute continuity condition is in fact a characterization of the measures that satisfy Pesin's entropy formula, provided the system is of class $C^{1+\alpha}$ with $\alpha > 0$. This characterization is crucial, serving as the cornerstone for subsequent research on the existence of measures satisfying Pesin's entropy formula. For instance, the existence of an SRB measure satisfying Pesin's entropy formula was proved in [6] for $C^{1+\alpha}$ diffeomorphisms with a dominated splitting. In [3], the authors established the existence and uniqueness of an SRB measure satisfying Pesin's entropy formula for Gibbs–Markov induced maps, which correspond to a piecewise $C^{1+\alpha}$ dynamics.

The Ledrappier–Young characterization of measures satisfying Pesin's entropy formula via absolute continuity along unstable submanifolds does not hold for C^1 systems that lack $C^{1+\alpha}$ regularity. Indeed, generic C^1 systems do not possess measures with this absolute continuity property [27, 7]. Nevertheless, under certain forms of hyperbolicity, C^1 systems can still admit measures satisfying Pesin's entropy formula. Specifically, A. Tahzibi [33] proved that generic two-dimensional C^1 systems possess an invariant measure that satisfies Pesin's entropy formula. Subsequently, Sun and Tian [32] extended Tahzibi's result to C^1 -generic volume-preserving diffeomorphisms in any dimension with a dominated splitting. In [14], it was shown that the pseudo-physical measures, which necessarily exist, satisfy Pesin's entropy formula for all C^1 systems with a dominated splitting in any dimension. A similar result was established in [16] for C^1 expanding maps in dimension one, and in [5] for C^1 non-uniformly expanding maps in any dimension.

1.3 Statement of the result for expanding maps.

Definition 1.9 (Expanding maps). A C^1 map $f : M \rightarrow M$ is (uniformly) *expanding* if there exists a constant $\lambda > 1$ such that

$$\|Df_x(v)\| \geq \lambda \|v\| \quad \forall (x, v) \in TM.$$

Recall Definition 1.6 of the Lyapunov exponents. For an expanding map, the norm of every vector in the tangent space grows by a factor greater than $\lambda > 1$ at each iteration. Consequently, the exponential growth rate of vectors in any direction is strictly bounded below by $\log \lambda > 0$. Thus, all Lyapunov exponents are strictly positive.

Theorem 1.10 (Liouville formula). *For any C^1 map f and for any f -invariant measure μ*

$$\int \sum_{i=1}^{k(x)} \chi_i(x) d\mu = \int \log |\det(Df)| d\mu,$$

where $\sum_{i=1}^{k(x)} \chi_i(x)$ is the sum of the Lyapunov exponents at the point x , counting each one as many times as its multiplicity.

For a proof of Liouville formula see for example [34].

Corollary 1.11. *If the C^1 map f is expanding then, for any f -invariant measure μ*

$$\int \sum_{i=1}^{m(x)} \chi_i^+(x) d\mu = \int \log |\det(Df)| d\mu,$$

where $\sum_{i=1}^{m(x)} \chi_i^+(x)$ is the sum of all the positive Liapounov exponents at the point x , counting each one as many times as its multiplicity.

Proof. Since the map is expanding, all the Lyapunov exponents are positive. Therefore, this corollary is a restatement of Liouville formula. $\square$

Proposition 1.12. *For a C^1 expanding map f , an invariant probability measure μ satisfies Pesin's entropy formula if and only if it is an equilibrium state for the potential $\psi = -\log |\det(Df)|$ and the pressure $P(f, \psi) = 0$.*

Proof. Due to Theorem 1.7 and Corollary 1.11, we have

$$P(f, \psi) = \sup_{\mu \in \mathbb{M}_f} (h_\mu(f) + \int \psi d\mu) = \sup_{\mu \in \mathbb{M}_f} (h_\mu(f) - \int \log |\det(Df)| d\mu) \leq 0.$$

Then, the pressure $P(f, \psi)$ is not positive.

Now, recalling Definition 1.8 of Pesin's entropy formula, and applying again Corollary 1.11, an f -invariant measure μ satisfies Pesin's formula if and only if

$$0 = h_\mu(f) - \int \log |\det(Df)| d\mu = h_\mu(f) + \int \psi d\mu = \sup_{\mu \in \mathbb{M}_f} (h_\mu(f) + \int \psi d\mu).$$

$\square$

The main result to be proved in this paper is the following Theorem:

Theorem 1.13. *Let $f : M \mapsto M$ be an expanding C^1 map on a compact Riemannian manifold M of finite dimension. Then, any (necessarily existing) pseudo-physical measure μ for f satisfies Pesin's entropy formula. Namely,*

$$h_\mu(f) = \int \sum_{i=1}^{m(x)} \chi_i^+(x) d\mu = \int \log |\det(Df)| d\mu.$$

Equivalently, μ is an equilibrium state for the potential

$$\psi = -\log |\det(Df)|,$$

and the pressure $P(f, \psi) = 0$.

Theorem 1.13 generalizes to any finite dimension the result previously established in [16] for the one-dimensional setting. The proof of Theorem 1.13 presented in this paper is part of the unpublished 2017 Master's thesis of F. Valenzuela [35]. In a result announced in 2017, Araujo and Santos [5, Corollary H] established a more general framework that holds not only for the C^1 (uniformly) expanding maps of Theorem 1.13 (as given in Definition 1.9), but also for non-uniformly expanding maps. Although Theorem 1.13 is a special case of [5, Corollary H], our proof introduces an alternative approach and provides a direct argument for not necessarily ergodic physical measures of uniformly expanding maps.

Organization of the paper. In Section 2, we establish a key topological property of expanding maps, namely expansiveness, which is required for the proof of Theorem 1.13. Section 3 defines the

metric entropy and recalls some of its well-known properties. In Section 4, we provide the proof of Theorem 1.13. Finally, Section 5 is devoted to illustrative examples.

ACKNOWLEDGEMENTS. Sections 2, 3, and 4 comprise an English translation, with minor modifications and additions, of the Master's thesis of Fernando Valenzuela Gómez [35]. The second author gratefully acknowledges the financial support of a Master's scholarship during his graduate studies at PEDECIBA (Programa de Desarrollo de las Ciencias Básicas), Área Matemática (Uruguay). The first author is supported by ANII (Agencia Nacional de Investigación e Innovación) and PEDECIBA, both in Uruguay.

2 Expanding maps are expansive.

To prove Theorem 1.13 we need an important topological property defined for continuous maps, called expansiveness. We need to show that, in particular, the C^1 expanding maps, according to Definition 1.9, are expansive.

Definition 2.1 (Expansive maps). A continuous map $f : M \mapsto M$ is *expansive in the future* if there exists a constant $\alpha > 0$, called the *expansivity constant*, such that if $x, y \in M$ satisfies

$$\text{dist}(f^n(x), f^n(y)) \leq \alpha \quad \forall n \geq 0,$$

then $x = y$.

The expansiveness is understood as the sensitivity to the initial condition. In fact, two orbits with different initial states, even if these initial states are arbitrarily close, separate more than α at some iteration in the future.

Proposition 2.2. *If a C^1 map $f : M \mapsto M$ on the compact Riemannian manifold M is expanding, then it is expansive in the future.*

Proof. Let $x \in M$, and $\delta > 0$. Denote by $B_\delta(0) \subset T_x M$ the open ball in the tangent space at x , centered at 0 with radius δ , i.e. the set of vectors in $T_x M$ with norm smaller than δ . Choose δ small enough such that the exponential map $\exp_x : B_\delta(0) \subset T_x M \rightarrow M$ is a diffeomorphism onto its image. Explicitly, for any $y \in M$ such that $\text{dist}(x, y) < \delta$, there exists a unique vector $v = \exp_x^{-1}(y) \in T_x M$, and this vector satisfies $\|v\| = \text{dist}(x, y) < \delta_1$. Since M is compact, we can choose a uniform $\delta > 0$. Namely, δ does not depend on x .

In the following we will denote $y - x$ to refer to the vector $v = \exp_x^{-1}(y) \in B_\delta(0) \subset T_x M$.

Since f is C^1 ,

$$\|f(y) - f(x)\| = \|Df_z(y - x)\|,$$

for some point z in the (convex) ball centered at x of radius δ .

Then

$$\|f(y) - f(x)\| \geq \lambda \|y - x\|,$$

where $\lambda > 1$ is the constant in Definition 2.1 of the expanding map f .

It is enough to prove that $\delta > 0$ is a constant of expansivity for f , as in Definition 2.1. Assume that

$$\text{dist}(f^j(x), f^j(y)) = \|f^j(y) - f^j(x)\| \leq \delta \quad \forall j \geq 0.$$

Then

$$\|f^{j+1}(y) - f^{j+1}(x)\| \geq \lambda \|f^j(y) - f^j(x)\| \quad \forall j \geq 0,$$

and therefore

$$\delta \geq \text{dist}(f^n(x), f^n(y)) = \|f^n(y) - f^n(x)\| \geq \lambda^n \|y - x\| \quad \forall n \geq 0.$$

Since $\lambda^n \rightarrow +\infty$ with n , while $\lambda^n \|y - x\| \leq \delta$ for all $n \geq 0$, we deduce that $x = y$. $\square$

3 The metric entropy.

In this section, we review the definition of the metric entropy of f with respect to an invariant measure μ , and state the properties required for the proof of Theorem 1.13.

A *finite measurable partition* $\mathcal{P}$ is a finite family of pairwise disjoint measurable sets $P \subset M$ whose union is M . The sets $P \in \mathcal{P}$ are called the *pieces* of $\mathcal{P}$. For brevity, we shall refer to a finite measurable partition simply as a *partition*.

The *boundary* $\partial\mathcal{P}$ of a partition is the union of the topological boundaries of its pieces. Namely,

$$\partial\mathcal{P} := \bigcup_{P \in \mathcal{P}} \partial P.$$

The *diameter* $\text{diam}(\mathcal{P})$ of a partition is the maximum diameter of its pieces:

$$\text{diam}(\mathcal{P}) = \max_{P \in \mathcal{P}} \text{diam}(P).$$

The *product* $\mathcal{P} \vee \mathcal{Q}$ of two partitions is the partition whose pieces are given by $P \cap Q$, where $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$. More generally, given a collection of $N \geq 1$ partitions $\{\mathcal{P}_i\}_{1 \leq i \leq N}$, we define their *product* as follows:

$$\bigvee_{i=1}^N \mathcal{P}_i = \left\{ \bigcap_{i=1}^N P_i : P_i \in \mathcal{P}_i \ \forall \ 1 \leq i \leq N \right\}.$$

Definition 3.1. The entropy $H(\mathcal{P}, \mu)$ of the partition $\mathcal{P}$ with respect to a probability measure μ is

$$H(\mathcal{P}, \mu) := - \sum_{P \in \mathcal{P}} \mu(P) \log \mu(P),$$

where we agree to take $0 \cdot \log 0 = 0$.

Now, let us introduce the dynamics of $f : M \mapsto M$ to the study of the entropy of the partitions with respect to a probability measure.

For any partition $\mathcal{P}$, we consider the following product partition:

$$\mathcal{P}_f^n := \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P} = \mathcal{P} \vee f^{-1}\mathcal{P} \vee \dots \vee f^{-(n-1)}\mathcal{P}, \quad (3.1)$$

where $f^{-i}\mathcal{P} := \{f^{-i}(P) : P \in \mathcal{P}\}$.

Proposition 3.2. If μ is f -invariant, then the following limit exists and satisfies the equality and inequality at right:

$$\lim_{n \rightarrow \infty} \frac{H(\mathcal{P}_f^n, \mu)}{n} = \inf_{n \geq 1} \frac{H(\mathcal{P}_f^n, \mu)}{n} \leq H(\mathcal{P}, \mu).$$

Proof. See for instance [36], Lemma 9.1.12. □

Definition 3.3. Let $\mathcal{P}$ be a partition, and μ be an f -invariant measure. We call the following expression $h_\mu(f, \mathcal{P})$ the *entropy of f with respect to the partition $\mathcal{P}$ and to the measure μ* :

$$h_\mu(f, \mathcal{P}) := \lim_{n \rightarrow \infty} \frac{H(\mathcal{P}_f^n, \mu)}{n} = \inf_{n \geq 1} \frac{H(\mathcal{P}_f^n, \mu)}{n}.$$

Definition 3.4 (The metric entropy). Let μ be an f -invariant probability measure. We call the following expression $h_\mu(f)$ the *metric entropy of f with respect to the measure μ* :

$$h_\mu(f) := \sup \{h(f, \mathcal{P}) : \mathcal{P} \text{ finite measurable partition}\}.$$

3.1 Properties of the entropy of partitions.

In this subsection, we state several properties of the entropy of partitions that will be employed in the proof of Theorem 1.13. For a more comprehensive account of the properties of entropy, we refer the reader to, for instance, the books [19, 36].

Proposition 3.5. *For any (finite measurable) partition $\mathcal{P}$ and any probability measure μ*

$$H(\mathcal{P}, \mu) \leq \log p,$$

where p is the number of pieces of $\mathcal{P}$.

