

On the convergence of a perturbed one dimensional Mann's process



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Abstract

We study a perturbed version of Mann's iterative process (DMP), defined by

$$x_{n+1} = (1 - \theta_n)x_n + \theta_n f(x_n) + r_n,$$

where $f : [0, 1] \rightarrow [0, 1]$ is a continuous function, $\{\theta_n\} \subset [0, 1]$ is a given sequence, and $\{r_n\}$ represents an error term. We prove that if the sequence $\{\theta_n\}$ converges sufficiently slowly to zero and the error term r_n is suitably small at infinity, then any sequence $\{x_n\} \subset [0, 1]$ generated by this process converges to a fixed point of f . In addition, we investigate the asymptotic behavior of the trajectories $x(t)$ as $t \rightarrow \infty$ for a continuous-time version of the process (DMP). We emphasize the parallels between the discrete and the continuous dynamics. Furthermore, through numerical experiments, we analyze the influence of the sequence $\{\theta_n\}$ and the error terms on the stability and the convergence rate of the discrete and the continuous processes. Notably, we observe that the (DMP) algorithm, when affected by stochastic and relatively large error terms, can outperform the bisection method in efficiently identifying the fixed point set of the function f .

Keywords: Discrete and continuous dynamical systems, Mann's iterative process, ordinary differential equation.

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1 Introduction and main results

Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. A simple application of the intermediate value theorem ensures that the function f has at least one fixed point. Since $f(0)(f(1) - 1) \leq 0$ then an application of the deterministic bisection method to the function $g(x) = f(x) - x$ leads to a numerical approximation of a particular fixed point of f with any prefixed tolerance $\epsilon > 0$. In order to discover more fixed points of f , it is worth to try to construct iterative processes that converge to a fixed point of f whatever the value of the initial data x_0 in $[0, 1]$. We might initially consider using the classical Banach iterative process $x_{n+1} = f(x_n)$, $n = 0, 1, \dots$. But in general this process does not converge. To see this it suffices to consider the case where $f(x) = 1 - x$ and the initial data $x_0 \neq \frac{1}{2}$. In fact, in this case the sequence $\{x_n\}$ is given explicitly by $x_n = \frac{1}{2} + (-1)^n(x_0 - \frac{1}{2})$. In a pioneering paper [10], W. Robert Mann introduced the following iterative process

$$x_{n+1} = \frac{n}{n+1}x_n + \frac{1}{n+1}f(x_n). \quad (1.1)$$

He then established the following result.

Theorem 1.1. *Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. If f has a unique fixed point $p \in [0, 1]$, then, for any initial data $x_0 \in [0, 1]$, the sequence $\{x_n\}$ generated by the process (1.1) converges towards p .*

Later, R.L. Franks and R.P. Marzec, in a brief and elegant paper [3], showed that the uniqueness condition for the fixed point of the function f in the previous theorem is not necessary. Precisely, they established the following result.

Theorem 1.2. *If $f : [0, 1] \rightarrow [0, 1]$ is a continuous function then, for any initial data $x_0 \in [0, 1]$, the sequence $\{x_n\}$ generated by the process (1.1) converges towards a fixed point of f*

The first objective of this paper is to examine the impact of small computational errors on the asymptotic behavior of the iterative process (1.1). In this context, we establish the following result.

Theorem 1.3. *Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function and $\{\theta_n\} \in [0, 1]$ a given real sequence. Let $\{x_n\} \in [0, 1]$ be a sequence satisfying the perturbed iterative process*

$$x_{n+1} = (1 - \theta_n)x_n + \theta_n f(x_n) + r_n, \quad (1.2)$$

where $\{r_n\}$ is a real sequence. If

1. $\theta_n \rightarrow 0$ as $n \rightarrow \infty$,
2. $\sum_{n=0}^{\infty} \theta_n$ diverges,
3. $\frac{r_n}{\theta_n} \rightarrow 0$ as $n \rightarrow \infty$,
4. $\sum_{n=0}^{\infty} r_n$ converges,

then the sequence $\{x_n\}$ converges towards a fixed point of f .

Remark 1.1. Let $f : [0, 1] \rightarrow [0, 1]$ be a nondecreasing function. It is well-known that f has at least one fixed point even in the absence of continuity. This naturally raises the question of whether iterative schemes such as the Banach process $x_{n+1} = f(x_n)$, the algorithm (1.1), or its general and perturbed form (1.2) converge to a fixed point of f , regardless of the choice of the initial value $x_0 \in [0, 1]$. To date, the author has not arrived at a solution of this problem.