Proof. See for instance [36], Lemma 9.1.3. □

Proposition 3.6. *Let $\mathcal{P}, Q$ be two partitions and μ any probability measure. Then,*

$$H(\mathcal{P}, \mu) \leq H(\mathcal{P} \vee Q, \mu) \leq H(\mathcal{P}, \mu) + H(Q, \mu).$$

Proof. See for instance [36], Section 9.1. □

Proposition 3.7. *For any partition $\mathcal{P}$ and any (not necessarily f -invariant) probability measure μ :*

$$H(\mathcal{P}_f^n, \mu) \leq \sum_{i=0}^{n-1} H(f^{-i}\mathcal{P}, \mu) = \sum_{i=0}^{n-1} H(\mathcal{P}, f^{*i}\mu).$$

Proof. Applying Equality (3.1) and Proposition 3.6 we have

$$H(\mathcal{P}_f^n, \mu) = H\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}, \mu\right) \leq \sum_{i=0}^{n-1} H(f^{-i}\mathcal{P}, \mu). \quad (3.2)$$

Besides, from Definition 3.1:

$$H(f^{-i}\mathcal{P}, \mu) = - \sum_{P \in \mathcal{P}} \mu(f^{-i}P) \log \mu(f^{-i}P),$$

and from the definition of the pull-back f^* , we have $\mu(f^{-i}P) = f^{*i}\mu(P)$. So,

$$H(f^{-i}\mathcal{P}, \mu) = H(\mathcal{P}, f^{*i}\mu).$$

Finally, substituting this last equality in (3.2), we deduce

$$H(\mathcal{P}_f^n, \mu) \leq \sum_{i=0}^{n-1} H(f^{-i}\mathcal{P}, \mu) = \sum_{i=0}^{n-1} H(\mathcal{P}, f^{*i}\mu),$$

as stated. □

Proposition 3.8. *For any partition $\mathcal{P}$ and any (not necessarily f -invariant) probability measure μ ,*

$$\frac{H(\mathcal{P}_f^n, \mu)}{n} \leq \frac{1}{n} \sum_{i=0}^{n-1} H(\mathcal{P}, f^{*i}\mu) \leq H\left(\mathcal{P}, \frac{1}{n} \sum_{i=0}^{n-1} f^{*i}\mu\right).$$

Proof. Consider the following continuous real function $\phi : [0, 1] \rightarrow \mathbb{R}$:

$$\phi(u) = -u \log u \text{ if } u \in (0, 1], \quad \phi(0) = 0.$$

It is easy to check that ϕ is C^∞ in $(0, 1)$, and that $\phi''(u) < 0$ for all $u \in (0, 1)$. Therefore, the graph of ϕ is above the secant line. Thus, the value of ϕ at the convex combination of n values $u_0, \dots, u_{n-1} \in [0, 1]$ (which is the ordinate of a point in the graph of ϕ), is larger or equal than

the convex combination of $\phi(u_0), \dots, \phi(u_{n-1})$ (which is the ordinate of a point in the secant line). Precisely, if $0 < \lambda_i < 1$ for all $0 \leq i \leq n-1$ and $\sum_{i=0}^{n-1} \lambda_i = 1$, then

$$\phi\left(\sum_{i=0}^{n-1} \lambda_i u_i\right) \geq \sum_{i=0}^{n-1} \lambda_i \phi(u_i).$$

Therefore

$$\begin{aligned} H\left(\mathcal{P}, \frac{1}{n} \sum_{i=0}^{n-1} f^{*i} \mu\right) &= \sum_{P \in \mathcal{P}} \phi\left(\frac{1}{n} \sum_{i=0}^{n-1} f^{*i} \mu(P)\right) \\ &\geq \sum_{P \in \mathcal{P}} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{*i} \mu(P)) \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \sum_{P \in \mathcal{P}} \phi(f^{*i} \mu(P)) = \frac{1}{n} \sum_{i=0}^{n-1} H(\mathcal{P}, f^{*i} \mu). \end{aligned}$$

Finally, using Proposition 3.7 the last expression is greater than or equal to $(1/n)H(\mathcal{P}_f^n, \mu)$, as claimed. $\square$

Proposition 3.9. *Let $\mathcal{P}$ be a (finite measurable) partition and μ a probability measure such that*

$$\mu(\partial P) = 0.$$

If $\{\mu_n\}_{n \geq 0}$ is a sequence of probabilities measures such that $\lim_{n \rightarrow +\infty}^* \mu_n = \mu$, then

$$\lim_{n \rightarrow +\infty} H(\mathcal{P}, \mu_n) = H(\mathcal{P}, \mu).$$

In other words, this proposition states the continuity at μ of the entropy of a partition $\mathcal{P}$ as a function of the measure μ , if the boundary of the partition has zero μ -measure.

To prove Proposition 3.9, we will use the following lemma

Lemma 3.10. *Let μ_n, μ be probability measures such that $\lim_{n \rightarrow +\infty}^* \mu_n = \mu$. The following holds:*

- (1) *If $K \subset M$ is compact, then $\limsup_{n \rightarrow +\infty} \mu_n(K) \leq \mu(K)$.*
- (2) *If $V \subset M$ is open, then $\liminf_{n \rightarrow +\infty} \mu_n(V) \geq \mu(V)$.*
- (3) *If A is a Borel set such that $\mu(\partial A) = 0$, then*

$$\lim_{n \rightarrow +\infty} \mu_n(A) = \mu(A).$$

Proof. (1) Let $\epsilon > 0$ and $V \subset M$ such that $K \subset V$ and $\mu(V \setminus K) < \epsilon$. Let $\phi : M \mapsto [0, 1]$ be a continuous function such that $\phi|_K = 1$ and $\phi|_{M \setminus V} = 0$. Then

$$\mu_n(K) \leq \int \phi d\mu_n \quad \forall n \geq 1.$$

From the continuity of ϕ , and the convergence in the weak* topology of μ_n to μ , we obtain

$$\begin{aligned} &\int \phi d\mu_n \\ &\longrightarrow \int \phi d\mu = \int_{\phi \neq 0} \phi d\mu \leq \int_V \phi d\mu \leq \int_V 1 d\mu = \mu(V) = \mu(K) + \mu(V \setminus K) < \mu(K) + \epsilon. \end{aligned}$$

Then

$$\mu_n(K) \leq \int \phi d\mu_n < \mu(K) + \epsilon \quad \forall n \geq 1.$$

Now, taking $\limsup$ we obtain

$$\limsup_{n \rightarrow +\infty} \mu_n(K) \leq \mu(K) + \epsilon.$$

Since the above inequality holds for all $\epsilon > 0$, we conclude that $\limsup_{n \rightarrow +\infty} \mu_n(K) \leq \mu(K)$.

(2) Let $K = M \setminus V$. We have

$$\mu_n(V) = 1 - \mu_n(K) \quad \forall n \geq 1.$$

Taking $\liminf$, we obtain

$$\liminf_{n \rightarrow +\infty} \mu_n(V) = 1 - \limsup_{n \rightarrow +\infty} \mu_n(K).$$

Since K is a closed set in the compact metric space M , it is compact. Applying property (1), we conclude

$$\liminf_{n \rightarrow +\infty} \mu_n(V) = 1 - \limsup_{n \rightarrow +\infty} \mu_n(K) \geq 1 - \mu(K) = \mu(V),$$

as stated.

(3) We consider the interior $\text{int}(A)$ of A and its closure $\bar{A}$. Each of these sets differs from A at the boundary of A , which has zero μ -measure. Thus

$$\mu(\text{int}(A)) = \mu(A) = \mu(\bar{A}).$$

For μ_n we have

$$\mu_n(\text{int}(A)) \leq \mu_n(A) \leq \mu_n(\bar{A}).$$

Applying properties (1) and (2), we obtain

$$\begin{aligned} \mu(\text{int}(A)) &\leq \liminf_{n \rightarrow +\infty} \mu_n(\text{int}(A)) \leq \liminf_{n \rightarrow +\infty} \mu_n(A) \leq \limsup_{n \rightarrow +\infty} \mu_n(A) \\ &\leq \limsup_{n \rightarrow +\infty} \mu_n(\bar{A}) \leq \mu(\bar{A}). \end{aligned}$$

Since $\mu(\text{int}(A)) = \mu(\bar{A}) = \mu(A)$, the last chain of inequalities is a chain of equalities. Therefore, $\limsup$ and $\liminf$ of $\mu_n(A)$ coincide and are equal to $\mu(A)$. We conclude $\lim_{n \rightarrow +\infty} \mu_n(A) = \mu(A)$ as stated. $\square$

Proof of Proposition 3.9. Consider the following continuous real function $\phi : [0, 1] \mapsto \mathbb{R}$:

$$\phi(u) = -u \log u \text{ if } 0 < u \leq 1, \quad \phi(0) = 0.$$

From Definition 3.1 we have

$$H(\mathcal{P}, \mu) = \sum_{P \in \mathcal{P}} \phi(\mu(P)), \quad H(\mathcal{P}, \mu_n) = \sum_{P \in \mathcal{P}} \phi(\mu_n(P))$$

Since $\mu(\partial \mathcal{P}) = 0$ we can apply part (3) of Lemma 3.10 to each piece $P \in \mathcal{P}$. Using also the continuity of the function ϕ , we obtain

$$\lim_{n \rightarrow +\infty} \phi(\mu_n(P)) = \phi(\mu(P)) \quad \forall P \in \mathcal{P}.$$

Finally we sum the above equality for all the finite number of pieces P of the partition $\mathcal{P}$, concluding that

$$\lim_{n \rightarrow +\infty} H(\mathcal{P}, \mu_n) = \lim_{n \rightarrow +\infty} \sum_{P \in \mathcal{P}} \phi(\mu_n(P)) = \sum_{P \in \mathcal{P}} \lim_{n \rightarrow +\infty} \phi(\mu_n(P)) = \sum_{P \in \mathcal{P}} \phi(\mu(P)) = H(\mathcal{P}, \mu),$$

as claimed. $\square$

3.2 The metric entropy for expansive maps.

For expansive maps, the Kolmogorov–Sinai Theorem (Theorem 3.12 and Corollary 3.13 in this section) ensures the existence of partitions that attain the supremum in Definition 3.4. In other words, the metric entropy of f can be computed as the entropy of f with respect to a single, concrete partition.

Definition 3.11. A partition $\mathcal{P}$ of M , is a generator (in the future) for $f : M \rightarrow M$ if the σ -algebra generated by $\left\{ \bigvee_{j=0}^k f^{-j}\mathcal{P} \right\}_{k \geq 0}$ is the Borel σ -algebra.

Recall Definitions 3.3 and 3.4 of $h_\mu(f, \mathcal{P})$ and $h_\mu(f)$.

Theorem 3.12 (Kolmogorov–Sinai). If the (finite measurable) partition $\mathcal{P}$ of M is a generator for $f : M \rightarrow M$, then for any f -invariant probability measure μ

$$h_\mu(f) = h_\mu(f, \mathcal{P}).$$

Proof. See for instance [19], Theorem 3.2.18. □

Corollary 3.13. If $f : M \rightarrow M$ is expansive in the future with expansivity constant α , and $\mathcal{P}$ is a partition with $\text{diam}(\mathcal{P}) < \alpha$ then for any f -invariant probability measure μ :

$$h_\mu(f) = h(f, \mathcal{P}) = \lim_{n \rightarrow +\infty} \frac{H(\mathcal{P}_f^n, \mu)}{n}.$$

Proof. Applying Theorem 3.12 it is enough to prove that $\mathcal{P}$ is a generator for f .

Let $k \geq 0$ and for each point $x \in M$ consider the piece $A_k(x)$ of the partition $\bigvee_{i=0}^k f^{-i}\mathcal{P}$ that contains x . We denote by $B_\delta(x)$ the ball centered at x with radius $\delta < 0$. We will first prove the following statement:

Assertion A. For all $0 < \delta < \alpha$ there exists $k = k(\delta)$ such that

$$A_k(x) \subset B_\delta(x) \quad \forall x \in M. \quad (3.3)$$

Suppose for a contradiction that there is $\delta \in (0, \alpha)$ such that for all $k \geq 0$ there exist points $x_k, y_k \in M$ satisfying

$$y_k \in A_k(x_k) \setminus B_\delta(x_k) \quad \forall k \geq 0.$$

Since M is compact, there are subsequences $\{x_{k_j}\}_j$ and $\{y_{k_j}\}_j$ that are convergent to the points x and y respectively. On the one hand, as $y_k \notin B_\delta(x_k)$ we have

$$\text{dist}(x, y) = \lim_{j \rightarrow +\infty} \text{dist}(x_{k_j}, y_{k_j}) \geq \delta. \quad (3.4)$$

On the other hand, as $y_k \in A_k(x_k) \in \bigvee_{i=0}^k f^{-i}\mathcal{P}$ and $\text{diam}(\mathcal{P}) < \alpha$, we obtain

$$\text{dist}(f^i(x_{k_j}), f^i(y_{k_j})) < \alpha \quad \forall 0 \leq i \leq k_j.$$

Taking the limit in the inequality above when $j \rightarrow +\infty$ with i fixed, we deduce

$$\text{dist}(f^i(x), f^i(y)) \leq \alpha \quad \forall i \geq 0.$$

Due to the expansiveness in the future of f , the inequality above implies $x = y$, contradicting the inequality (3.4). This ends the proof of Assertion A.

Any open set $V \subset M$ can be written as the union of open balls $B_{\delta(x)}(x) \subset V$ with $\delta(x) < \alpha$. Using Assertion A, each point x of V is inside a piece $A_{k(x)}(x) \subset B_{\delta(x)}(x)$ for some $k(x) \geq 0$. Therefore V is the union of a family of pieces

$$A_{k(x)}(x) \in \bigcup_{k \geq 0} \left\{ \bigvee_{i=0}^k f^{-i}\mathcal{P} : k \geq 0 \right\}.$$

Since the family of all such pieces is countable (because they are the pieces of a countable union of finite partitions), we deduce that the σ -algebra generated by them includes all the open subsets of M . Thus, it includes the Borel σ -algebra. Conversely, any piece $A_k(x)$ is a Borel-measurable set. Therefore, the σ -algebra generated by them is included in the Borel σ -algebra. We conclude that both σ -algebras coincide, as stated. $\square$

4 Proof of Theorem 1.13.

To prove Theorem 1.13, we will fix some notation:

For the C^1 expanding map $f : M \mapsto M$ denote

$$\psi(x) := -\log |\det Df_x| < 0, \quad \forall x \in M. \quad (4.1)$$

Recall that $\psi : M \rightarrow \mathbb{R}$ is the potential in the statement of Theorem 1.13.

In the space $\mathcal{M}$ of all the Borel probability measures on the manifold M , we fix the following weak* metric:

$$\text{dist}^*(\mu, \nu) := \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \int \phi_i d\nu \right|, \quad (4.2)$$

where $\phi_0 := \psi$ and $\{\phi_i\}_{i \geq 1}$ is a countable family of continuous functions that is dense in the space $C^0(M, [0, 1])$.

Lemma 4.1. *For any $\mu \in \mathcal{M}$ and any $\epsilon > 0$, the ball $\mathcal{B} := \{\nu \in \mathcal{M} : \text{dist}^*(\mu, \nu) < \epsilon\}$ is convex.*

Proof. Let $\nu_1, \nu_2 \in \mathcal{B}$. We will prove that if $\lambda_1, \lambda_2 \in [0, 1]$ are such that $\lambda_1 + \lambda_2 = 1$, then $\lambda_1 \nu_1 + \lambda_2 \nu_2 \in \mathcal{B}$. Using the triangle inequality, we have

$$\begin{aligned} \text{dist}(\lambda_1 \nu_1 + \lambda_2 \nu_2, \mu) &= \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \int \phi_i d(\lambda_1 \nu_1 + \lambda_2 \nu_2) \right| \\ &\leq \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \lambda_1 \int \phi_i d\mu - \lambda_1 \int \phi_i d\nu_1 \right| + \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \lambda_2 \int \phi_i d\mu - \lambda_2 \int \phi_i d\nu_2 \right| \\ &= \lambda_1 \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \int \phi_i d\nu_1 \right| + \lambda_2 \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \int \phi_i d\nu_2 \right| \\ &< (\lambda_1 + \lambda_2) \epsilon = \epsilon \end{aligned}$$

as claimed. $\square$

Now, state and prove a series of lemmas that we will use in the proof of Theorem 1.13.

Recall Definition 3.1 of the entropy $H(\mathcal{P}, \mu)$ of a partition with respect to a probability measure μ , and the equality (3.1) defining the product partition $\mathcal{P}_f^n$.

Lemma 4.2. *Let μ be an f -invariant probability measure. Let $\mathcal{P}$ be any finite partition of the manifold M into measurable sets. If $\mu(\partial \mathcal{P}) = 0$, then, for all $\epsilon > 0$, and for all $q \geq 1$ there exists $\epsilon^* > 0$ such that*

$$\left| \frac{H(\mathcal{P}_f^q, \rho)}{q} - \frac{H(\mathcal{P}_f^q, \mu)}{q} \right| < \epsilon \quad (4.3)$$

for any probability measure ρ such that $\text{dist}(\rho, \mu) < \epsilon^*$.