Remark 1.2. Variate version of the Mann's process (DMP) are used in many scientific research works to approximate the fixed points of different kinds of function defined on closed and convex subsets of a Hilbert spaces or a uniformly convex Banach spaces. See for instance [5, 9, 12, 13, 15].

Inspired by the works [7], [4], [6], [11] and [14], we also study the asymptotic behavior of the trajectories $x(t)$ of a continuous version of the discrete dynamical system (1.2). We obtain the following result.

Theorem 1.4. *Let $f : [0, 1] \rightarrow [0, 1]$ and $\theta : [0, \infty) \rightarrow [0, \infty)$ be two continuous functions and let $x : [0, \infty) \rightarrow [0, 1]$ be a continuous differentiable function that satisfies the perturbed differential equation:*

$$x'(t) + \theta(t)x(t) = \theta(t)f(x(t)) + r(t), t \geq 0, \quad (1.3)$$

where $r : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function. If

1. $\lim_{t \rightarrow \infty} \frac{r(t)}{\theta(t)} = 0$,
2. $\int_0^{\infty} \theta(t) dt$ diverges,
3. $\int_0^{\infty} r(t) dt$ converges,

then $x(t)$ converges towards a fixed point of f as $t \rightarrow \infty$.

The rest of paper is organized as follows: In §2, we provide a detailed proof of Theorem 1.3. The third section is devoted to the proof of Theorem 1.4. In the fourth section, we study through some numerical experiments the effect of the sequence $\{\theta_n\}$ and the stochastic error term $\{r_n\}$ on the stability and the rate of convergence of the discrete process (1.2). We also study the effect of the asymptotic behavior of the function $\theta(t)$ and the error term $r(t)$ on the speed of the convergence of the trajectories of the continuous process (1.3). The final section of the paper is devoted to a complete and detailed proof of Mann's convergence original result (Theorem 1.1).

2 On the convergence of the perturbed discrete dynamical system (1.2)

In this section, we provide a detailed proof of Theorem 1.3. The proof is greatly inspired by the original paper [3]. It essentially relies on the classical notion of the omega limit set of a real sequence. We recall here briefly this notion and some of its main properties.

Definition 2.1. Let $\{x_n\}$ be a real sequence. The omega limit set of the sequence $\{x_n\}$ is the set $\omega(\{x_n\})$ of real numbers z such that $z = \lim_{n \rightarrow \infty} x_{n_k}$ for some subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$.

Lemma 2.1. Let $\{x_n\}$ be a real sequence. Then

1. $\omega(\{x_n\}) = \bigcap_{n \in \mathbb{N}} \overline{\{x_m : m \geq n\}}^{\mathbb{R}}$ where $\overline{\{x_m : m \geq n\}}^{\mathbb{R}}$ denotes the closure of the set $\{x_m : m \geq n\}$ in $\mathbb{R}$.
2. If $\{x_n\}$ is bounded then $\omega(\{x_n\})$ is a nonempty compact subset of $\mathbb{R}$.
3. If $\{x_n\}$ is bounded then $\{x_n\}$ converges in $\mathbb{R}$ if and only if the set $\omega(\{x_n\})$ is reduced to a single element.
4. If $\lim_{n \rightarrow \infty} x_{n+1} - x_n = 0$ then $\omega(\{x_n\})$ is a connected subset of $\mathbb{R}$.

The proof of this lemma is classical and easy. Now we are in position to prove our first main result (Theorem 1.3).

Proof. Since the sequence $\{x_n\}$ is in $[0, 1]$ and satisfies

$$x_{n+1} - x_n = \theta_n[f(x_n) - x_n] + r_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then, in view of the previous lemma, there exist two real number $0 \leq a \leq b \leq 1$ such that $\omega(\{x_n\}) = [a, b]$. Let us suppose that $a < b$. We claim that in this case $f(w) = w$ for any $w \in (a, b)$. Let $w \in (a, b)$. Suppose for the sake of contradiction that $f(w) > w$. Then thanks to the continuity of f there exist two positive real numbers δ and ε such that $[w - \delta, w + \delta] \subset (a, b)$ and $f(x) \geq x + \varepsilon$ for any $x \in [w - \delta, w + \delta]$. On the other hand, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$|x_{n+1} - x_n| < \delta, \quad (2.1)$$

$$r_n \geq -\varepsilon\theta_n. \quad (2.2)$$

Moreover, there exist two positive integers $n_2 > n_1 > n_0$ such that $x_{n_1} \in (w + \delta, b)$ and $x_{n_2} \in (a, w - \delta)$. Let m be the greatest integer between n_1 and n_2 such that $x_m \geq w - \delta$. Therefore, in view of (2.1), we have $x_{m+1} < w - \delta \leq x_m < w$. Now by going back to the dynamical system (1.2) and using the fact that $f(x_m) \geq x_m + \varepsilon$, we get

$$\begin{aligned} x_{m+1} &\geq x_m + \theta_m \varepsilon + r_m \\ &\geq x_m \quad (\text{in view of (2.2)}). \end{aligned}$$

This contradicts the fact that $x_{m+1} < x_m$. Similarly, we can show that the assumption $f(w) < w$ leads to a contradiction. We therefore conclude $f(w) = w$ for any $w \in (a, b)$ if $a < b$. We continue working under the assumption $a < b$. Let $c = \frac{b+a}{2}$ and set $\alpha = \frac{b-a}{8}$. There exists a positive integer m_0 such that

$$|x_{m_0} - c| < \alpha, \quad (2.3)$$

$$\left| \sum_{n=m_0}^m r_n \right| < \alpha, \forall m \geq m_0. \quad (2.4)$$

Let $m_1 > m_0$ be the smallest positive integer such that $x_{m_1} \notin [c - 2\alpha, c + 2\alpha]$ (such integer exists since $[c - 2\alpha, c + 2\alpha] \subset (a, b)$). Therefore, for any integer $n \in [m_0, m_1 - 1]$, we have $f(x_n) = x_n$ and

$$x_{n+1} = x_n + r_n.$$

Summing up the previous equalities, we obtain

$$x_{m_1} = x_{m_0} + \sum_{n=m_0}^{m_1-1} r_n.$$

Combining this identity with the estimations (2.3) and (2.4), we get $|x_{m_1} - c| \leq 2\alpha$ which contradicts the definition of x_{m_1} . We then conclude that $a = b$. Hence, in view of Lemma 2.1, we infer that the sequence $\{x_n\}$ converges in $\mathbb{R}$; let x_∞ be its limit. Clearly $x_\infty \in [0, 1]$. Let us prove by contradiction that $f(x_\infty) = x_\infty$. We suppose for instance that $f(x_\infty) > x_\infty$. Thanks to the continuity of the function f , there exists a positive real number $\beta > 0$ and a positive integer p_0 such that, for any $n \geq p_0$,

$$\begin{aligned} f(x_n) &\geq x_n + 2\beta, \\ r_n &\geq -\beta\theta_n. \end{aligned}$$

Hence, for any integer $n \geq p_0$,

$$x_{n+1} - x_n = \theta_n(f(x_n) - x_n) + r_n \geq 2\beta\theta_n + r_n \geq \beta\theta_n.$$

Summing up these inequalities yields

$$\beta \sum_{n=p_0}^{\infty} \theta_n \leq x_\infty - x_{p_0}.$$

This contradicts the assumption on the sequence $\{\theta_n\}$. Similarly, we can show that $f(x_\infty)$ can not be less than x_∞ . We therefore conclude that $f(x_\infty) = x_\infty$ which completes the proof. $\square$

3 On the convergence of the perturbed continuous dynamical system (1.3)

This section is devoted to the proof of Theorem 1.4. Before starting the proof of this theorem, let us recall the definition and some simple proprieties of the omega limit set of a continuous real valued function defined on the interval $[0, \infty)$. (For more details on the notion of the omega limit in a more general context, we refer the readers to the books [4], [6], and [7].)

Definition 3.1. *let $u : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The omega limit set associated to the function u (denoted by $\omega(u(t))$) is the set of real numbers z such that $\lim_{n \rightarrow \infty} u(t_n) = z$ for some positive sequence $\{t_n\}$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$.*

The following lemma gathers some simple and well known properties of the omega limit set associated to a continuous and bounded real valued function u defined on the interval $[0, \infty)$.

Lemma 3.1. *let $u : [0, \infty) \rightarrow \mathbb{R}$ be a continuous and bounded function. Then*

1. $\omega(u(t)) = \cap_{k \in \mathbb{N}} \overline{u([k, \infty))}^{\mathbb{R}}$, where $\overline{u([k, \infty))}^{\mathbb{R}}$ denotes the closure of the set $u([k, \infty))$ in $\mathbb{R}$.
2. $\omega(u(t))$ is a nonempty compact interval of $\mathbb{R}$.
3. $u(t)$ converges to some real number u_∞ as $t \rightarrow \infty$ if and only if $\omega(u(t)) = \{u_\infty\}$.