Proof. Assume by contradiction that for all $\epsilon^* > 0$ there exists a probability measure ρ , whose distance to μ is smaller than ϵ^* , and that does not satisfy inequality (4.3). Thus, in particular for $\epsilon^* = 1/m$ where $m \in \mathbb{N}$, there exists ρ_m such that

$$\text{dist}(\rho_m, \mu) < \frac{1}{m},$$

$$\left| \frac{H(\mathcal{P}_f^q, \rho_m)}{q} - \frac{H(\mathcal{P}_f^q, \mu)}{q} \right| \geq \epsilon \quad \forall m \geq 1. \quad (4.4)$$

We have $\lim_{m \rightarrow +\infty} \rho_m = \mu$ in the weak* topology and $\mu(\partial \mathcal{P}) = 0$. Note that $\mu(\partial \mathcal{P}_f^q) = 0$ because μ is f -invariant. Applying part (iii) of Lemma 3.10, $\lim_{m \rightarrow +\infty} \rho_m(Y) = \mu(Y)$ for any piece $Y \in \mathcal{P}_f^q$. By Proposition 3.9, for fixed $q \geq 1$, we have:

$$\lim_{m \rightarrow +\infty} \frac{H(\mathcal{P}_f^q, \rho_m)}{q} = \frac{H(\mathcal{P}_f^q, \mu)}{q},$$

contradicting inequality (4.4). $\square$

Recall Definition 2.1 of the expansive maps, and Proposition 2.2. Let $\alpha > 0$ be an expansivity constant for f . From Corollary 3.13 of the Kolmogorov-Sinai Theorem for expansive maps, for any f -invariant probability measure μ we have

$$h_\mu(f) = \lim_{q \rightarrow +\infty} \frac{H(\mathcal{P}_f^q, \mu)}{q} \quad \text{if } \text{diam}(\mathcal{P}) < \alpha. \quad (4.5)$$

Lemma 4.3. *Let $f : M \mapsto M$ be a C^1 expanding map on the compact manifold M . Let $\alpha > 0$ be an expansivity constant for f . For all $0 < \delta < \alpha$, for all $\epsilon > 0$, and for any f -invariant measure μ , there exists a finite partition $\mathcal{P}$ of M , a real number $\epsilon^* > 0$, and a natural number $n_0 \geq 1$, such that:*

(i) $\text{diam}(\mathcal{P}) < \delta < \alpha$,

(ii) $\mu(\partial \mathcal{P}) = 0$,

(iii) *For any sequence of non necessarily invariant probabilities ν_n , if $\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f^j)^* \nu_n$, and if*

$\text{dist}(\mu_n, \mu) < \epsilon^*$, $\forall n \geq 1$, then

$$\frac{1}{n} H(\mathcal{P}_f^n, \nu_n) \leq h_\mu(f) + \epsilon, \quad \forall n \geq n_0.$$

Proof. Recall Equality (3.1) that defines the product partition $\mathcal{P}_f^n$ and Definition 3.4 of the metric entropy $h_\mu(f)$ of f . For simplicity, since we will not change the map f , we will write h_μ instead of $h_\mu(f)$ and $\mathcal{P}^n$ instead of $\mathcal{P}_f^n$. Take any finite covering $\mathcal{U} = \{Y_1, \dots, Y_p\}$ of M with open balls with radius smaller than $\delta/2$. Denote $\partial \mathcal{U} := \bigcup_{i=1}^p \partial Y_i$. Since the family of boundaries of the balls with radius $r > 0$ is not countable when changing r , but these boundaries may have positive μ -measure only for at most a countable subfamily, the radius of each ball $Y_i \in \mathcal{U}$ can be chosen such that $\mu(\partial Y_i) = 0$. Thus $\mu(\partial \mathcal{U}) = 0$. Therefore, the partition $\mathcal{P} = \{X_i\}_{1 \leq i \leq p}$ defined by $X_1 := Y_1 \in \mathcal{U}$, $X_{i+1} := Y_{i+1} \setminus (\bigcup_{j=1}^i X_j)$, satisfies the assertions (i) and (ii).

Next, for any given $\epsilon > 0$, let us find $\epsilon^* > 0$ and $n_0 \geq 1$ such that assertion (iii) holds. Let us fix two integer numbers $q \geq 1$ and $n \geq q$. Write $n = Nq + j$ where N, j are integer numbers such that $0 \leq j \leq q - 1$. Fix a (not necessarily invariant) probability ν . Applying Propositions 3.6 and 3.7, we obtain:

$$\begin{aligned} H(\mathcal{P}^n, \nu) &= H(\mathcal{P}^{Nq+j}, \nu) \\ &\leq H(\bigvee_{i=0}^{j-1} f^{-(Nq+i)} \mathcal{P}, \nu) + H(\bigvee_{i=0}^{N-1} f^{-iq} \mathcal{P}^q, \nu) \\ &\leq \sum_{i=0}^{j-1} H(f^{-(Nq+i)} \mathcal{P}, \nu) + \sum_{i=0}^{N-1} H(f^{-iq} \mathcal{P}^q, \nu) \\ &= \sum_{i=0}^{j-1} H(\mathcal{P}, (f^{Nq+i})^* \nu) + \sum_{i=0}^{N-1} H(\mathcal{P}^q, (f^{iq})^* \nu). \end{aligned}$$

From the above inequality, using Proposition 3.5, and recalling that $j \leq q - 1 < q$, we obtain

$$H(\mathcal{P}^n, \nu) \leq q \log p + \sum_{i=0}^{N-1} H(\mathcal{P}^q, (f^{iq})^* \nu) \quad \forall q \geq 1, \quad n \geq q,$$

where p is the number of pieces of the partition $\mathcal{P}$.

The inequality above holds also for $f^{-l}\mathcal{P}$ instead of $\mathcal{P}$, for any $l \geq 0$, because it holds for any partition with exactly p pieces. Thus:

$$H(f^{-l}\mathcal{P}^n, \nu) \leq q \log p + \sum_{i=0}^{N-1} H(f^{-l}\mathcal{P}^q, (f^{iq})^*\nu) = q \log p + \sum_{i=0}^{N-1} H(\mathcal{P}^q, (f^{iq+l})^*\nu).$$

Adding the above inequalities for $0 \leq l \leq q-1$, we obtain:

$$\sum_{l=0}^{q-1} H(f^{-l}\mathcal{P}^n, \nu) \leq q^2 \log p + \sum_{l=0}^{q-1} \sum_{i=0}^{N-1} H(\mathcal{P}^q, (f^{iq+l})^*\nu)$$

Therefore, on the one hand we have:

$$\sum_{l=0}^{q-1} H(f^{-l}\mathcal{P}^n, \nu) \leq q^2 \log p + \sum_{i=0}^{Nq-1} H(\mathcal{P}^q, (f^i)^*\nu). \quad (4.6)$$

On the other hand, applying the inequality of Proposition 3.6 and Equality (3.1), for all $0 \leq l \leq q-1$ we have

$$H(\mathcal{P}^n, \nu) \leq H(\mathcal{P}^n \vee f^{-n}\mathcal{P} \vee f^{-(n+1)}\mathcal{P} \vee \dots \vee f^{-(n+l-1)}\mathcal{P}, \nu) = H(\mathcal{P}^{n+l}, \nu) = H((\bigvee_{i=0}^{l-1} f^{-i}\mathcal{P}) \vee (f^{-l}\mathcal{P}^n)).$$

Therefore,

$$H(\mathcal{P}^n, \nu) \leq H(\mathcal{P}^{n+l}, \nu) \leq \left(\sum_{i=0}^{l-1} H(f^{-i}\mathcal{P}, \nu) \right) + H(f^{-l}\mathcal{P}^n, \nu).$$

So,

$$H(\mathcal{P}^n, \nu) \leq q \log p + H(f^{-l}\mathcal{P}^n, \nu).$$

Adding the above inequalities for $0 \leq l \leq q-1$ and joining with the inequality (4.6), we obtain:

$$qH(\mathcal{P}^n, \nu) \leq q^2 \log p + \sum_{i=0}^{q-1} H(f^{-l}\mathcal{P}^n, \nu) \leq 2q^2 \log p + \sum_{i=0}^{Nq-1} H(\mathcal{P}^q, (f^i)^*\nu).$$

Recall that $n = Nq + j$ with $0 \leq j \leq q-1$. So $n-1 = Nq + j-1 \geq Nq-1$ and then

$$qH(\mathcal{P}^n, \nu) \leq 2q^2 \log p + \sum_{i=0}^{n-1} H(\mathcal{P}^q, (f^i)^*\nu)$$

Now we put $\nu = \nu_n$ and divide by n . Recalling that $\mu_n = (1/n) \sum_{j=0}^{n-1} (f^j)^*\nu_n$ and applying Proposition 3.8, we obtain

$$\frac{qH(\mathcal{P}^n, \nu_n)}{n} \leq \frac{2q^2 \log p}{n} + \frac{1}{n} \sum_{i=0}^{n-1} H(\mathcal{P}^q, (f^i)^*\nu_n) \leq \frac{2q^2 \log p}{n} + H(\mathcal{P}^q, \mu_n).$$

For any fixed $\epsilon > 0$ (and the natural number $q \geq 1$ still fixed), take $n \geq n(q) := \max\{q, 6q \log p / \epsilon\}$ in the inequality above. We deduce:

$$\frac{q}{n} H(\mathcal{P}^n, \nu_n) \leq \frac{q\epsilon}{3} + H(\mathcal{P}^q, \mu_n) \quad \forall n \geq n(q) \quad \forall q \geq 1,$$

from where we obtain

$$\frac{1}{n} H(\mathcal{P}^n, \nu_n) \leq \frac{\epsilon}{3} + \frac{H(\mathcal{P}^q, \mu_n)}{q} \quad \forall n \geq n(q) \quad \forall q \geq 1. \quad (4.7)$$

The inequality above holds for any fixed $q \geq 1$ and for any n large enough, depending on q .

By hypothesis, μ is f -invariant. So, after Equality (4.5), there exists $q \geq 1$ such that

$$\frac{H(\mathcal{P}^q, \mu)}{q} \leq h_\mu + \frac{\epsilon}{3}. \quad (4.8)$$

Fix such a value of q . Since $\mu(\partial(\mathcal{P})) = 0$ due to the construction of $\mathcal{P}$ (depending on the given measure μ), we can apply Lemma 4.2 to find $\epsilon^* > 0$ such that

$$\frac{H(\mathcal{P}^q, \rho)}{q} \leq \frac{H(\mathcal{P}^q, \mu)}{q} + \frac{\epsilon}{3} \text{ if } \text{dist}(\rho, \mu) < \epsilon^*.$$

To prove assertion (iii) we assume $\text{dist}(\mu_n, \mu) < \epsilon^*$ for all $n \geq 1$. We deduce

$$\frac{H(\mathcal{P}^q, \mu_n)}{q} \leq \frac{H(\mathcal{P}^q, \mu)}{q} + \frac{\epsilon}{3} \quad \forall n \geq 1.$$

Joining this latter assertion with inequalities (4.7) and (4.8) we obtain

$$\frac{1}{n} H(\mathcal{P}^n, \nu_n) \leq h_\mu + \epsilon \quad \forall n \geq n(q).$$

Thus, after denoting $n_0 := n(q)$, assertion (iii) is proved. $\square$

Notation. Recall Equality (4.1) that defines the continuous real function $\psi : M \rightarrow \mathbb{R}$ that is the potential in the statement of Theorem 1.13. For any real number $r \geq 0$ construct

$$\mathcal{K}_r := \{\nu \in \mathcal{M}_f : \int \psi d\nu + h_\nu \geq -r\}. \quad (4.9)$$

(We note that, a priori, the set $\mathcal{K}_r$ of f -invariant probabilities may be empty.)

For any integer $n \geq 1$ and for any point $x \in M$ recall Definition 1.1 of the empirical probability $\sigma_n(x)$ and of the $p\omega$ -limit set $p\omega(x)$ in the set $\mathcal{M}$ of Borel probabilities. We also recall the weak* metric dist^* in the space $\mathcal{M}$ of probability measures constructed by equality (4.2).

Lemma 4.4. *Let f be a C^1 expanding map on M . Let m be the Lebesgue measure on M . Fix $r > 0$ and let $\mathcal{K}_r$ be defined by Equality (4.9). Then, for all $0 < \epsilon < r/2$, and for all $\mu \in \mathcal{M}_f$ such that $\mu \notin \mathcal{K}_r$, there exist $n_0 \geq 1$ and $0 < \epsilon^* \leq \epsilon/3$ such that*

$$m(\{x \in M : \text{dist}(\sigma_n(x), \mu) < \epsilon^*\}) < e^{n(\epsilon-r)} < e^{-nr/2} \quad \forall n \geq n_0. \quad (4.10)$$

Proof. As in the proof of Lemma 4.3, for simplicity, we write $\mathcal{P}^n$ instead of $\mathcal{P}_f^n$, and h_μ instead of $h_\mu(f)$. From Proposition 2.2, f is expansive in the future. Let $\alpha > 0$ be a expansivity constant for f . For the given value of $\epsilon > 0$, fix a uniform continuity modulus $0 < \delta < \alpha$ for $\epsilon/3$ of the function $\psi = -\log |\det(Df)|$. Namely,

$$|\psi(x) - \psi(y)| < \epsilon/3 \text{ if } \text{dist}(x, y) < \delta. \quad (4.11)$$

For such a value of δ , for the given measure $\mu \in \mathcal{M}_f$, and for $\epsilon/3$ instead of ϵ , apply Lemma 4.3 to construct the partition $\mathcal{P}$ in M , and the numbers $\epsilon^* > 0$ and $n_0 \geq 1$, such that assertions (i), (ii) and (iii) hold. In particular, assertion (iii) states that for any sequence of probability measures ν_n , if $\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f^j)^* \nu_n$, satisfies $\text{dist}(\mu_n, \mu) < \epsilon^* \quad \forall n \geq 1$, then

$$\frac{1}{n} H(\mathcal{P}^n, \nu_n) \leq h_\mu + \frac{\epsilon}{3} \quad \forall n \geq n_0. \quad (4.12)$$

It is not restrictive to assume that $\epsilon^* \leq \epsilon/3$. Denote, for all $n \geq 1$

$$C_n := \{x \in M : \text{dist}(\sigma_n(x), \mu) < \epsilon^*\} \quad (4.13)$$

To prove the Lemma we must prove that

$$m(C_n) \leq e^{n(\epsilon-r)} \quad \forall \quad n \geq n_0 \quad (\text{to be proved}) \quad (4.14)$$

Since f is C^1 expanding, its derivative Df_x is invertible for all $x \in M$. Thus, by the local inverse map theorem, f is a local diffeomorphism. The compactness of M implies that there exists a uniform value $\delta_1 > 0$ such that f restricted to any ball of radius δ_1 is a diffeomorphism onto its image. Therefore, if the diameter of the partition $\mathcal{P}$ is chosen small enough, the restricted map $f^n|_X : X \mapsto f^n(X)$ is a diffeomorphism for all $X \in \mathcal{P}^n$ and for all $n \geq 1$. Thus, recalling that $\psi = -\log|\det Df|$, we deduce the following equality for all $X \in \mathcal{P}^n$

$$m(X \cap C_n) = \int_{f^n(X \cap C_n)} |\det Df^{-n}| dm = \int_{f^n(X \cap C_n)} e^{\sum_{j=0}^{n-1} \psi \circ f^j} dm.$$