The proof of Theorem 1.4 uses also the following technical lemma which is inspired by the papers and [1] and [8].

Lemma 3.2. *let $u : [0, \infty) \rightarrow \mathbb{R}$ be a continuous and bounded function. If $\omega(u(t)) = [a, b]$ with $a < b$ then for any real numbers c and d such that $a < c < d < b$ and any $T_0 > 0$ there exist four real numbers s_1, s_2, τ_1 and τ_2 such that:*

1. $\tau_i > s_i > T_0$, $i = 1, 2$.
2. $u(t) \in [c, d]$ for any $t \in [s_i, \tau_i]$, $i = 1, 2$.

3. $u(s_1) = c$ and $u(\tau_1) = d$.

4. $u(s_2) = d$ and $u(\tau_2) = c$.

Proof. Let $T_0 > 0$ and c and d two real numbers such that $a < c < d < b$. From the definition of $\omega(u(t))$, there exist two real numbers s_0 and τ_0 such that $\tau_0 > s_0 > T_0$, $u(s_0) < c$ and $u(\tau_0) > d$. Let $s_1 = \sup\{t \in [s_0, \tau_0] : u(t) < c\}$. Using the continuity of u we can easily verify that $s_0 < s_1 < \tau_0$, $u(s_1) = c$ and $u(t) \geq c$ for any $t \in [s_1, \tau_0]$. Let $\tau_1 = \inf\{t \in [s_1, \tau_0] : u(t) > d\}$. Again the continuity of u ensures that $\tau_0 > \tau_1 > s_1$, $u(\tau_1) = d$ and $u(t) \leq d$ for every $t \in [s_1, \tau_1]$. This completes the construction of s_1 and τ_1 . The construction of s_2 and τ_2 can be done similarly. In fact, there exist two real numbers s'_0 and τ'_0 such that $\tau_0 > s_0 > T_0$, $u(\tau'_0) < c$ and $u(s'_0) > d$. Now let $s_2 = \sup\{t \in [s'_0, \tau'_0] : u(t) > d\}$ and $\tau_2 = \inf\{t \in [s_2, \tau'_0] : u(t) < c\}$. Again by using the continuity of u we can easily verify that s_2 and τ_2 satisfy all the required proprieties. $\square$

Now we are in position to prove Theorem 1.4.

Proof. From Lemma 3.1, there exist two real numbers $0 \leq a \leq b \leq 1$ such that $\omega(x(t)) = [a, b]$. First, for the sake of a contradiction, we suppose that $a < b$. Now let w be an arbitrary element of the open interval (a, b) . Let us suppose that $f(w) > w$. Then there exist $\varepsilon, \delta > 0$ such that $[w - \delta, w + \delta] \subset (a, b)$ and $f(z) > z + \varepsilon$ for any $z \in [w - \delta, w + \delta]$. Let $T_0 > 0$ such that $r(t) \geq -\varepsilon\theta(t)$, $\forall t \geq T_0$. According to Lemma 3.2, there exist $\tau_2 > s_2 > T_0$ such that $x(s_2) = w + \delta$, $x(\tau_2) = w - \delta$, and $x(t) \in [w - \delta, w + \delta]$ for any $t \in [s_2, \tau_2]$. Therefore, for every $t \in [s_2, \tau_2]$,

$$x'(t) = \theta(t)(f(x(t)) - x(t)) + r(t) \geq \varepsilon\theta(t) + r(t) \geq 0.$$

Hence by integrating on the interval $[s_2, \tau_2]$, we get the contradiction $-2\delta \geq 0$. Similarly, by using the times s_1 and τ_1 instead of s_2 and τ_2 , we can verify that the assumption $f(w) < w$ leads also to a contradiction. We then conclude that if $a < b$ then $f(w) = w$ for any $w \in [a, b]$. We continue working under the assumption $a < b$. Let $c = \frac{a+b}{2}$ and $\alpha = \frac{b-a}{4}$. Let $T_1 > 0$ be greater enough such that

$$\left| \int_s^\tau r(t)dt \right| \leq \alpha, \quad \forall \tau > s > T_1. \quad (3.1)$$

According to Lemma 3.2, there exist $\tau_1 > s_1 > T_1$ such that $x(s_1) = c - \alpha$, $x(\tau_1) = c + \alpha$, and $x(t) \in [c - \alpha, c + \alpha]$ for any $t \in [s_1, \tau_1]$. Therefore, for any $t \in [s_1, \tau_1]$,

$$\begin{aligned} x'(t) &= \theta(t)(f(x(t)) - x(t)) + r(t) \\ &= r(t). \end{aligned}$$