Therefore

$$m(C_n) = \sum_{X \in \mathcal{P}^n} \int_{f^n(X \cap C_n)} e^{\sum_{j=0}^{n-1} \psi \circ f^j} dm. \quad (4.15)$$

Either $C_n = \emptyset$, and Assertion (4.14) becomes trivially proved, or the finite family of pieces $\{X \in \mathcal{P}^n : X \cap C_n \neq \emptyset\} = \{X_1, \dots, X_N\}$ has $N = N(n) \geq 1$ pieces. In the latter case, choose a single point $y_k \in X_k \cap C_n$ for each $k = 1, \dots, N$. Denote by $Y(n) = \{y_1, \dots, y_N\}$ the collection of such points. Due to the construction of $\delta > 0$ according to Equation (4.11), and since the partition $\mathcal{P}$ has a diameter smaller than δ (because it satisfies (i) of Lemma 4.3), we deduce:

$$\sum_{j=0}^{n-1} \psi(f^j(y)) \leq \sum_{j=0}^{n-1} (\psi(f^j(y_k)) + \epsilon/3) \quad \forall \quad y, y_k \in X_k, \quad \forall \quad k = 1, \dots, N.$$

Therefore, substituting in Equality (4.15),

$$m(C_n) \leq e^{n\epsilon/3} \sum_{k=1}^N e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))} m(f^n(X_k \cap C_n)).$$

Thus

$$m(C_n) \leq e^{n\epsilon/3} \sum_{k=1}^N e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))}.$$

Define

$$L := \sum_{k=1}^N e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))}, \quad \lambda_k := \frac{1}{L} e^{\sum_{j=0}^{n-1} \psi(f^j(y_k))} \in (0, 1). \quad (4.16)$$

Then $\sum_{k=1}^N \lambda_k = 1$, and

$$m(C_n) \leq e^{(n\epsilon/3) + \log L}, \quad (4.17)$$

where

$$\log L = \left(\sum_{k=1}^N \lambda_k \sum_{j=0}^{n-1} \psi(f^j(y_k)) \right) - \left(\sum_{k=1}^N \lambda_k \log \lambda_k \right). \quad (4.18)$$

(To prove the equality above, take log in the equality at right in (4.16), multiply by λ_k and take the sum for $k = 1, \dots, N$.)

Define the probability measures

$$\nu_n := \sum_{k=1}^N \lambda_k \delta_{y_k}, \quad (4.19)$$

$$\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f^j)^*(\nu_n) = \sum_{k=1}^N \lambda_k \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(y_k)} = \sum_{k=1}^N \lambda_k \sigma_n(y_k). \quad (4.20)$$

To see the right equality above one can recall Definition 1.1 of the empirical probability measures $\sigma_n(y_k)$. Then,

$$\sum_{k=1}^N \lambda_k \sum_{j=0}^{n-1} \psi(f^j(y_k)) = n \int \psi d\mu_n. \quad (4.21)$$

Recall that for any piece $X_k \in \mathcal{P}^n$ such that $C_n \cap X_k \neq \emptyset$ we have chosen a single point $y_k \in C_n \cap X_k$. Then $\nu_n(X_k) = \lambda_k \delta_{y_k}(X_k) = \lambda_k$, and we deduce that

$$-\sum_{k=1}^N \lambda_k \log \lambda_k = H(\mathcal{P}^n, \nu_n). \quad (4.22)$$

Therefore, combining Equations (4.17), (4.18), (4.21) and (4.22), we deduce

$$m(C_n) \leq \exp\left(\frac{n\epsilon}{3} + \log L\right) = \exp\left(n\left(\frac{\epsilon}{3} + \int \psi d\mu_n + \frac{H(\mathcal{P}^n, \nu_n)}{n}\right)\right). \quad (4.23)$$

Now, we claim

$$\text{dist}(\mu_n, \mu) < \epsilon^* \leq \frac{\epsilon}{3} \quad \forall n \geq 1. \quad (4.24)$$

In fact, by construction $y_k \in C_n$ for all $k = 1, \dots, N$, Thus, from Equality (4.13), we have

$$\text{dist}(\sigma_n(y_k), \mu) < \epsilon^*.$$

Recalling Lemma 4.1, the ball in $\mathcal{M}$ of center μ and radius ϵ^* is convex. Thus, any convex combination of the measures $\sigma_n(y_k)$ belongs to that ball. From Equality (4.20) at right, μ_n is a convex combination of the measures $\sigma_n(y_k)$. We deduce that μ_n belongs to that ball. Hence, inequality (4.24) is proved. So equation (4.12) holds.

Combining Equations (4.12) and (4.23), we deduce

$$m(C_n) \leq \exp\left(n\left(\frac{2 \cdot \epsilon}{3} + \int \psi d\mu_n + h_\mu\right)\right) \quad \forall n \geq n_0.$$

Besides, from inequality (4.24) and the construction of the weak*-metric dist^* in $\mathcal{M}$ with $\phi_0 = \psi$ (recall Equality (4.2)), we deduce that

$$\left| \int \psi d\mu_n - \int \psi d\mu \right| < \frac{\epsilon}{3}, \quad \int \psi d\mu_n < \int \psi d\mu + \frac{\epsilon}{3}.$$

Therefore, we obtain

$$m(C_n) \leq \exp\left(n\left(\epsilon + \int \psi d\mu + h_\mu\right)\right) \quad \forall n \geq n_0. \quad (4.25)$$

Finally, by hypothesis $\mu \notin \mathcal{K}_r$. Thus, $\int \psi d\mu + h_\mu < -r$. Substituting this latter inequality in (4.25), we conclude (4.14), ending the proof. $\square$

The following lemma is a well-known elementary result in Probability Theory. We will apply it in the particular case where M is a compact Riemannian manifold and $\mathcal{B}$ is the Borel σ -algebra of subsets of M .

Lemma 4.5 (Borel-Cantelli). *Let μ be a probability measure on a measurable space $(M, \mathcal{B})$. Let $\{C_n\}_{n \geq 1}$ be a sequence of measurable subsets $C_n \subset M$. Then*

$$\sum_{n=1}^{+\infty} \mu(C_n) < +\infty \implies \mu\left(\bigcap_{N \geq 1} \bigcup_{n \geq N} C_n\right) = 0$$

Proof. The sequence $\left\{\bigcup_{n \geq N} C_n\right\}_{N \geq 1}$ is (not necessarily strictly) decreasing with N . Then

$$\mu\left(\bigcap_{N \geq 1} \bigcup_{n \geq N} C_n\right) = \lim_{N \rightarrow +\infty} \mu\left(\bigcup_{n \geq N} C_n\right) \leq \lim_{N \rightarrow +\infty} \sum_{n=N}^{+\infty} \mu(C_n).$$

Finally, $\lim_{N \rightarrow +\infty} \sum_{n=N}^{+\infty} \mu(C_n) = 0$ because $\sum_{n=N}^{+\infty} \mu(C_n)$ is the tail of the convergent series $\sum_{n=1}^{+\infty} \mu(C_n)$. $\square$

4.1 End of the proof of Theorem 1.13

Proof. We will prove that any pseudo-physical measure μ satisfies Pesin's Entropy Formula. Namely, for the C^1 -expanding map f , according to Proposition 1.12, we will prove that:

$$h_\mu(f) + \int \psi d\mu = 0 \quad \text{where } \psi := -\log |\det Df|.$$

For any $r > 0$ consider the compact set $\mathcal{K}_r \subset \mathcal{M}$ defined by Equality (4.9). Since $\{\mathcal{K}_r\}_r$ is decreasing when r decreases, we have

$$\mathcal{K}_0 = \bigcap_{r>0} \mathcal{K}_r \quad \text{where } \mathcal{K}_0 := \left\{\mu \in \mathcal{M}_f : \int \psi d\mu + h_\mu(f) \geq 0\right\}.$$

By Margulis-Ruelle's inequality (see Theorem 1.7) and Corollary 1.11 applied to C^1 expanding maps, we have

$$h_\mu(f) \leq \int \log |\det Df| d\mu = - \int \psi d\mu \quad \forall \mu \in \mathcal{M}_f. \quad (4.26)$$

Therefore, the (a-priori maybe empty) set $\mathcal{K}_0$ is composed by all the invariant measures μ such that

$$\int \psi d\mu + h_\mu(f) = 0,$$

or, in other words, $\mathcal{K}_0$ is the set of invariant measures μ that satisfy Pesin's Entropy Formula. So, to prove that any pseudo-physical measure μ satisfies Pesin's Entropy Formula, we must prove that $\mu \in \mathcal{K}_r$ for all $r > 0$.

Assume by contradiction that there exists $r > 0$ such that the pseudo-physical measure μ does not belong to $\mathcal{K}_r$. From Lemma 4.4, there exists $n_0 \geq 1$ and $\epsilon^* > 0$ such that

$$m\left(\{x \in M : \text{dist}^*(\sigma_n(x), \mu) < \epsilon^*\}\right) \leq e^{-nr/2}, \quad \forall n \geq n_0, \quad (4.27)$$

where m denotes the Lebesgue measure.

From Definition 1.4 of pseudo-physical measure, for any $\epsilon^* > 0$ the set

$$A = \{x \in M : \text{dist}(p\omega(x), \mu) < \epsilon^*\}$$

has positive Lebesgue measure: $m(A) > 0$. For each $n \geq 1$, denote

$$C_n := \{x \in M : \text{dist}(\sigma_n(x), \mu) < \epsilon^*\}.$$

Apply Definition 1.1 of the set $p\omega(x)$ in $\mathcal{M}$, composed by the weak*-limits of all the convergent subsequences of $\{\sigma_n(x)\}$. Therefore,

$$A \subset \bigcap_{N \geq 1} \bigcup_{n \geq N} C_n,$$

We deduce the following inequality:

$$m\left(\bigcap_{N \geq 1} \bigcup_{n \geq N} C_n\right) \geq m(A) > 0. \quad (4.28)$$

But inequality (4.27) implies that $\sum_{n=1}^{\infty} m(C_n) < +\infty$; hence, applying Borel-Cantelli Lemma (see Lemma 4.5), it follows that $m\left(\bigcap_{N \geq 1} \bigcup_{n \geq N} C_n\right) = 0$, which contradicts inequality (4.28). $\square$

5 Examples

Definition 5.1 ($C^{1+\alpha}$ -maps). We say that a map $f : M \mapsto M$ is C^1 plus Hölder, and denote $f \in C^{1+\alpha}$, if f is differentiable, with continuous derivative and besides there exists $\alpha > 0$ such that the derivative of f is α -Hölder continuous. Namely, for some constant $K > 0$, the following inequality holds:

$$\|Df_x - Df_y\| \leq K(\text{dist}(x, y))^\alpha \quad \text{for all } x, y \in M.$$

As stated in the introduction, the existence theory of physical measures (see Definition 1.2) for $C^{1+\alpha}$ expanding maps (see Definition 1.9) is well established. Therefore, we focus exclusively on examples of maps f that are of class C^1 but fail to be $C^{1+\alpha}$ for any $\alpha > 0$.

More precisely, we construct examples of $C^1 \setminus C^{1+\alpha}$ expanding maps on the circle S^1 and on the 2-torus T^2 , and investigate their pseudo-physical measures (see Definition 1.4). In the specific examples constructed throughout this section, the pseudo-physical measures are, in fact, physical. By applying Theorem 1.13, we deduce that all these measures satisfy Pesin's entropy formula (see Definition 1.8).

For the examples presented in this section, the invariant physical measures are absolutely continuous with respect to the Lebesgue measure. In particular, these examples do not belong to the C^1 -generic family of maps studied in [7].

First, we construct two examples on S^1 , followed by an example on $T^2 = S^1 \times S^1$ given by the product of the previous S^1 constructions. The two examples on S^1 are adapted from Subsection 2.1 of [5].

Since the circle S^1 can be obtained by identifying the endpoints of a closed interval, we construct our examples via interval maps consisting of finitely many C^1 -expanding, orientation-preserving pieces of continuity. We require these maps to be surjective on each continuity piece, and ensure that both the maps and their derivatives match smoothly at the boundaries of these pieces so that they induce well-defined C^1 maps on S^1 . Specifically, for the example illustrated in Figure 1, we fix a real number $0 < b_0 < 1$ and construct a piecewise C^1 map $f : [-1, 1] \rightarrow [-1, 1]$ satisfying the following conditions, thereby defining a C^1 -expanding and orientation-preserving map on the circle $S^1 = [-1, 1]/\sim$ where $-1 \sim 1$:

$$f(-1) = f(-b_0^+) = f(b_0^+) = -1, \quad f(1) = f(-b_0^-) = f(b_0^-) = 1,$$

where $f(x_0^+)$ and $f(x_0^-)$ denote the right-hand and left-hand lateral limits of $f(x)$ as $x \rightarrow x_0$, respectively. Furthermore, we assume that

$$f'(x) > 1 \quad \forall x \in (-1, -b_0) \cup (-b_0, b_0) \cup (b_0, 1),$$

and that the following lateral limits exist, match, and are finite:

$$f'(-1^+) = f'(1^-), \quad f'(-b_0^+) = f'(-b_0^-), \quad f'(b_0^+) = f'(b_0^-).$$

5.1 Dynamically Defined Cantor Sets

Before presenting the examples, we recall the construction of a Cantor set that is dynamically defined on an interval by an expanding map restricted to two closed disjoint subintervals.

To fix ideas, we will use the example $f : [-1, 1] \rightarrow [-1, 1]$ described above (see Figure 1).

Let $I_0 := [-1, -b_0]$, $I_1 := [b_0, 1]$, and $I_c^* := (-b_0, b_0)$. We denote the restrictions by $G := f|_{I_0 \cup I_1}$, $G_0 := f|_{I_0}$, and $G_1 := f|_{I_1}$ (see Figure 2). We then construct the following compact set:

$$K := \bigcap_{n=1}^{\infty} G^{-n}([-1, 1]) \subset I_0 \cup I_1. \quad (5.1)$$

Each point $x \in K$ has an *itinerary*, which is defined as the sequence of 0's and 1's given by

$$\underline{a} = \underline{a}(x) := a_0, a_1, \dots, a_n, \dots \in \{0, 1\}^{\mathbb{N}}, \quad \text{where } G^n(x) \in I_{a_n} \text{ for all } n \geq 0.$$

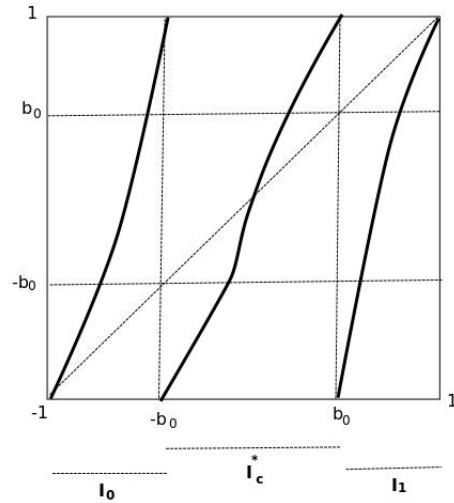


Fig. 1: An expanding map f on the circle S^1 with index 3.

For a fixed $n \geq 1$, we denote by $\underline{a}_n$ the following word of length n composed of 0's and 1's:

$$\underline{a}_n := a_0, a_1, \dots, a_{n-1} \in \{0, 1\}^n.$$

Furthermore, we define the interval

$$I_{\underline{a}_n} := G_{a_0}^{-1} \circ G_{a_1}^{-1} \circ \dots \circ G_{a_{n-1}}^{-1}([-1, 1]). \quad (5.2)$$

Note that for each fixed word $\underline{a}_n$, the set $I_{\underline{a}_n}$ is a closed interval because G_0 and G_1 are strictly increasing and continuous maps. By construction, each interval $I_{\underline{a}_n}$ consists of all points in $I_0 \cup I_1$ that share the same finite initial itinerary of length n , from time 0 to time $n-1$.

From the above construction, letting $\underline{a}_n(x)$ be the finite word of length $n \geq 1$ truncated from the sequence $\underline{a}(x)$ for a fixed $x \in K$, we have

$$x \in \bigcap_{n=1}^{\infty} I_{\underline{a}_n(x)}.$$

Since G_0 and G_1 are C^1 -expanding, the length $|I_{\underline{a}_{n+1}(x)}|$ of the interval $I_{\underline{a}_{n+1}(x)}$ satisfies

$$|I_{\underline{a}_{n+1}(x)}| \leq \lambda |I_{\underline{a}_n(x)}| \quad \text{for all } n \geq 1, \quad \text{with } 0 < \lambda < 1,$$

where $\lambda := (\min_{u \in I_0 \cup I_1} G'(u))^{-1}$. We conclude that $|I_{\underline{a}_n(x)}| \leq \lambda^n \rightarrow 0$ as $n \rightarrow +\infty$, which implies

$$\{x\} = \bigcap_{n=1}^{\infty} I_{\underline{a}_n(x)}.$$

Taking all possible finite words $\underline{a}_n \in \{0, 1\}^n$, we obtain for each time $n \geq 1$ the equality

$$G^{-n}([-1, 1]) = \bigcup_{\underline{a}_n \in \{0, 1\}^n} I_{\underline{a}_n},$$

which, combined with Equation (5.1), yields

$$K = \bigcap_{n=1}^{\infty} \bigcup_{\underline{a}_n \in \{0, 1\}^n} I_{\underline{a}_n}. \quad (5.3)$$

Since $|I_{\underline{a}_n(x)}| \rightarrow 0$ as $n \rightarrow +\infty$, it follows that K is a Cantor set.

Definition 5.2 (Dynamically defined Cantor set). We say that K is the *dynamically defined Cantor set* associated with the map G if it satisfies Equation (5.1) (and hence also Equation (5.3)).

Definition 5.3 (Atoms generating the Cantor set). For a fixed $n \geq 1$, the 2^n pairwise disjoint closed intervals $I_{\underline{a}_n}$ ($\underline{a}_n \in \{0, 1\}^n$) defined by Equation (5.2) and satisfying Equation (5.3) are called the *atoms of generation n* generating the Cantor set K .

We note that for a fixed $\underline{a}_n \in \{0, 1\}^n$, the atom $I_{\underline{a}_n}$ of generation n contains exactly two atoms of generation $n + 1$, namely $I_{a_0 a_1 \dots a_{n-1} 0}$ and $I_{a_0 a_1 \dots a_{n-1} 1}$, situated on the left and right respectively. These are obtained from $I_{\underline{a}_n}$ by removing an open interval $I_{\underline{a}_n}^*$ (see Figure 3). This matches the classical construction of a Cantor set on an interval, even when it is not dynamically defined.

Indeed, for $n = 1$, recall that G_0 and G_1 are continuous, strictly increasing, and map surjectively onto $[-1, 1] = I_0 \cup I_c^* \cup I_1$ (see Figure 2). Applying Equation (5.2), the two atoms of generation 2 contained in I_0 are I_{00} and I_{01} , which are the preimages under G_0 of I_0 and I_1 , respectively (see Figure 2). They are closed intervals situated on the left and right of I_0 , obtained after removing the open interval

$$I_0^* = G_0^{-1}(I_c^*). \quad (5.4)$$

Namely, $I_0 = I_{00} \cup I_0^* \cup I_{01}$. Analogously, $I_1 = I_{10} \cup I_1^* \cup I_{11}$, where

$$I_1^* = G_1^{-1}(I_c^*). \quad (5.5)$$

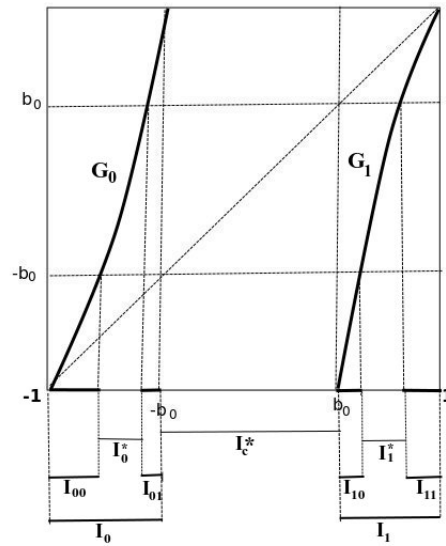


Fig. 2: The expanding map G defined in the intervals I_0 and I_1 , where $G_0 = G|_{I_0}$, $G_1 = G|_{I_1}$. The atoms of generation 1 of the Cantor set dynamically defined by G are $I_0 = G_0^{-1}([-1, 1])$ and $I_1 = G_1^{-1}([-1, 1])$. Inside them there are open intervals, the gaps of generation 1: $I_0^* = G_0^{-1}(-b_0, b_0)$ and $I_1^* = G_1^{-1}(-b_0, b_0)$, respectively. Removing these gaps from the atoms of generation 1, we obtain the atoms of generation 2: $I_{00} = G_0^{-1}(I_0)$, $I_{01} = G_0^{-1}(I_1)$, $I_{10} = G_1^{-1}(I_0)$, $I_{11} = G_1^{-1}(I_1)$.

By induction on $n \geq 2$, the continuity and strictly increasing property of the maps G_0 and G_1 ensure that there are exactly two atoms of generation $n + 1$ inside $I_{a_0 a_1 \dots a_{n-1}}$. These are obtained by taking the preimage under G_{a_0} of the closed subinterval

$$I_{a_1 \dots a_{n-1}} = I_{a_1 \dots a_{n-1} 0} \cup I_{a_1 \dots a_{n-1}}^* \cup I_{a_1 \dots a_{n-1} 1}.$$

Specifically, these two atoms of generation $n + 1$ are the preimages under G_{a_0} of the closed subintervals $I_{a_1 \dots a_{n-1} 0}$ and $I_{a_1 \dots a_{n-1} 1}$, respectively. They are obtained by removing the open interval

$$I_{a_0 a_1 \dots a_{n-1}}^* := G_{a_0}^{-1}(I_{a_1 \dots a_{n-1}}^*) \quad (5.6)$$

from $I_{a_0 a_1 \dots a_{n-1}}$, leaving the two subintervals situated on the left and right, respectively (see Figure 3). Namely,

$$I_{a_0 a_1 \dots a_{n-1}} = I_{a_0 a_1 \dots a_{n-1} 0} \cup I_{a_0 a_1 \dots a_{n-1}}^* \cup I_{a_0 a_1 \dots a_{n-1} 1}. \quad (5.7)$$

Definition 5.4 (Gaps of the Cantor set). The open interval I_c^* (see Figure 2) is called the *gap of generation 0* of the Cantor set K . For a fixed $n \geq 1$, the 2^n pairwise disjoint open intervals $I_{\underline{a}_n}^*$ contained respectively in the atoms $I_{\underline{a}_n}$ —as defined above for $\underline{a}_n \in \{0, 1\}^n$ by Equations (5.4), (5.5) and (5.6)—are called the *gaps of generation n* of the Cantor set K .

Observe that the countably infinite collection of all the gaps across all generations consists of pairwise disjoint open intervals.

From the construction of the atoms and gaps of K , we deduce that the union of the atoms of generation $n + 1$ is obtained from the union of the atoms of generation n by removing all the gaps of generation n . Consequently, using Equation (5.3), we obtain

$$K = [-1, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{\underline{a}_n \in \{0, 1\}^n} I_{\underline{a}_n}^*, \quad (5.8)$$

with the convention $\{0, 1\}^0 = \{c\}$. In other words, the Cantor set K is obtained from the interval $[-1, 1]$ by removing its countably infinite collection of pairwise disjoint gaps.

5.2 Bowen's construction of a dynamical defined Cantor set with positive Lebesgue measure.

The examples of $C^1 \setminus C^{1+\alpha}$ -expanding maps on S^1 and on T^2 that we construct in the following subsections are based on Bowen's method for constructing dynamically defined Cantor sets on an interval with positive Lebesgue measure. Throughout this subsection, we follow the specific construction introduced by Bowen in [10].

The construction is carried out in two steps. First, we construct a Cantor set K in the interval $[-1, 1]$ contained within two disjoint closed subintervals I_0 and I_1 . At this stage, K is not yet dynamically defined, but it possesses positive Lebesgue measure. Second, we construct a map G as illustrated in Figure 2 such that the restrictions $G_0 := G|_{I_0}$ and $G_1 := G|_{I_1}$ are $C^1 \setminus C^{1+\alpha}$ expanding, thereby ensuring that the Cantor set from the first step becomes dynamically defined by G in accordance with Definition 5.2.

Step 1: Construction of the Cantor set K with positive Lebesgue measure. We adopt the notation from Subsection 5.1.

Fix $0 < b_0 < 1$, and define the gap of generation 0 to be $I_c^* := (-b_0, b_0) \subset [-1, 1]$, which leaves the two atoms of generation 1, namely $I_0 := [-1, -b_0]$ and $I_1 := [b_0, 1]$.

Choose a sequence of positive real numbers $\alpha_n > 0$ for all $n \geq 1$ satisfying

$$\sum_{n=1}^{\infty} \alpha_n < 2(1 - b_0), \quad \alpha_{n+1} < \alpha_n \text{ for all } n \geq 1, \quad \lim_{n \rightarrow +\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1, \quad \alpha_1 < 2b_0 \quad (5.9)$$

(for instance, one may take $\alpha_n := (kn^2)^{-1}$ for a sufficiently large constant $k > 0$).

We now construct the two gaps I_0^* and I_1^* of generation 1, alongside the four atoms I_{00} , I_{01} , I_{10} , and I_{11} of generation 2. For each $a_0 \in \{0, 1\}$, let $I_{a_0}^* \subset I_{a_0}$ be the open interval centered at the midpoint of I_{a_0} with length $\alpha_1/2$. The atoms $I_{a_0 0}$ and $I_{a_0 1}$ of generation 2 are then the closed intervals obtained from I_{a_0} by removing the gap $I_{a_0}^*$ of generation 1, situated to the left and right of $I_{a_0}^*$, respectively.

By induction on $n \geq 1$, for every word $\underline{a}_n \in \{0, 1\}^n$, we construct the gaps $I_{\underline{a}_n}^*$ of generation n and the atoms $I_{\underline{a}_n 0}$ and $I_{\underline{a}_n 1}$ of generation $n + 1$ as follows: $I_{\underline{a}_n}^*$ is the open interval contained in the atom $I_{\underline{a}_n}$ of generation n , centered at the midpoint of $I_{\underline{a}_n}$ with length $\alpha_n/2^n$. The atoms $I_{\underline{a}_n 0}$ and $I_{\underline{a}_n 1}$ of generation $n + 1$ are the closed intervals obtained from $I_{\underline{a}_n}$ by removing the gap $I_{\underline{a}_n}^*$, situated to the left and right of $I_{\underline{a}_n}^*$, respectively (see Figure 3).

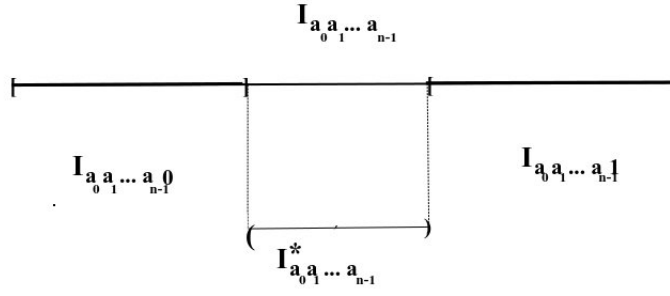


Fig. 3: The atom of generation n named $I_{\underline{a}_n}$ where $\underline{a}_n = a_0 a_1 \dots a_{n-1} \in \{0, 1\}^n$. Inside it, there is a gap of generation n named $I_{\underline{a}_n}^*$. Removing this gap from the atom $I_{\underline{a}_n}$, the two atoms of generation $n+1$ are obtained: $I_{\underline{a}_n 0}$ at left of the gap, and $I_{\underline{a}_n 1}$ at right.

We compute below the Lebesgue measure of the Cantor set K defined by Equation (5.3), or equivalently by Equation (5.8). By Equation (5.8), K is the complement in the interval $[-1, 1]$ of the union of all gaps. Therefore,

$$\text{Leb}(K) = \text{Leb}([-1, 1]) - \sum_{n=0}^{\infty} \sum_{\underline{a}_n \in \{0, 1\}^n} \text{Leb}(I_{\underline{a}_n}^*)$$

(recall the convention $\{0, 1\}^0 := \{c\}$). The gap I_c^* of generation 0 has length $2b_0$, and for all $n \geq 1$, the 2^n gaps of generation n all possess the same length $\alpha_n/2^n$. It follows that

$$\text{Leb}(K) = 2 - 2b_0 - \sum_{n=1}^{\infty} 2^n \frac{\alpha_n}{2^n} = 2(1 - b_0) - \sum_{n=1}^{\infty} \alpha_n > 0$$

(recall the left-hand inequality in (5.9)).

Step 2. Construction of the map G that renders K dynamically defined.

The map $G : I_0 \cup I_1 \rightarrow [-1, 1]$ to be constructed must satisfy Equations (5.4), (5.5), and (5.6). Accordingly, we will define G on each gap of K to adhere to these relations.

We begin with the gaps of generation 1: for each $a_0 \in \{0, 1\}$, we construct the restriction $G|_{I_{a_0}^*} : I_{a_0}^* \rightarrow I_c^*$ as a C^1 -expanding, orientation-preserving, and surjective map satisfying

$$\lim_{x \rightarrow \partial I_{a_0}^*} G'(x) = 2. \quad (5.10)$$

Here, the notation $x \rightarrow \partial I_{a_0}^*$ indicates that x approaches the boundaries from within the interior of the interval $I_{a_0}^*$.

Such a construction is feasible because $|I_c^*|/|I_{a_0}^*| = 2b_0/(\alpha_1/2) = 2 \cdot (2b_0)/\alpha_1 > 2$ (recall the right-hand inequality in (5.9)).