Hence, by integrating the last equality between s_1 and τ_1 we get

$$2\alpha = \int_{s_1}^{\tau_1} r(t)dt,$$

which contradicts (3.1). We therefore conclude that $a = b$ which means, thanks to Lemma 3.1, that $x(t)$ converges as $t \rightarrow \infty$ to some the real number $x_\infty = a = b$. It is clear that $x_\infty \in [0, 1]$, let us show that it is a fixed point of f . For the sake of absurdity, we assume for instance that $f(x_\infty) < x_\infty$. The continuity of $x(t)$ and f and the assumption on $r(t)$ ensure the existence of $\varepsilon > 0$ and T_2 such that for any $t \geq T_2$

$$\begin{aligned} x(t) - f(x(t)) &\geq 2\varepsilon, \\ r(t) &\leq \varepsilon\theta(t). \end{aligned}$$

Therefore, for every $t \geq T_2$,

$$x'(t) = \theta(t)(f(x(t)) - x(t)) + r(t) \leq -\varepsilon\theta(t).$$

Integrating the last inequality between T_2 and $T > T_2$ and then letting $T \rightarrow \infty$, we get the inequality

$$\varepsilon \int_{T_2}^{\infty} \theta(t)dt \leq x(T_2) - x_\infty$$

that contradict the assumption on θ . Similarly, we can show that $f(x_\infty)$ can not be greater than x_∞ . We therefore conclude that $f(x_\infty) = x_\infty$. $\square$

4 A numerical study of the rate of convergence of the processes (1.2) and (1.3)

In the first part of this section, we numerically investigate the effect of the speed of the vanishing of the sequence $\{\theta_n\}$ and the amplitude of the error term $\{r_n\}$ on the rate of convergence of the sequence $\{x_n\}$ generated by the perturbed process (1.2) to a fixed point of f . We consider the case where the sequence $\{\theta_n\}$ is defined by $\theta_n = \frac{1}{(n+1)^\alpha}$ where $0 < \alpha \leq 1$, and the objective function $f : [0, 1] \rightarrow [0, 1]$ is given by:

$$f(x) = \begin{cases} 2(\frac{1}{4} - x), & 0 \leq x \leq \frac{1}{4}, \\ 4(x - \frac{1}{4}), & \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 4(\frac{3}{4} - x), & \frac{1}{2} \leq x \leq \frac{3}{4}, \\ 2(x - \frac{3}{4}), & \frac{3}{4} \leq x \leq 1. \end{cases} \quad (4.1)$$

A simple calculation shows that f has exactly three fixed points: $a = \frac{1}{6}$, $b = \frac{1}{3}$, and $c = \frac{3}{5}$ (see

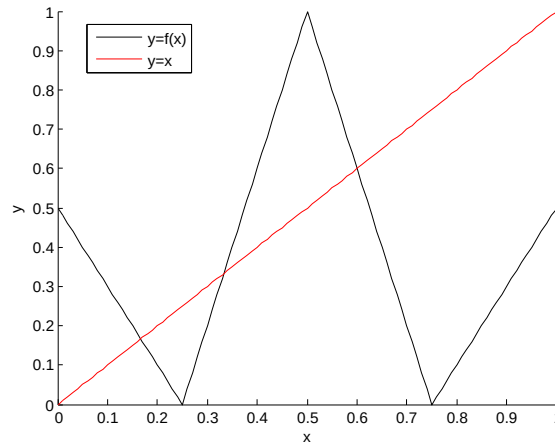


Fig. 1: The graph of the function f and its fixed points

Figure 1). We can now introduce our example of the process (1.2): Let x_0 be an arbitrary element of $[0, 1]$. The sequence $\{x_n\}$ is generated by the iterative stochastic process

$$x_{n+1} = \Pi((1 - \theta_n)x_n + \theta_n f(x_n) + A \frac{M_n}{1 + n^2}), \quad (4.2)$$

where $A > 0$ is a constant and $\{M_n\}$ is a sequence of i.i.d random variables such that each M_n follows the uniform distribution $U([-1, 1])$. The function $\Pi : \mathbb{R} \rightarrow [0, 1]$ is the metric projection on the interval $[0, 1]$ which is defined explicitly by:

$$\Pi(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 \leq x \leq 1, \\ 1, & x \geq 1. \end{cases} \quad (4.3)$$