Next, we define G on the gaps of generation $n+1$ for all $n \geq 1$: for each $\underline{a}_{n+1} = a_0 a_1 \dots a_{n-1} a_n \in \{0, 1\}^{n+1}$, we construct $G|_{I_{\underline{a}_{n+1}}^*} : I_{a_0 a_1 \dots a_{n-1} a_n}^* \rightarrow I_{a_1 \dots a_{n-1} a_n}^*$ as a C^1 -expanding, orientation-preserving, and surjective map satisfying

$$\lim_{x \rightarrow \partial I_{\underline{a}_{n+1}}^*} G'(x) = 2. \quad (5.11)$$

This construction is possible because $|I_{a_1 \dots a_{n-1} a_n}^*|/|I_{a_0 a_1 \dots a_{n-1} a_n}^*| = (\alpha_n/2^n)/(\alpha_{n+1}/2^{n+1}) = 2 \cdot \alpha_n/\alpha_{n+1} > 2$ (recall the second inequality from the left in (5.9)).

Furthermore, as n tends to infinity, we construct G on each gap of generation $n+1$ such that, for some sequence $\epsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, the following estimate holds:

$$|G'(x) - 2| < \epsilon_n \quad \text{for all } x \in I_{\underline{a}_{n+1}}^* \text{ and all } \underline{a}_{n+1} \in \{0, 1\}^{n+1}. \quad (5.12)$$

This condition is achievable since $\lim_{n \rightarrow +\infty} \alpha_{n+1}/\alpha_n = 1$ by the final condition in (5.9). Consequently,

$$\lim_{n \rightarrow +\infty} \frac{|I_{a_1 \dots a_{n-1} a_n}^*|}{|I_{a_0 a_1 \dots a_{n-1} a_n}^*|} = \lim_{n \rightarrow +\infty} 2 \cdot \frac{\alpha_n}{\alpha_{n+1}} = 2.$$

Having defined the map G on all gaps of the Cantor set K contained in $I_0 \cup I_1$ according to conditions (5.4), (5.5), and (5.6), we extend it continuously to K . This yields a continuous map G on $I_0 \cup I_1$ as in Figure 2. This map G is C^1 -expanding on the union of all gaps. Due to Equations (5.10) and (5.11), and to Equality (5.12), the map G extends to a C^1 map on K as well, making it expanding on both intervals I_0 and I_1 with $G'(x) = 2$ for all $x \in K$.

Since G satisfies Equations (5.4), (5.5), and (5.6), and the Cantor set K constructed in Step 1 satisfies Equations (5.3) and (5.8), it follows that K also satisfies Equation (5.1). Thus, K is the dynamically defined Cantor set associated with G , as claimed.

Assertion. *The map G constructed above is not $C^{1+\alpha}$ for any $\alpha > 0$.*

In fact, since $\text{Leb}(K) > 0$, this assertion is a restatement of the following proposition:

Proposition 5.5. *Let $G : I_0 \cup I_1 \mapsto [-1, 1]$ be a $C^{1+\alpha}$ -expanding map such that $G_0 = G|_{I_0}$ and $G_1 = G|_{I_1}$ are surjective on $[-1, 1]$ as in Figure 2. Then, the Cantor set dynamically defined by G according to Definition 5.2, has zero Lebesgue measure.*

Proposition 5.5 is a well known result of the ergodic theory of $C^{1+\alpha}$ dynamical systems. For the sake of completeness, we include its proof here.

Proof. For fixed $\underline{a}_n = a_0 a_1 \dots a_{n-1} \in \{0, 1\}^n$, $n \geq 1$, recall the construction of the atom $I_{\underline{a}_n}$ of the Cantor set K given by Equality (5.2). Since G is C^1 , let us compute the length (the Lebesgue measure) of the closed interval $I_{\underline{a}_n}$. We apply the Mean Value Theorem of the differential calculus. We obtain

$$|[-1, 1]| = (G^n)'(x_{\underline{a}_n}) \cdot |I_{\underline{a}_n}|$$

for some point $x_{\underline{a}_n} \in I_{\underline{a}_n}$.

Now, consider the gap $I_{\underline{a}_n}^* \subset I_{\underline{a}_n}$ as in Equality (5.7). Using Equalities (5.4), (5.5) and (5.6), we obtain

$$I_c^* = G^n(I_{\underline{a}_n}^*)$$

and, applying again the Mean Value Theorem, there exists some point $y_{\underline{a}_n} \in I_{\underline{a}_n}^* \subset I_{\underline{a}_n}$ such that

$$|I_c^*| = (G^n)'(y_{\underline{a}_n}) \cdot |I_{\underline{a}_n}^*|$$

We obtain

$$\frac{|I_{\underline{a}_n}^*|}{|I_{\underline{a}_n}|} = \frac{(G^n)'(x_{\underline{a}_n})}{(G^n)'(y_{\underline{a}_n})} \cdot r_0 \text{ where } 0 < r_0 := \frac{|I_c^*|}{|[-1, 1]|} < 1. \quad (5.13)$$

Note that the points $x_{\underline{a}_n}$ and $y_{\underline{a}_n}$ have the same itinerary up to time n because both belong to the same atom $I_{\underline{a}_n}$. Therefore we can apply the following well known lemma that holds for $C^{1+\alpha}$ -expanding maps:

Bounded Distortion Lemma. *If G is a $C^{1+\alpha}$ expanding map as in Figure 2, then there exists a constant $c > 0$ such that for all $n \geq 1$, if x and y have the same itinerary up to time n , the following inequality holds:*

$$e^{-c} < \frac{(G^n)'(x)}{(G^n)'(y)} < e^c$$

The proof of the Bounded Distortion Lemma is, for example, in [25, Thm 1, Chap 4, p.58].

We substitute the inequality of the Bounded Distortion Lemma in Equality (5.13) and deduce

$$\frac{|I_{\underline{a}_n}^*|}{|I_{\underline{a}_n}|} > e^{-c} \cdot r_0.$$

Then

$$\sum_{\underline{a}_n \in \{0,1\}^n} |I_{\underline{a}_n}^*| > e^{-c} \cdot r_0 \cdot \sum_{\underline{a}_n \in \{0,1\}^n} |I_{\underline{a}_n}|. \quad (5.14)$$

Denote

$$A_n := \bigcup_{\underline{a}_n \in \{0,1\}^n} I_{\underline{a}_n}, \quad U_n := \bigcup_{\underline{a}_n \in \{0,1\}^n} I_{\underline{a}_n}^*.$$

More precisely, A_n is the union of all the atoms of generation n , and U_n is the union of all the gaps of generation n , one gap inside each atom of the same generation. From the construction of the Cantor set dynamically defined by G , we have

$$A_{n+1} = A_n \setminus U_n, \text{ hence } m(A_{n+1}) = m(A_n) - m(U_n),$$

where m denotes the Lebesgue measure.

The atoms of generation n are pairwise disjoint, then the gaps of generation n contained in them are also pairwise disjoint and we obtain the following result using inequality (5.14):

$$m(U_n) = \sum_{\underline{a}_n \in \{0,1\}^n} |I_{\underline{a}_n}^*| > e^{-c} \cdot r_0 \cdot \sum_{\underline{a}_n \in \{0,1\}^n} |I_{\underline{a}_n}| = e^{-c} \cdot r_0 \cdot m(A_n).$$

Then

$$m(A_{n+1}) < m(A_n)(1 - e^{-c} \cdot r_0) = \lambda \cdot m(A_n) \text{ where } 0 < \lambda := 1 - e^{-c} \cdot r_0 < 1.$$

Since the above inequality holds for all $n \geq 1$ we deduce

$$m(A_n) < \lambda^{n-1} m(A_1) \text{ for all } n \geq 1.$$

Applying Equality (5.3) and taking into account that $A_{n+1} \subset A_n$ for all $n \geq 1$ we conclude

$$m(K) = m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow +\infty} m(A_n) \leq \lim_{n \rightarrow +\infty} \lambda^{n-1} m(A_1) = 0,$$

completing the proof of Proposition 5.5. $\square$

5.3 First example on S^1 .

In this subsection we construct a $C^1 \setminus C^{1+\alpha}$ -expanding map on the circle S^1 that possesses a pseudo-physical measure μ_K that is actually physical. It is supported on a Cantor set $K \subset S^1$ of positive Lebesgue measure. The physical measure μ_K is Bernoulli (and hence ergodic) and absolutely continuous with respect to the Lebesgue measure though not equivalent to it (although μ_K is equivalent to the restriction of the Lebesgue measure to K). By Theorem 1.13, μ_K satisfies Pesin's entropy formula.

The construction of this example is based on Subsections 5.1 and 5.2, and is adapted from the first part of Example 2.1 in [5].

Consider the construction from Subsection 5.2 of the $C^1 \setminus C^{1+\alpha}$ -expanding map $G : I_0 \cup I_1 \rightarrow [-1, 1]$ illustrated in Figure 2, alongside the dynamically defined Cantor set K satisfying $m(K) > 0$, where m denotes the Lebesgue measure. We construct an orientation-preserving, surjective, and C^1 -expanding map $G_c : \overline{I_c^*} \rightarrow [-1, 1]$ such that

$$\lim_{x \rightarrow -b_0^+} G_c'(x) = \lim_{x \rightarrow b_0^-} G_c'(x) = 2.$$

We then define $f : [-1, 1] \rightarrow [-1, 1]$, a piecewise $C^1 \setminus C^{1+\alpha}$ -expanding map with three pieces of continuity (corresponding to degree 3 as an expanding map on the circle), which is orientation-preserving and surjective on each piece (see Figure 1), as follows:

$$f(x) := \begin{cases} G_0(x) = G|_{I_0}(x) & \text{if } x \in I_0 = [-1, -b_0], \\ G_c(x) & \text{if } x \in I_c^* = (-b_0, b_0), \\ G_1(x) = G|_{I_1}(x) & \text{if } x \in I_1 = [b_0, 1]. \end{cases}$$

As noted at the beginning of this section, f induces a well-defined $C^1 \setminus C^{1+\alpha}$ -expanding map on the circle S^1 .

5.3.1 The invariant probability measure

We define the probability measure μ_K by

$$\mu_K(B) := \frac{m(B \cap K)}{m(K)} \quad \text{for any Borel set } B \subset S^1.$$

By construction, μ_K is supported on the Cantor set K and satisfies $\mu_K \ll m$. However, m is not absolutely continuous with respect to μ_K because the Lebesgue measure of the gaps is strictly positive, whereas their μ_K -measure is zero.

Proposition 5.6. *The probability measure μ_K is f -invariant.*

Proof. Since the atoms of all generations intersected with the Cantor set K generate the σ -algebra of K , to prove that μ_K is f -invariant it is enough to prove that $\mu_K(f^{-1}(A_{\underline{a}_n})) = \mu_K(A_{\underline{a}_n})$ for all $\underline{a}_n \in \{0, 1\}^n$. By construction of the Cantor set K dynamically defined by $G = f|_{I_0 \cup I_1}$, all the 2^n atoms of generation n have the same Lebesgue measure. We conclude that $\mu_K(A_{\underline{a}_n}) = 1/2^n$. Also $f^{-1}(A_{\underline{a}_n}) = G^{-1}(A_{a_0 a_1 \dots a_{n-1}}) = A_{0a_0 a_1 \dots a_{n-1}} \cup A_{1a_0 a_1 \dots a_{n-1}}$, which are two disjoint atoms of generation $n+1$. Since all the 2^{n+1} atoms of generation $n+1$ have the same Lebesgue measure, we obtain

$$\mu_K(f^{-1}(A_{\underline{a}_n})) = \mu_K(A_{0a_0 a_1 \dots a_{n-1}}) + \mu_K(A_{1a_0 a_1 \dots a_{n-1}}) = \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} = \mu_K(A_{\underline{a}_n}),$$

as needed. $\square$

5.3.2 The Bernoulli property.

Proposition 5.7. *The probability measure μ_K is Bernoulli for f . Precisely, it is equivalent, through a bimeasurable conjugation $h : K \mapsto 2^{\mathbb{N}}$, to the Bernoulli measure ν on the shift space $2^{\mathbb{N}}$ of all the sequences of 0's and 1's that assigns the same weight $1/2$ to each symbol 0 or 1.*

Proof. Consider the shift space $2^{\mathbb{N}}$ of two symbols composed by all the sequences of 0's and 1's. Precisely,

$$2^{\mathbb{N}} := \{\underline{a} = a_0 a_1 \dots a_n \dots : a_n \in \{0, 1\} \text{ for all } n \geq 0\}.$$

Consider the shift map to the right: $\sigma : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ defined by

$$\sigma(a_0 a_1 \dots a_n \dots) = a_1 \dots a_n \dots$$

and the σ -invariant Bernoulli measure ν in $2^{\mathbb{N}}$ giving to each digit 0 or 1 the same weight $1/2$.

Define $h : K \rightarrow 2^{\mathbb{N}}$, assigning to each point $x \in K$ its itinerary $a_0 a_1 \dots a_n \dots$, namely $f^n(x) \in I_{a_n}$ for all $n \geq 0$. As seen in Subsection 5.1 h is invertible. Besides, from the construction of the atoms of K when applying $G = f|_{I_0 \cup I_1}$ to an atom $I_{a_0 a_1 \dots a_{n-1}}$ of generation $n \geq 2$, one obtains the atom $I_{a_1 \dots a_{n-1}}$ of generation $n-1$. So,

$$h \circ f(x) = \sigma \circ h(x) \quad \text{for all } x \in K,$$

and h is a measurable conjugation, with measurable inverse, between $f|_K$ and the shift σ . To prove that μ_K is Bernoulli, we will prove that

$$\mu_K = h^* \nu, \quad (5.15)$$

where h^* is the pull forward defined by $h^* \nu(B) = \nu(h(B))$ for any Borel set $B \subset K$.

Since the atoms of all generations, intersected with K , generate the Borel σ -algebra of K , it is enough to prove Equality (5.15) for $B = I_{\underline{a}_n}$ for any $\underline{a}_n \in \{0, 1\}^n$ and any $n \geq 1$.

On the one hand, as said at the beginning, the measure μ_K of any atom of generation n is $1/2^n$. On the other hand, $h(I_{\underline{a}_n})$ is the cylinder $C_{\underline{a}_n} \subset 2^{\mathbb{N}}$, where $\underline{a}_n = a_0 a_1 \dots a_{n-1}$ is fixed. Precisely, the cylinder $C_{\underline{a}_n}$ is the set of all the sequences of 0's and 1's such that their first n terms are fixed and equal to $a_0 a_1 \dots a_{n-1}$.