Clearly, the sequence $\{x_n\}$ belongs to the interval $[0, 1]$ and satisfies the process (1.2) with an error term $r_n = \Pi(y_n + A \frac{M_n}{1 + n^2}) - y_n$ where $y_n = (1 - \theta_n)x_n + \theta_n f(x_n)$. Since $y_n \in [0, 1]$, we have $\Pi(y_n) = y_n$. Then thanks to the fact that Π is non expansive,

$$|r_n| \leq \frac{A}{1 + n^2}.$$

Therefore, according to Theorem 1.3, the sequence $\{x_n\}$ converges to a fixed point x_∞ of f . For the two chosen values of the amplitude of the error term $A = 0.1$ and 0.001 and some chosen values of $\alpha \in [0, 1]$ and $\epsilon > 0$, we conduct $K_{\max} = 100$ times the process (4.2) with arbitrary initial data x_0 in $[0, 1]$ under the stopping criteria $|f(x_n) - x_n| < \epsilon$. Let $N(A, \alpha, \epsilon)$ be the average of number

α/ϵ	$\epsilon = 0.1$	$\epsilon = 0.01$	$\epsilon = 0.001$	$\epsilon = 0.0001$
$\alpha = 0.1$	4.27	77.07	117.41	133.64
$\alpha = 0.2$	5.83	24.40	36.46	41.72
$\alpha = 0.4$	5.05	12.83	17.03	22.50
$\alpha = 0.6$	4.04	7.65	11.84	20.54
$\alpha = 0.8$	3.53	6.47	11.20	21.63
$\alpha = 0.9$	3.65	5.83	10.73	25.07
$\alpha = 1$	3.40	6.14	11.44	28.89

Tab. 1: $N(A = 0.1, \alpha, \epsilon)$

α/ϵ	$\epsilon = 0.1$	$\epsilon = 0.01$	$\epsilon = 0.001$	$\epsilon = 0.0001$
$\alpha = 0.1$	4.88	76.14	119.15	134.84
$\alpha = 0.2$	5.14	22.94	35.09	38.39
$\alpha = 0.4$	4.97	11.52	16.49	17.81
$\alpha = 0.6$	3.95	7.39	9.49	10.85
$\alpha = 0.8$	3.46	5.87	6.68	8.63
$\alpha = 0.9$	3.57	5.72	6.70	9.20
$\alpha = 1$	3.28	4.89	6.60	8.78

Tab. 2: $N(A = 0.001, \alpha, \epsilon)$

of iterations needed to achieve the stopping criteria. The following two tables give $N(0.1, \alpha, \epsilon)$ and $N(0.001, \alpha, \epsilon)$ for some values of α and ϵ .

We notice that the process (1.2) converges faster if α is close to 1 (i.e., the sequence $\{\theta_n\}$ converges relatively quickly to zero) and A is small (i.e., the perturbation term $\{r_n\}$ is relatively weak). In this case the speed of the convergence of the process (1.2) is slightly better than the rate of the convergence of the classical bisection method applied for the function $f(x) - x$ for which it is known that $N(\epsilon)$, the number of needed iterations to achieve the precision ϵ , is equal to the entire part of $\frac{-\log(\epsilon)}{\log(2)}$ where $0 < \epsilon < 1$, [2]. Now we study the effect of the amplitude A of the error term on the the limit of the sequences $\{x_n\}$ generated by the process (4.2). We take $\alpha = 1$ and we consider the four cases corresponding to x_0 equals 0 or 1 and A equals 0.01 or 0.5. For each case we conduct the process (4.2) $N_{max} = 1000$ times under the stopping criteria $|f(x_k) - x_k| < 0.0001$. The results of these experiments are given in the following table where p (respectively q and r) represents the probability that the process converges to the fixed point $a = \frac{1}{6}$ (respectively $b = \frac{1}{3}$ and $c = 0.6$) of f . We observe that, in contrast to the deterministic bisection method, the process (4.2) with a

x_0, A	p	q	r
$x_0 = 0, A = 0.01$	0	0	1
$x_0 = 0, A = 0.5$	0.339	0	0.661
$x_0 = 1, A = 0.001$	0	0	1
$x_0 = 1, A = 0.5$	0.349	0	0.651

Tab. 3: The effect of the stochastic perturbation term on the convergence of the process (4.2)

significant stochastic perturbation term $e(t)$ is capable of identifying a broader set of fixed points of the objective function f .