So $h^* \nu(I_{\underline{a}_n}) = \nu(h(I_{\underline{a}_n})) = \nu(C_{\underline{a}_n}) = p(a_0) \cdot p(a_1) \cdot \dots \cdot p(a_{n-1})$, where $p(a_i)$ is the weight of the symbol a_i of the Bernoulli measure ν . Since both symbols 0 and 1 are equally weighted by $1/2$ we conclude that $h^* \nu(I_{\underline{a}_n}) = 1/2^n = \mu_K(I_{\underline{a}_n})$, as stated. $\square$

Proposition 5.8. *The measure μ_K is physical.*

Proof. Recall Definition 1.1 of the basin of statistical attraction of a probability measure. Since μ_K is Bernoulli, it is ergodic and the sequence of empiric probabilities $\sigma_n(x)$ converges to μ_K for μ_K -almost every point x . In other words, the basin $B(\mu_K)$ of statistical attraction of μ_K contains μ_K -almost all the points. Since μ_K is absolutely continuous with respect to the Lebesgue measure m , we conclude that $m(B(\mu_K)) > 0$ and applying Definition 1.2, μ_K is physical. $\square$

5.4 Second example on S^1

In this subsection, we construct a $C^1 \setminus C^{1+\alpha}$ -expanding map on the circle S^1 that possesses finitely many pseudo-physical measures μ_{K_i} ($i = 1, 2, \dots, m$) that are actually physical. They are supported on pairwise disjoint Cantor sets $K_i \subset S^1$ of positive Lebesgue measure. All these physical measures μ_{K_i} are Bernoulli (and hence ergodic) and absolutely continuous with respect to the Lebesgue measure. Since they are physical, Theorem 1.13 implies that each measure μ_{K_i} satisfies Pesin's entropy formula.

The construction of this example is based on the model in Subsection 5.3 and is adapted from Example 2.1 of [5]. The method consists of modifying the map f from the example in Subsection 5.3 inside a gap of the Cantor set K dynamically defined by f , by introducing an additional piece of continuity within this gap. Subsequently, we construct a new Cantor set with positive Lebesgue measure inside this gap by reapplying Bowen's construction detailed in Subsection 5.2. This procedure can be repeated finitely many times by choosing a different gap from the previously constructed Cantor sets at each step. However, this process cannot be iterated infinitely many times because each application increases the number of continuity pieces in the interval $[-1, 1]$ (corresponding to the degree of the induced map on the circle S^1).

To fix ideas, we explain in detail the construction of a second Cantor set with positive Lebesgue measure and a new piece of continuity inside the gap $I_c^* = (-b_0, b_0)$ shown in Figure 1.

Let f_1 be the expanding map on the circle from the example in Subsection 5.3, and denote by K_1 its dynamically defined Cantor set contained in the intervals $I_0 \cup I_1$ as in Figure 1, with I_c^* being its gap of generation 0. Choose two real numbers c_1 and b_1 such that

$$0 < b_1 < c_1 < b_0 < 1.$$

We replace the restriction $f_1|_{I_c^*}$ on the gap I_c^* of generation 0 with a pair of C^1 -expanding, orientation-preserving, and surjective maps $g_0 : [-b_0, 0] \rightarrow [-1, 1]$ and $g_1 : [0, b_0] \rightarrow [-1, 1]$ as illustrated in Figure 4, satisfying

$$\begin{aligned} g_0(-c_1) &= -c_1, & g_0(-b_1) &= c_1, & g_1(b_1) &= -c_1, & g_1(c_1) &= c_1, \\ \lim_{x \rightarrow -b_0^+} g_0'(x) &= \lim_{x \rightarrow b_0^-} g_1'(x) = 2, & \lim_{x \rightarrow 0^-} g_0'(x) &= \lim_{x \rightarrow 0^+} g_1'(x). \end{aligned}$$

Construct $g : [-1, 1] \mapsto [-1, 1]$ with four order preserving, surjective and expanding pieces, as follows (see Figure 4):

$$g(x) := \begin{cases} f_1|_{I_0}(x) = G_0(x) & \text{if } x \in I_0 = [-1, -b_0] \\ g_0(x) & \text{if } x \in (-b_0, 0] \\ g_1(x) & \text{if } x \in (0, b_0) \\ f_1|_{I_1}(x) = G_1(x) & \text{if } x \in I_1 = [b_0, 1] \end{cases}$$

We have that $-c_1 \in (-b_0, 0)$ is the fixed point of g_0 and $c_1 \in (0, b_0)$ is the fixed point of g_1 . Observe that, in the interval $[-c_1, c_1]$ there are two disjoint closed subintervals J_0 and J_1 separated by an open interval (a gap) J_c^* , defined by:

$$J_0 := [-c_1, -b_1] = g_0^{-1}([-c_1, c_1]), \quad J_1 := [b_1, c_1] = g_1^{-1}([-c_1, c_1]),$$

$$J_c^* := (-b_1, b_1) = [-c_1, c_1] \setminus (J_0 \cup J_1).$$

As seen in Subsection 5.1, the C^1 -expanding maps $g_0|_{J_0} : J_0 \rightarrow [-c_1, c_1]$ and $g_1|_{J_1} : J_1 \rightarrow [-c_1, c_1]$ dynamically define a Cantor set contained in $J_0 \cup J_1$.

Now, we replace $g_0|_{J_0}$ and $g_1|_{J_1}$ with $C^1 \setminus C^{1+\alpha}$ -expanding, orientation-preserving, and surjective maps

$$H_0 : J_0 \mapsto [-c_1, c_1], \quad H_1 : J_1 \mapsto [-c_1, c_1]$$

respectively, such that the Cantor set K_2 dynamically defined by H_0 and H_1 in $J_0 \cup J_1$ possesses positive Lebesgue measure. These maps H_0 and H_1 , alongside the Cantor set K_2 , are constructed by applying Bowen's method detailed in Subsection 5.2, where we replace the interval $[-1, 1]$ with $[-c_1, c_1]$, the atoms of generation 1, namely I_0 and I_1 , with J_0 and J_1 respectively, and the gap $I_c^* = (-b_0, b_0)$ with the gap $J_c^* = (-b_1, b_1)$.

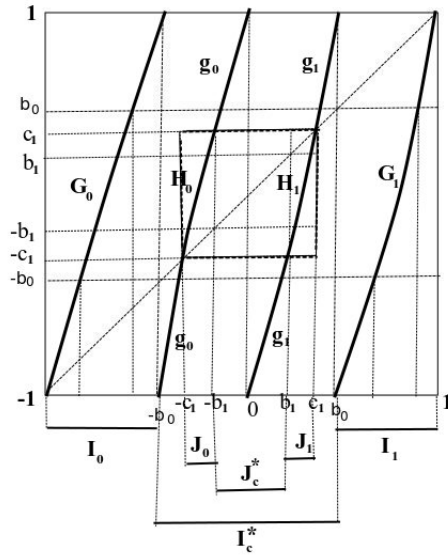


Fig. 4: The maps $G_0 = f|_{I_0} : I_0 \rightarrow [-1, 1]$ and $G_1 = f|_{I_1} : I_1 \rightarrow [-1, 1]$ dynamically define the first Cantor set $K_1 \subset I_0 \cup I_1$ with positive Lebesgue measure, whose atoms of generation 1 are I_0 and I_1 . The maps $H_0 = f|_{J_0} : J_0 \rightarrow [-c_1, c_1]$ and $H_1 = f|_{J_1} : J_1 \rightarrow [-c_1, c_1]$ dynamically define the second Cantor set $K_2 \subset J_0 \cup J_1$, also with positive Lebesgue measure, whose atoms of generation 1 are J_0 and J_1 .

Finally, modify g_0 (we will still call it g_0) in a small left neighborhood of $-c_1$ and in a small right neighborhood of $-b_1$ so it C^1 -glues well with H_0 at $-c_1$ and $-b_1$. Analogously modify g_1 (we will still call it g_1) in a small left neighborhood of b_1 and in a small right neighborhood of c_1 so it C^1 -glues well with H_1 at b_1 and c_1 (see Figure 4).

Define

$$f(x) := \begin{cases} f_1|_{I_0}(x) = G_0(x) & \text{if } x \in I_0 = [-1, -b_0] \\ f_1|_{I_1}(x) = G_1(x) & \text{if } x \in I_1 = [b_0, 1] \\ H_0(x) & \text{if } x \in J_0 = [-c_1, -b_1] \\ H_1(x) & \text{if } x \in J_1 = [b_1, c_1] \\ g_0(x) & \text{if } x \in (-b_0, -c_1) \cup (-b_1, 0] \\ g_1(x) & \text{if } x \in (0, b_1) \cup (c_1, b_0) \end{cases}$$

By construction f dynamically defines two Cantor sets $K_1 \subset I_0 \cup I_1$ and $K_2 \subset J_0 \cup J_1$ with positive Lebesgue measures.

Construct the following two probability measures supported on K_1 and K_2 , respectively:

$$\mu_{K_1}(B) := \frac{m(B \cap K_1)}{m(K_1)}, \quad \mu_{K_2}(B) := \frac{m(B \cap K_2)}{m(K_2)},$$

for any Borel set $B \subset [-1, 1]$, where m denotes the Lebesgue measure.

By construction μ_{K_1} and μ_{K_2} are absolutely continuous with respect to the Lebesgue measure. As proved in Subsection 5.3, μ_{K_1} and μ_{K_2} are f -invariant, Bernoulli and physical. Besides, as proved in that subsection, the basins of statistical attractions $B(\mu_{K_1})$ and $B(\mu_{K_2})$ contain Lebesgue almost all points of K_1 and K_2 respectively.

5.5 Example on T^2 .

In this subsection we construct a $C^1 \setminus C^{1+\alpha}$ -expanding map on the 2-torus T^2 that possesses finitely many pseudo-physical measures μ_{K_i} ($i = 1, 2, \dots, k$). They are actually physical and supported on pairwise disjoint Cantor sets $K_i \subset T^2$ of positive Lebesgue measure. All these physical measures μ_{K_i} are Bernoulli (and hence ergodic) and absolutely continuous with respect to the Lebesgue measure, though not equivalent to it. By Theorem 1.13, each measure μ_{K_i} satisfies Pesin's entropy formula.

The construction is based on the examples detailed in Subsections 5.3 and 5.4.

Since $T^2 = S^1 \times S^1$, we construct $f : T^2 \rightarrow T^2$ by choosing two maps $f_1, f_2 : S^1 \rightarrow S^1$ and defining

$$f(x, y) := (f_1(x), f_2(y)) \quad \text{for all } (x, y) \in T^2. \quad (5.16)$$

Specifically, we choose f_1 to be the expanding map on the circle constructed in the example of Subsection 5.3, and f_2 to be the expanding map on the circle constructed in the example of Subsection 5.4.

Proposition 5.9. *The map f defined by Equality (5.16) is $C^1 \setminus C^{1+\alpha}$ -expanding on the torus T^2 .*

Proof. First, f is $C^1 \setminus C^{1+\alpha}$ because f_1 and f_2 also are. Second, let us prove that f is expanding. Recall Definition 1.9 of expanding maps on manifolds. Take $(x, y) \in T^2$ and $v = (v_1, v_2)$ in the tangent space $T_{(x,y)}T^2$.

We have

$$\begin{aligned} \|Df_{(x,y)}v\| &= \|Df_{(x,y)}(v_1, v_2)\| = \|f'_1(x)v_1, f'_2(y)v_2\| = \\ &= \sqrt{(f'_1(x)v_1)^2 + (f'_2(y)v_2)^2} \geq \gamma \sqrt{v_1^2 + v_2^2} = \gamma \|v\|, \end{aligned}$$

where $\gamma = \min\{\min_{x \in S^1} f'_1(x), \min_{y \in S^1} f'_2(y)\}$. As $\gamma > 1$ because f_1 and f_2 are expanding maps on the circle, we conclude that f is expanding on the torus, as stated. $\square$

Let m_{T^2} denote the Lebesgue measure on the torus, and m_{S^1} the Lebesgue measure on the circle. We have

$$m_{T^2} = m_{S^1} \times m_{S^1}, \quad m_{T^2}(B) = m_{S^1}(B_1) \cdot m_{S^1}(B_2) \quad (5.17)$$

for any measurable rectangle $B = B_1 \times B_2$ in the torus, where B_1 and B_2 are Borel sets in the circle.

Analogously, given two finite measures μ_{M_1} and μ_{M_2} on the measurable spaces M_1 and M_2 , respectively, the product measure $\mu := \mu_{M_1} \times \mu_{M_2}$ on the product space $M_1 \times M_2$ satisfies

$$\mu(B) = \mu_{M_1}(B_1) \cdot \mu_{M_2}(B_2) \quad (5.18)$$

for any rectangle $B = B_1 \times B_2$, where $B_1 \subset M_1$ and $B_2 \subset M_2$ are measurable sets.

On the one hand, from the construction in Subsection 5.3, the map f_1 on S^1 dynamically defines a Cantor set K_{S^1} of positive Lebesgue measure, which supports a f_1 -invariant measure

$$\mu_{K_{S^1}} \ll m_{S^1}.$$

This measure is Bernoulli and physical for f_1 .

On the other hand, from the construction in Subsection 5.4, we know that the map f_2 on S^1 dynamically defines $k \geq 1$ pairwise disjoint Cantor sets $K_{1,S^1}, K_{2,S^1}, \dots, K_{k,S^1}$, all possessing positive Lebesgue measure, which support k distinct f_2 -invariant measures

$$\mu_{K_{1,S^1}}, \mu_{K_{2,S^1}}, \dots, \mu_{K_{k,S^1}} \ll m_{S^1}.$$

These measures are Bernoulli and physical for f_2 .

We construct the following k pairwise disjoint Cantor sets on the torus:

$$K_{i,T^2} := K_{S^1} \times K_{i,S^1} \quad \text{for all } 1 \leq i \leq k.$$

They all have positive Lebesgue measure on the torus. Indeed, applying Equation (5.17), we have

$$m_{T^2}(K_{i,T^2}) = m_{S^1}(K_{S^1}) \cdot m_{S^1}(K_{i,S^1}) > 0 \quad \text{for all } 1 \leq i \leq k.$$

Recall from the constructions in Subsections 5.3 and 5.4 that the following probability measures on S^1 are f_1 - and f_2 -invariant, respectively, Bernoulli and physical:

$$\begin{aligned} \mu_{K_{S^1}}(B_1) &:= \frac{m_{S^1}(K_{S^1} \cap B_1)}{m_{S^1}(K_{S^1})} \quad \forall \text{ Borel set } B_1 \subset S^1, \\ \mu_{K_{i,S^1}}(B_2) &:= \frac{m_{S^1}(K_{i,S^1} \cap B_2)}{m_{S^1}(K_{i,S^1})} \quad \forall \text{ Borel set } B_2 \subset S^1, \quad \forall i \in \{1, 2, \dots, k\}. \end{aligned}$$

We then construct the following probability measures on T^2 :

$$\mu_{K_{i,T^2}} := \mu_{K_{S^1}} \times \mu_{K_{i,S^1}} \quad \forall i \in \{1, 2, \dots, k\}.$$

It is immediate to verify that $\mu_{K_{i,T^2}}$ is supported on the Cantor set K_{i,T^2} and is f -invariant. In fact, it suffices to check that $\mu_{K_{i,T^2}}(f^{-1}(B)) = \mu_{K_{i,T^2}}(B)$ for any measurable rectangle $B = B_1 \times B_2$.

Proposition 5.10. *For all $i \in \{1, 2, \dots, k\}$ and for all Borel set $B \subset T^2$ the following equality holds:*

$$\mu_{K_{i,T^2}}(B) = \frac{m_{T^2}(K_{i,T^2} \cap B)}{m_{T^2}(K_{i,T^2})}.$$

Hence $\mu_{K_{i,T^2}} \ll m_{T^2}$.