The last part of this section is devoted to the numerical study of the asymptotic behavior of the trajectories $x(t)$ of a example of the continuous process (1.3). Let θ be the function defined on $[0, \infty)$ by $\theta(t) = \frac{1}{(1+t)^\alpha}$ with $0 < \alpha \leq 1$ and the error term $e(t)$ given by $e(t) = A \frac{\sin(t)}{1+t^2}$ with $A > 0$. We now consider the system:

$$\begin{cases} x'(t) + x(t) = \Pi((1 - \theta(t))x(t) + \theta(t)f(x(t)) + e(t)) \\ x(0) = x_0 \end{cases} \quad (4.4)$$

where Π is given by (4.3) and $f : [0, 1] \rightarrow [0, 1]$ is here a general continuous function. We first prove the following existence result.

Proposition 4.1. *For any initial data $x_0 \in [0, 1]$, the system has a solution $x : [0, \infty) \rightarrow [0, 1]$ that satisfies in addition the continuous process (1.3) with an error term $r(t)$ satisfying*

$$|r(t)| \leq \frac{A}{1+t^2}, \quad \forall t \geq 0.$$

Proof. Since the function $F : \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(x) = f(\Pi(x))$ is continuous, then from the classical Peano's existence theorem, the differential equation

$$x'(t) + x(t) = \Pi((1 - \theta(t))x(t) + \theta(t)F(x(t)) + e(t))$$

has at least one maximal solution $x : [0, T^*) \rightarrow \mathbb{R}$ that satisfies the initial condition $x(0) = x_0$. For any $t \in [0, T^*)$,

$$x(t) = e^{-t}x_0 + \int_0^t e^{-(t-s)}h(s)ds,$$

with $h(s) = \Pi((1 - \theta(s))x(s) + \theta(s)F(x(s)) + e(s))$. Since $0 \leq h \leq 1$, then for any $t \in [0, T^*)$,

$$0 \leq \int_0^t e^{-(t-s)}h(s)ds \leq 1 - e^{-t},$$

and as a consequence $0 \leq x(t) \leq 1$.

We therefore infer that the maximal solution $x(t)$ is in fact global (i.e., $T^* = \infty$) and satisfies the system (4.4) since $F(x(t)) = f(x(t))$ for any $t \geq 0$. Finally, if we set

$$y(t) = (1 - \theta(t))x(t) + \theta(t)f(x(t)),$$

then

$$x'(t) + \theta(t)x(t) = \theta(t)f(x(t)) + r(t),$$

with

$$r(t) = \Pi(y(t) + e(t)) - y(t).$$

Since $y(t) \in [0, 1]$ and the projection operator Π is non expansive, $|r(t)| \leq |e(t)| \leq \frac{A}{1+t^2}$. $\square$

We now consider the particular case of the system (4.4) in which $x_0 = \frac{1}{2}$ and f is the function given by (4.1). In this case, since f is a Lipschitz function, the solution $x(t)$ given by the previous Proposition is unique. According to Theorem 1.4, the trajectory $x(t)$ converges as $t \rightarrow \infty$ to a fixed point of f . The graphs (Figure 2) and (Figure 3) give an idea about the effect of the parameters α and A on the speed of the convergence of $x(t)$. We observe that the rate at which the function $\theta(t)$ vanishes as $t \rightarrow \infty$ has little to no effect on the convergence speed of the trajectories $x(t)$. Additionally, the trajectories tend to converge slightly faster when the error term $e(t)$ is sufficiently small.

5 Appendix

It seems that the original proof of Theorem 1.1 due to Mann (see the proof of [10, Theorem 4]) is incomplete. Hence, in this section, we provide a complete and detailed proof of this result.

Proof. The function $g(x) = f(x) - x$ defined on the interval $[0, 1]$ is continuous, satisfies $g(0) \leq 0$ and $g(1) \geq 0$, and has p as a unique root, then from the intermediate value theorem, $g(x) > 0$ on the interval $[0, p)$ and $g(x) < 0$ on $(p, 1]$. Now let $\varepsilon > 0$ be an arbitrary but a fixed real number. There exists $\delta > 0$ such that

$$(x \in [0, 1] \wedge |x - p| \geq \varepsilon) \Rightarrow |g(x)| \geq \delta. \quad (5.1)$$