Proof. Since the measurable rectangles generate the Borel σ -algebra of T^2 it is enough to check the equality of the proposition for $B = B_1 \times B_2$, where $B_1, B_2 \subset S^1$ are Borel sets in the circle. In fact, applying Equalities (5.17) and (5.18), we obtain:

$$\begin{aligned} \frac{m_{T^2}(K_{i,T^2} \cap (B_1 \times B_2))}{m_{T^2}(K_{i,T^2})} &= \frac{m_{T^2}((K_{S^1} \times K_{i,S^1}) \cap (B_1 \times B_2))}{m_{T^2}(K_{S^1} \times K_{i,S^1})} = \\ \frac{m_{T^2}((K_{S^1} \cap B_1) \times (K_{i,S^1} \cap B_2))}{m_{S^1}(K_{S^1}) \cdot m_{S^1}(K_{i,S^1})} &= \frac{m_{S^1}(K_{S^1} \cap B_1) \cdot m_{S^1}(K_{i,S^1} \cap B_2)}{m_{S^1}(K_{S^1}) \cdot m_{S^1}(K_{i,S^1})} = \\ \mu_{K_{S^1}}(B_1) \cdot \mu_{K_{i,S^1}}(B_2) &= \mu_{K_{i,T^2}}(B_1 \times B_2). \end{aligned}$$

□

Proposition 5.11. *For all $i \in \{1, 2, \dots\}$ the probability measure $\mu_{K_{i,T^2}}$ is Bernoulli. Precisely, it is equivalent to the Bernoulli measure ν on the shift space $4^{\mathbb{N}}$ of all the sequences of symbols 0, 1, 2 or 3 that assigns the same weight $1/4$ to each symbol.*

Proof. From Proposition 5.7 the measures $\mu_{K_{S^1}}$ and $\mu_{K_{i,S^1}}$ on the circle are equivalent, through bimeasurable conjugations $h_1 : K_{S^1} \mapsto 2^{\mathbb{N}}$ and $h_{2,i} : K_{i,S^1} \mapsto 2^{\mathbb{N}}$, to the Bernoulli measure $\nu_{(1/2,1/2)}$ on the shift space $2^{\mathbb{N}}$ of all the sequences of 0's and 1's that assigns the same weight $1/2$ to each symbol 0 or 1. Namely,

$$h_1 \circ f_1(x) = \sigma_{2^{\mathbb{N}}} \circ h_1(x) \quad \text{for all } x \in K_{S^1} \subset S^1,$$

$$h_{2,i} \circ f_2(y) = \sigma_{2^{\mathbb{N}}} \circ h_{2,i}(y) \quad \text{for all } y \in K_{i,S^1} \subset S^1,$$

where $\sigma_{2^{\mathbb{N}}}$ is the shift to the right in the space $2^{\mathbb{N}}$, and from Equality (5.15):

$$\mu_{K_{S^1}} = h_1^* \nu_{(1/2,1/2)}, \quad \mu_{K_{i,S^1}} = h_{2,i}^* \nu_{(1/2,1/2)}.$$

So

$$\mu_{K_{i,T^2}} := \mu_{K_{S^1}} \times \mu_{K_{i,S^1}} = h_1^* \nu_{(1/2,1/2)} \times h_{2,i}^* \nu_{(1/2,1/2)} \quad (5.19)$$

For any point $x \in K_{S^1}$ consider the sequence $h_1(x) = a_0, a_1, \dots, a_n, \dots \in 2^{\mathbb{N}}$ and denote $(h_1(x))_n := a_n$ for all $n \geq 0$. Analogously, for any point $y \in K_{i,S^1}$ consider the sequence $h_{2,i}(y) = b_0, b_1, \dots, b_n, \dots \in 2^{\mathbb{N}}$ and denote $(h_{2,i}(y))_n := b_n$ for all $n \geq 0$.

Define $h : K_{i,T^2} \mapsto 4^{\mathbb{N}}$ as follows:

$$h(x, y) = c_0, c_1, \dots, c_n, \dots \text{ such that } c_n \in \{0, 1, 2, 3\} \text{ is}$$

the symbol 0 if $(h_1(x))_n (h_{2,i}(y))_n = 00$,

the symbol 1 if $(h_1(x))_n (h_{2,i}(y))_n = 01$,

the symbol 2 if $(h_1(x))_n (h_{2,i}(y))_n = 10$,

the symbol 3 if $(h_1(x))_n (h_{2,i}(y))_n = 11$.

We denote such a correspondence as $(h(x, y))_n = (h_1(x))_n (h_{2,i}(y))_n$

So, we have the following equalities for any point $(x, y) \in K_{i,T^2}$

$$\begin{aligned} (h \circ f(x, y))_n &= (h(f_1(x), f_2(y)))_n = (h_1(f_1(x))_n (h_{2,i}(f_2(y))_n))_n = \\ &= (\sigma_{2^{\mathbb{N}}}(h_1(x)))_n (\sigma_{2^{\mathbb{N}}}(h_{2,i}(y)))_n = \sigma_{4^{\mathbb{N}}}((h_1(x))_n (h_{2,i}(y))_n) = \sigma_{4^{\mathbb{N}}}((h(x, y))_n), \end{aligned}$$

where $\sigma_{4^{\mathbb{N}}}$ is the shift to the right in the space $4^{\mathbb{N}}$. We have proved that

$$h \circ f = \sigma_{4^{\mathbb{N}}} \circ h,$$

or, in other words, f and the shift are conjugated by h . Besides, h is invertible because h_1 and $h_{2,i}$ also are.

What is left to prove is to check that

$$\mu_{K_{i,T^2}} = h^* \nu, \quad (5.20)$$

where ν is the Bernoulli measure on the shift space $4^{\mathbb{N}}$ that assigns to each symbol the weight $1/4$.

Since the rectangles in $T^2 = S^1 \times S^1$ generate the Borel σ -algebra of T^2 , let us check the above equality for any $B = B_1 \times B_2 \subset T^2$, where B_1 and B_2 are Borel sets in S^1 . In fact

$$h^* \nu(B_1 \times B_2) = \nu(h(B_1 \times B_2)) = \nu(h_1(B_1) \times h_{2,i}(B_2))$$

In the shift spaces, the Bernoulli measure ν in $4^{\mathbb{N}} = 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, which assigns to each of the four symbols the same weight $1/4$, is the product measure $\nu_{(1/2,1/2)} \times \nu_{(1/2,1/2)}$.

Therefore, taking into account that $h_1(B_1) \times h_{2,i}(B_2)$ is a rectangle in $4^{\mathbb{N}}$, and applying Equality (5.19), we obtain

$$\begin{aligned} h^* \nu(B_1 \times B_2) &= \nu(h_1(B_1) \times h_{2,i}(B_2)) = \nu_{(1/2,1/2)}(h_1(B_1)) \cdot \nu_{(1/2,1/2)}(h_{2,i}(B_2)) = \\ &= h_1^* \nu_{(1/2,1/2)}(B_1) \cdot h_{2,i}^* \nu_{(1/2,1/2)}(B_2) = \\ &= (h_1^* \nu_{(1/2,1/2)} \times h_{2,i}^* \nu_{(1/2,1/2)})(B_1 \times B_2) = \mu_{K_{i,T^2}}(B_1 \times B_2), \end{aligned}$$

ending the proof of Equality (5.20). $\square$

Proposition 5.12. *For all $i \in \{1, 2, \dots, k\}$, the probability measure $\mu_{K_{i,T^2}}$ is physical.*

Proof. Repeat the proof of Proposition 5.8 using the measure $\mu_{K_{i,T^2}}$ on the torus instead of the measure μ_K on the circle. $\square$

References

- [1] ALVAREZ, C.F., CANTARINO, M. *Equilibrium states for partially hyperbolic maps with one-dimensional center.* Journ. Stat. Phys. **190** (2023), Article number 194, 1–21
- [2] ALVES, J.F., DIAS, C.L., LUZZATTO, S., PINHEIRO, V., *SRB measures for partially hyperbolic systems whose central direction is weakly expanding.* Journ Europ. Math. Soc. **19** (10) (2017), 2911–2946
- [3] ALVES, J.F.; MESQUITA, D., *Entropy formula for systems with inducing schemes.* Trans. Amer. Math. Soc. **376** (2023), 1263–1298
- [4] ALVES, J.F.; OLIVEIRA, K., SANTANA, E., *Equilibrium states for hyperbolic potentials via inducing schemes.* Nonlinearity **37** (9) (2024), 095030
- [5] ARAUJO, V., SANTOS, F., *Pesin’s Entropy Formula for C^1 non-uniformly expanding maps.* Mathematische Zeitschrift **291** (3-4) (2019), 1451–1494
- [6] ARAUJO, V., TAHZIBI, A., *Physical measures at the boundary of hyperbolic maps.* Disc. Cont. Dyn. Sys. A **20** (2008), 849–876
- [7] AVILA, A.; BOCHI, J., *Generic C^1 expanding maps without absolutely continuous invariant σ -finite measure.* Math. Research Letters **14** (2007), 721–730
- [8] BARREIRA, L., PESIN, YA. B. *Lyapunov exponents and smooth ergodic theory*, Univ. Lecture Series **23** Amer. Math. Soc., Providence, 2002.
- [9] BOUKHECHAM, H.: EXISTENCE IF SRB MEASURES FOR HYPERBOLIC MAPS WITH WEAK REGULARITY, Dyn. Sys. **39** (1) (2024) 62–78
- [10] BOWEN, R., *A horseshoe with positive measure.* Invent. Math. **29** (1975)
- [11] BOWEN, R., *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lect. Notes in Math. **470**, Springer-Verlag, New York-Heidelberg-Berlin 1975
- [12] BOWEN, R., RUELLE, D., *The ergodic theory of Axiom A flows.* Invent. Math. **29** (1975) 181–202
- [13] CAMPBELL, J., QUAS, A., *A generic C^1 expanding map has a singular SRB measure.* Commun. Math. Phys. **349** (2001), 221–335
- [14] CATSIGERAS, E; CERMINARA, M; ENRICH, H., *Pesin’s Entropy Formula for C^1 Diffeomorphisms with Dominated Splitting* Erg. Th. & Dyn. Sys. **35** (2015), 737–761.
- [15] CATSIGERAS, E., ENRICH, H., *SBR-like measures for C^0 dynamics.* Bull. Polish Acad. Sci. Math. **59** (2011), 151–164
- [16] CATSIGERAS, E, ENRICH, H., *Equilibrium States and SRB-like measures of C^1 Expanding Maps of the Circle*, Portugaliae Mathematica, **69** (3) (2012) 193–212
- [17] CLIMENHAGA, V., LUZZATTO, S., PESIN, YA., *SRB measures and Young towers for surface diffeomorphisms* Ann. Henry Poincaré (2022) 1–87
- [18] HEMENWAY, G., *Equilibrium states for non-uniformly expanding skew products* Erg. Th. & Dyn. Sys. **44** (9) (2024) 2599–2620
- [19] KELLER, G., *Equilibrium states in ergodic theory.* London Mathematics Society Student Texts. Cambridge University Press, Cambridge, 1998

- [20] LEDRAPPIER, F., YOUNG, L. S., The metric entropy of diffeomorphisms. Part I: Characterization of measures satisfying Pesin's entropy formula. *Annals of Mathematics* **122** (1985), 509–574
- [21] MARGULIS, G. A., Unpublished proof, 1966. Cited in Hasselblatt, B. Pesin, Ya. B. Pesin's Entropy formula *Scholarpedia* **3** (3) (2008): 3733.
- [22] MISIUREWICZ, M., *Ergodic natural measures* in *Contemporary Mathematics* **385** Algebraic and topological dynamics, eds: Kolyada, S.; Manin, Y.; Ward, T., pp. 1–6 Amer. Math. Soc. Providence, R.I., 2005
- [23] OBATA, D., *Open sets of partially hyperbolic skew products having a unique SRB measure* *Advances in Math.* **427** (2023) 109136
- [24] PACIFICO, M.J., YANG, F., YANG, J. , *Existence and uniqueness of equilibrium states for systems with specification at a fixed scale: an improved Climenhaga-Thompson criterion.* *Non-linearity* **35** (12) (2022), 5963
- [25] PALIS, J., TAKENS, F., *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations: Fractal Dimensions and Infinitely Many Attractors.* Cambridge University Press, Cambridge, 1993 ISBN 13 978-0-521-39064-4
- [26] PESIN, YA. B., *Characteristic Lyapunov exponents and smooth ergodic theory.* *Russian Math. Surveys* **32** (1977), 55–112.
- [27] PUGH, C., *The $C^{1+\alpha}$ hypothesis in Pesin theory.*, *Publ. Math. Inst. Hautes Études Sci.* **59** (1984), 143–161.
- [28] QIU, H., *Existence and uniqueness of SRB measure on C^1 generic hyperbolic attractors.* *Commun. Math. Phys.* **302** (2011), 345–357
- [29] RUELLE, D., *A measure associated with Axiom A attractors.* *Amer. J. Math.* **98** (1976), 619–654
- [30] RUELLE, D., *An inequality of the entropy of differentiable maps.* *Bol. Sc. Bra. Mat.* **9** (1978), 83–87
- [31] SINAI, YA. G., *Gibbs measure in ergodic theory*, *Russ. Math. Surveys* **27** (1972), 21–69.
- [32] SUN, W.; TIAN, X., *Dominated Splitting and Pesin's Entropy Formula.* *Disc. Cont. Dyn. Sys. A* **32** (2012) 1421–1434
- [33] TAHZIBI, A., *C^1 -generic Pesin's Entropy Formula.* *C.R. Mathématique Acad. Sci. Paris. Series I* **335** (12) (2002), 1057–1062
- [34] TESCHL, G. *Ordinary Differential Equations and Dynamical Systems*, Graduate Studies in Mathematics **140** Amer. Math. Soc., Providence, 2012.
- [35] VALENZUELA, F., *Medidas tipo-SRB y Fórmula de Pesin en Sistemas C^1 expansores.* Tesis de Maestría en Matemática PEDECIBA, Universidad de la República, Montevideo, Uruguay (2017) <https://www.cmat.edu.uy/biblioteca/monografias-y-tesis/tesis-de-maestria/>
- [36] VIANA, M.; OLIVEIRA, K., *Foundations of ergodic theory.* Cambridge University Press, Cambridge, U.K. (2016)
- [37] L.S. YOUNG, *What are SRB measures, and which dynamical systems have them?*, *Journ. Stat. Physics* **108** (2002), 733–754.

Eleonora Catsigeras and Fernando Valenzuela Gómez

INSTITUTO DE MATEMÁTICA Y ESTADÍSTICA,

“PROF. RAFAEL LAGUARDIA”, UNIVERSIDAD DE LA REPÚBLICA, URUGUAY.

POSTAL ADDRESS: AV. HERRERA Y REISSIG 565, C.P.11300 MONTEVIDEO, URUGUAY.

E-mail address: eleonora@fing.edu.uy

E-mail address: fagomez7@hotmail.com