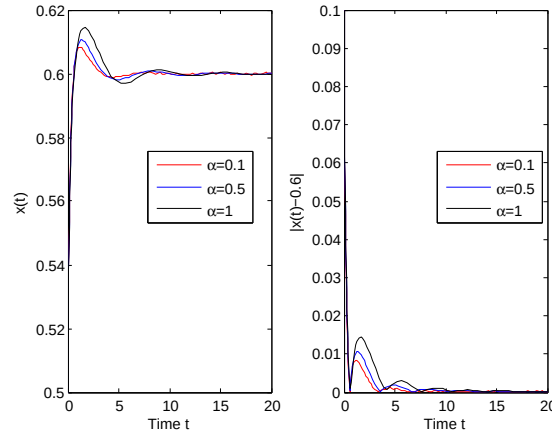


Fig. 2: Behavior of some trajectories $x(t)$ of the system (4.4) when $A = 0.1$

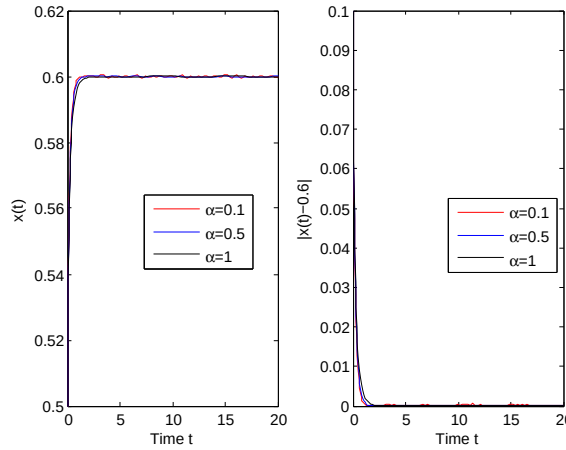


Fig. 3: Behavior of some trajectories $x(t)$ of the system (4.4) when $A = 0.001$

On the other hand since,

$$|x_{n+1} - x_n| = \frac{1}{n+1} |g(x_n)| \leq \frac{1}{n+1},$$

there exists a positive integer n_0 such that

$$|x_{n+1} - x_n| < \varepsilon, \forall n \geq n_0. \quad (5.2)$$

Now let us suppose that for any integer $n \geq n_0$, $x_n \notin [p - \varepsilon, p + \varepsilon]$. This implies that either $x_n > p + \varepsilon$ for any $n \geq n_0$ or $x_n < p - \varepsilon$ for any $n \geq n_0$, because otherwise there exists $m \geq n_0$ such that $x_m > p + \varepsilon$ and $x_{m+1} < p - \varepsilon$ which is impossible in view of (5.2). Let us assume for instance that $x_n > p + \varepsilon$ for any $n \geq n_0$. Then, from (5.1) and the fact that $g(x) < 0$ on $(p, 1]$, we deduce that

$$x_{n+1} - x_n = \frac{1}{n+1} g(x_n) \leq -\frac{\delta}{n+1}, \quad \forall n \geq n_0.$$

Therefore, for any $n > n_0$, we have the inequality

$$x_n - x_{n_0} \leq -\delta \sum_{k=n_0}^{n-1} \frac{1}{k+1}$$

which implies $\lim_{n \rightarrow \infty} x_n = -\infty$ contradicting the fact that $\{x_n\}$ is bounded. We therefore conclude that there exists $n_1 \geq n_0$ such that $x_{n_1} \in [p - \varepsilon, p + \varepsilon]$. We will now prove that for any $n \geq n_0$, if $x_n \in [p - \varepsilon, p + \varepsilon]$ then $x_n \in [p - \varepsilon, p + \varepsilon]$. Let $n \geq n_0$ such that $x_n \in [p - \varepsilon, p + \varepsilon]$. We consider the two possible cases:

The first case $x_n \in [0, p + \varepsilon]$. We have $g(x_n) \leq 0$, then $x_{n+1} = x_n + \theta_n g(x_n) \leq x_n$. Combining this inequality with the fact that $|x_{n+1} - x_n| < \varepsilon$, we deduce that $x_{n+1} \in [p - \varepsilon, p + \varepsilon]$.

The second case $x_n \in [p - \varepsilon, 0]$. We have $g(x_n) \geq 0$, then $x_{n+1} \geq x_n$. Hence, by using again the fact $|x_{n+1} - x_n| < \varepsilon$, we also deduce that $x_{n+1} \in [p - \varepsilon, p + \varepsilon]$. We deduce by induction that for any $n \geq n_1$, $x_n \in [p - \varepsilon, p + \varepsilon]$. This means that $x_n \rightarrow p$ as $n \rightarrow \infty$ and therefore completes the proof of the theorem. $\square$

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